Drawing Outer 1-planar Graphs with Few Slopes*

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Abstract. A graph is outer 1-planar if it admits a drawing where each vertex is on the outer face and each edge is crossed by at most another edge. Outer 1-planar graphs are a superclass of the outerplanar graphs and a subclass of the partial 3-trees. We show that an outer 1-planar graph G of bounded degree Δ admits an outer 1-planar straight-line drawing that uses $O(\Delta)$ different slopes, which extends a previous result by Knauer *et al.* about the planar slope number of outerplanar graphs (CGTA, 2014). We also show that $O(\Delta^2)$ slopes suffice to construct a crossing-free straight-line drawing of G; the best known upper bound on the planar slope number of planar partial 3-trees of bounded degree Δ is $O(\Delta^5)$ and is proved by Jelínek *et al.* (Graphs and Combinatorics, 2013).

1 Introduction

The *slope number* of a graph G is defined as the minimum number of distinct edge slopes required to construct a straight-line drawing of G. Minimizing the number of slopes used in a straight-line graph drawing is a desirable aesthetic requirement and an interesting theoretical problem which has received considerable attention since its first definition by Wade and Chu [21]. Let Δ be the maximum degree of a graph G and let m be the number of edges of G, clearly the slope number of G is at least $\frac{\Delta}{2}$ and at most m.

For non-planar graphs, there exist graphs with $\Delta \geq 5$ whose slope number is unbounded (with respect to Δ) [3,19], while the slope number of graphs with $\Delta = 4$ is unknown, and the slope number of graphs with $\Delta = 3$ is four [18].

Concerning planar graphs, the *planar slope number* of a planar graph G is defined as the minimum number of distinct slopes required by any planar straight-line drawing of G (see, e.g., [9]). Keszegh, Pach and Pálvölgyi [14] prove that $O(2^{O(\Delta)})$ is an upper bound and that $3\Delta - 6$ is a lower bound for the planar graphs of bounded degree Δ. The gap between upper and lower bound has been reduced for special families of planar graphs with bounded degree. Knauer, Micek and Walczak [15] prove that an outerplanar graph of bounded degree $\Delta \geq 4$ admits an outerplanar straight-line drawing that uses at most $\Delta - 1$ distinct edge slopes, and this bound is tight. Jelínek *et al.* [13] prove that the slope number of the planar partial 3-trees of bounded degree Δ is $O(\Delta^5)$, while in [17] it is proved that all partial 2-trees of bounded degree Δ have $O(\Delta)$ slope number. Di Giacomo *et al.* [7] show that planar graphs of bounded degree $\Delta \leq 3$ and at least five vertices have planar slope number four, which is worst case optimal.

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The research in this paper is motivated by the following observations. The fact that the best known upper bound on the slope number is $O(\Delta^5)$ for planar partial 3-trees while it is $O(\Delta)$ for partial 2-trees suggests to further investigate the planar slope number of those planar graphs whose treewidth is at most three. Also, the fact that non-planar drawings may require a number of slopes that is unbounded in Δ while the planar slope number of planar graphs is bounded in Δ, suggests to study how many slopes may be needed to construct straight-line drawings that are "nearly-planar" in some sense, i.e. where only some types of edge crossing are allowed.

We study *outer 1-planar graphs* that are graphs which admit drawings where each edge is crossed at most once and each vertex is on the boundary of the outer face (see, e.g., [2,5,11]). In 2013, Auer *et al.* [2], and independently Hong *et al.* [11], presented a linear-time algorithm to test outer 1-planarity. Both algorithms produce an outer 1-planar embedding of the graph if it exists. Given an outer 1-planar graph G, we define the *outer 1-planar slope number* of G, as the minimum number of distinct slopes required by any outer 1-planar straight-line drawing of G. We prove the following results.

1. The outer 1-planar slope number of outer 1-planar graphs with maximum degree Δ is at most $6\Delta + 12$ (Section 3). Since outerplanar drawings are a special case of the outer 1-planar drawings, this result extends the above mentioned upper bound on the planar slope number of outerplanar graphs [15].
2. Outer 1-planar drawings are known to be planar graphs and they have treewidth at most three [2]. We study crossing-free straight-line drawings of outer 1-planar graphs of bounded degree Δ and show an $O(\Delta^2)$ upper bound to the planar slope number (Section 4). Hence, for this special family, we are able to reduce the general $O(\Delta^5)$ upper bound [13].

Our results are constructive and give rise to linear-time drawing algorithms. Also, it may be worth recalling that the study of the 1-planar graphs, i.e. those graphs that can be drawn with at most one crossing per edge, has received a lot of interest in the recent graph drawing literature (see, e.g., [1,4,8,10,12,16,20]).

In Section 2 we introduce preliminaries. Section 5 lists some open problems. For reasons of space some proofs are sketched or omitted.

2 Preliminaries and Basic Definitions

A *drawing* Γ of a graph $G = (V,E)$ is a mapping of the vertices in V to points of the plane and of the edges in E to Jordan arcs connecting their corresponding endpoints but not passing through any other vertex. Also, no two edges that share an endpoint cross. Γ is a *straight-line drawing* if every edge is mapped to a straight-line segment. Γ is a *planar drawing* if no edge is crossed; it is a *1-planar drawing* if each edge is crossed at most once. A *planar graph* is a graph that admits a planar drawing; a *1-planar graph* is a graph that admits a 1-planar drawing.

A planar drawing of a graph partitions the plane into topologically connected regions, called *faces*. The unbounded region is called the *outer face*. A *planar embedding* of a planar graph is an equivalence class of planar drawings that define the same set of faces. The concept of planar embedding can be extended to 1-planar drawings as follows. In a

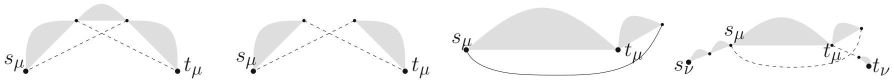

Fig. 1. Illustration of Properties 2– 4. The pertinent graph of: (a) an *R*-node μ; (b) a *P*-node μ (case (ii) of Property 3);. (c) a *P*-node μ that is *AOS* with respect to s_μ; (d) An *S*-node ν with a child μ that is *AOS* with respect to s_μ. Dashed edges cross in the embedding of the graph.

1-planar drawing Γ of a graph G each crossed edge is divided into two *edge fragments*. Also in this case, Γ partitions the plane into topologically connected regions, which we call faces. A 1*-planar embedding* of a 1-planar graph is an equivalence class of 1-planar drawings that define the same set of faces. An *outer 1-planar drawing* is a 1-planar drawing with all vertices on the outer face. An *outer 1-plane graph G* is a graph with a given outer 1-planar embedding.

The slope s of a line ℓ is the angle that an horizontal line needs to be rotated counter-clockwise in order to make it overlap with ℓ. The slope of a segment representing an edge in a straight-line drawing is the slope of the supporting line containing the segment.

Our drawing techniques use *SPQR*-trees, whose definition can be found in [6].

Properties of Outer 1-planar Graphs. The structural properties of outer 1-planar graphs have been studied in [2,11]. In this paragraph we derive properties that hold in the fixed outer 1-planar embedding setting and that easily follow from the results in [11]. In Section 4 we will use the same properties explaining how to adapt them to the planar embedding setting. The following property can be found as Lemma 1 in [11].

Property 1. Let G be an outer 1-plane graph. If G is triconnected, then it is isomorphic to K_4 and it has exactly one crossing.

In what follows we consider a biconnected outer 1-plane graph G and its *SPQR*-tree T. Let μ be a node of T, the *pertinent graph* G_μ of μ is the subgraph of G whose *SPQR*-tree (with respect to the reference edge e of μ) is the subtree of T rooted at μ. Notice that the edge e is not part of G_μ. From now on we assume G_μ to be an outer 1-plane graph using the embedding induced from G. We give the following definition [11].

Definition 1. *A node μ of T is one sided with respect to its poles s_μ and t_μ, or simply OS, if the edge (s_μ, t_μ) is on the outer face of G_μ.*

Furthermore, we consider T to be rooted at a Q-node ρ whose (only) child is denoted by ξ. In particular, we choose ρ to be associated with an edge that is not crossed and that belongs to the boundary of the outer face of G. It can be shown that such an edge always exists. This choice implies that ξ is *OS* by definition. The next property derives from Lemma 5 in [11] and defines the structure of the skeleton of *R*-nodes, see also Figure 1(a).

Property 2. Let μ be an *R*-node of T. Then: (i) The skeleton $\sigma(\mu)$ is isomorphic to K_4 and it has one crossing; (ii) The children of μ are all *OS*; (iii) Two children of μ are Q-nodes whose associated edges cross each other in G_μ.

Observe that if μ is an R-node of T, then it is always OS. In order to handle P-nodes, we first need to define a special kind of S-nodes [11].

Definition 2. *Let μ be an S-node of T. Let η be the unique child of μ having s_μ as a pole, and let η' be the unique child of μ having t_μ as a pole. Node μ has a tail at s_μ (t_μ), if η (η') is a Q-node.*

The next property derives from Lemma 6 in [11], see also Figure 1(b).

Property 3. Let μ be an OS P-node of T. One of the following cases holds: (i) μ has two children one of which is a Q-node and the other one is OS; (ii) μ has two children and none of them is a Q-node. Then both are OS S-nodes, one of them has a tail at s_μ, and the other one has a tail at t_μ. Also, the two edges associated with these two tails cross each other in G; (iii) μ has three children and one of them is a Q-node. For the remaining two children case (ii) applies.

Property 3 is restricted to P-nodes that are OS. However, an internal P-node μ (different from ξ) might not have the edge (s_μ, t_μ) on the outer face of G_μ [11], see also Figure 1(c) for an illustration.

Definition 3. *Let μ be a P-node of T different from ξ. Node μ is almost one sided with respect to s_μ (t_μ), or simply AOS with respect to s_μ (t_μ), if μ has $2 \leq k \leq 4$ children, one of them is an S-node with a tail at s_μ (t_μ), and for the remaining children one of the following cases applies: (i) If $k = 2$, then the other child is OS; (ii) If $k > 2$, all and only the cases in Property 3 can apply for the remaining $k-1$ children.*

Let μ be AOS with respect to s_μ (t_μ), then, in order to guarantee that the graph is outer 1-planar, the edge associated with the tail at s_μ (t_μ) crosses another edge, represented by a Q-node ψ in T, having t_μ (s_μ) as an end-vertex. This implies that in fact, μ and ψ are two children of an S-node ν in T [11] (see also Figure 1(d)). This observation will be used in Section 3 and in the next property, that is derived from Lemma 7 in [11].

Property 4. Let μ be an S-node of T. Let $\eta_1, \eta_2, \ldots, \eta_k$ be the k children of μ in T, such that $t_{\eta_{i-1}} = s_{\eta_i}$, for $i = 2, \ldots, k$. For each $1 \leq i \leq k$, one of the following cases applies: (i) η_i is OS; (ii) η_i is AOS with respect to s_{η_i} and η_{i+1} ($i < k$) is a Q-node; (iii) η_i is AOS with respect to t_{η_i} and η_{i-1} ($i > 1$) is a Q-node.

An immediate observation from these properties is that every node μ of T different from ξ is OS if it is an S- or R-node, while it is either OS or AOS if it is a P-node.

3 The Outer 1-planar Slope Number

In this section we first present an algorithm, called BO1P-DRAWER, that takes as input a biconnected outer 1-plane graph G with maximum degree Δ, and returns a straight-line drawing Γ of G that uses at most 6Δ slopes. This result is then extended to simply connected graphs with a number of slopes equal to $6\Delta + 12$.

A Universal Set of Slopes. We define a universal set of slopes used by algorithm BO1P-DRAWER to draw every biconnected outer 1-plane graph G with maximum degree Δ. Let $\alpha = \frac{\pi}{2\Delta}$ and observe that $0 < \alpha \leq \frac{\pi}{6}$ when $\Delta \geq 3$. We call *blue slopes* the set of slopes defined as $b_i = (i-1)\alpha$, for $i = 1,2,\ldots,2\Delta$. For each of the 2Δ blue slopes, we also define two *red slopes* as $r_i^- = b_i - \varepsilon$ and $r_i^+ = b_i + \varepsilon$, for $i = 1,2,\ldots,2\Delta$, where the value of ε only depends on Δ. The union of the blue and red slopes defines the universal set of slopes $\mathscr{S}_\Delta$ of size 6Δ. We choose ε as follows: $\varepsilon = \alpha - \arctan\left(\frac{\tan(\alpha)}{1+2\tan(2\alpha)\tan(\alpha)-2\tan(\alpha)\tan(\alpha)}\right)$. The reason of this choice will be clarified in the proof of Lemma 3. Clearly, ε depends only on Δ and it is possible to see that it is a positive value.

Algorithm Overview. Algorithm BO1P-DRAWER exploits *SPQR*-trees and the structural properties presented in Section 2. It takes as input a biconnected outer 1-plane graph G with maximum degree Δ and returns a straight-line drawing Γ of G that uses only slopes in $\mathscr{S}_\Delta$. We first construct the *SPQR*-tree T rooted at a *Q*-node ρ, whose (only) child is denoted by ξ. Moreover, the edge associated with ρ is not crossed and belongs to the boundary of the outer face of G. Then we draw G by visiting T bottom-up, handling ρ and ξ together as a special case. At each step we process an internal node μ of T and compute a drawing Γ_μ of its pertinent graph G_μ by properly combining the already computed drawings of the pertinent graphs of the children of μ. Let s_μ and t_μ be the poles of μ. With a slight overload of notation for the symbol Δ, we denote by $\Delta(s_\mu)$ and $\Delta(t_\mu)$ the degree of s_μ and t_μ in G_μ, respectively. For each drawing Γ_μ we aim at maintaining the following three invariants. **I1.** Γ_μ is outer 1-plane with respect to the embedding of G_μ. **I2.** Γ_μ uses only slopes in $\mathscr{S}_\Delta$. **I3.** Γ_μ is contained in a triangle τ_μ such that s_μ and t_μ are placed at the corners of its base. Also, $\beta_\mu < (\Delta(s_\mu)+1)\alpha$ and $\gamma_\mu < (\Delta(t_\mu)+1)\alpha$, where β_μ and γ_μ are the internal angles of τ_μ at s_μ and t_μ.

We now explain how to compute a drawing Γ_μ of G_μ, by combining the drawings $\Gamma_{\eta_1}, \Gamma_{\eta_2}, \ldots, \Gamma_{\eta_h}$ of the pertinent graphs $G_{\eta_1}, G_{\eta_2}, \ldots, G_{\eta_h}$ of the children $\eta_1, \eta_2, \ldots, \eta_h$ of μ. To this aim, the drawings $\Gamma_{\eta_1}, \Gamma_{\eta_2}, \ldots, \Gamma_{\eta_h}$ are possibly manipulated. First, observe that the triangle τ_{η_j} $(1 \leq j \leq h)$ can be arbitrarily scaled without modifying the slopes used in Γ_{η_j}. Furthermore, due to the symmetric choice of the blue and red slopes, if we rotate τ_{η_j} by an angle $c \cdot \alpha$, with c integer, the resulting drawing maintains invariant $I2$. Namely each blue slope b_i, for $i = 1,2,\ldots,2\Delta$, used in τ_{η_j} will be transformed in another blue slope $b_{i+c} = b_i + c \cdot \alpha = (i-1+c)\alpha$, where $i+c$ is considered modulo 2Δ. Similarly, any red slope will be transformed into another red slope. Moreover, let η_1 and η_2 be two children of μ. When we draw G_{η_1} and G_{η_2}, although they may share one or both the poles, we consider each graph to have its own copy of its poles. Then, when computing Γ_μ, we say that we *attach* Γ_{η_1} to Γ_{η_2} if they share either two poles (this is always true when μ is a *P*-node) or one pole (this may happen when μ is either an *S*- or *R*-node), meaning that we may scale, shift and rotate Γ_{η_1} or Γ_{η_2} in such a way that the points representing the shared poles on the drawing coincide.

As observed in Section 2, all the internal nodes of T are *OS* except for some *P*-nodes which are *AOS*. Let μ be any of these *P*-nodes, we know that μ is one of the children of an *S*-node, say ν, and it shares a pole with a *Q*-node, denoted by η (also a children of ν). We replace μ and η in T with a new node φ, that, for the sake of description, is called an *S^*-node*. Also, the children of μ become children of φ. If μ and η were

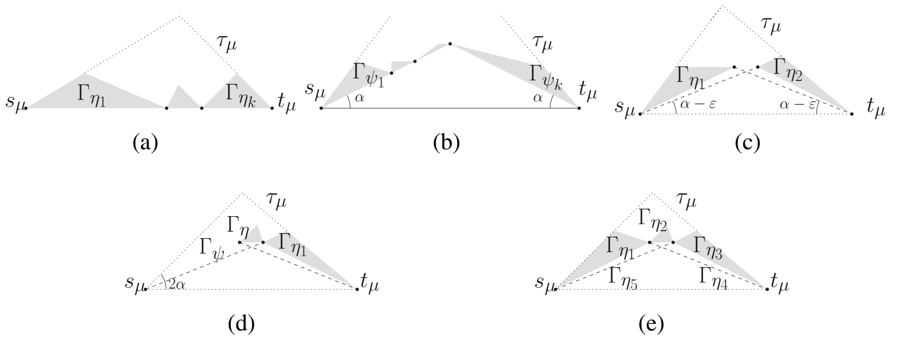

Fig. 2. The drawing of the pertinent graph of: (a) an *S*-node; (b) a *P*-node with two children such that one is a *Q*-node and the other one is an *S*-node; (c) a *P*-node with two children such that none of them is a *Q*-node; (d) an S^*-node; (e) an *R*-node. Edges drawn with red slopes are dashed.

the only two children of ν, then we also replace ν with φ. The pertinent graph of φ is $G_\varphi = G_\mu \cup G_\eta$, while the reference edge of φ is (s_μ, t_η), if μ is *AOS* with respect to s_μ, or (s_η, t_μ), if μ is *AOS* with respect to t_μ. It is easy to see that φ is *OS*. By means of this transformation we can consider only *P*-nodes that are *OS*. Similarly we can handle just *S*-nodes whose children are *OS*. In what follows we distinguish between *S*-, *P*-, S^*-, and *R*-nodes different from ξ.

Lemma 1. *Let μ be an S-node different from ξ. Then G_μ admits a straight-line drawing Γ_μ that respects Invariants* ***I1.**, **I2.** and **I3**.*

Proof sketch: The drawings of the pertinent graphs of the children $\eta_1, \eta_2, \ldots, \eta_k$ of μ are attached to each other as shown in Figure 2(a). Clearly all invariants hold. □

Lemma 2. *Let μ be a P-node different from ξ. Then G_μ admits a straight-line drawing Γ_μ that respects Invariants* ***I1.**, **I2.** and **I3**.*

Proof sketch: Recall that, thanks to the definition of S^*-nodes, here we need to only handle only *P*-nodes that are *OS*. By Property 3, one of the following cases applies: (i) μ has two children one of which is a *Q*-node and the other one is *OS*. (ii) μ has two children and none of them is a *Q*-node. Then both are *OS* *S*-nodes, one of them has a tail at s_μ, and the other one has a tail at t_μ. Also, the two edges associated with these two tails cross each other in *G*. (iii) μ has three children and one of them is a *Q*-node. For the remaining two children case (ii) applies.

Case (i) can be easily handled as shown in Figure 2(b). Consider case (ii) and let η_1 be the child of μ that is an *S*-node with a tail at t_μ, and η_2 be the child of μ that is an *S*-node with a tail at s_μ. Refer to Figure 2(c). Recall that $s_{\eta_1} = s_{\eta_2} = s_\mu$ and $t_{\eta_1} = t_{\eta_2} = t_\mu$. We modify the drawing Γ_{η_1} as follows. We first rotate Γ_{η_1} so that the segment $\overline{s_{\eta_1} t_{\eta_1}}$ uses the blue slope b_2. Then we redraw the tail of η_1 using the red slope $r^+_{2\Delta} = b_{2\Delta} + \varepsilon$ and so that s_{η_1} and t_{η_1} are horizontally aligned. Similarly, we modify the drawing Γ_{η_2}. We rotate Γ_{η_2} so that the segment $\overline{s_{\eta_2} t_{\eta_2}}$ uses the blue slope $b_{2\Delta}$ and redraw the tail of η_2

using the red slope $r_2^- = b_2 - \varepsilon$ and so that s_{η_2} and t_{η_2} are horizontally aligned. Finally, we attach Γ_{η_1} and Γ_{η_2} (possibly scaling one of them). Invariants **I1.** and **I2.** hold by construction. Also, Γ_μ is contained in a triangle τ_μ such that s_μ and t_μ are placed at the corners of its base. Moreover, we have that $\Delta(s_\mu) = \Delta(s_{\eta_1}) + 1$, and $\beta_\mu = \beta_{\eta_1} + \alpha < \Delta(s_{\eta_1} + 1)\alpha + \alpha = \Delta(s_{\eta_1} + 2)\alpha = \Delta(s_\mu + 1)\alpha$. Similarly, $\Delta(t_\mu) = \Delta(t_{\eta_2}) + 1$, and $\gamma_\mu = \gamma_{\eta_2} + \alpha < \Delta(t_{\eta_2} + 1)\alpha + \alpha = \Delta(t_{\eta_2} + 2)\alpha = \Delta(t_\mu + 1)\alpha$. Hence, Invariant **I3.** holds. In case (iii) we can use the same construction as in case (ii). Notice that the edge (s_μ, t_μ) can be safely drawn using the horizontal blue slope b_1. All invariants hold. □

Lemma 3. *Let μ be an S^*-node different from ξ. Then G_μ admits a straight-line drawing Γ_μ that respects Invariants **I1.**, **I2.** and **I3.***

Proof. Refer to Figure 2(d). Denote by η the child of μ that is an S-node with a tail at either s_μ or t_μ. Suppose that η has a tail at t_μ (the case when the tail is at s_μ is symmetric). Denote by ψ the child of μ that is a Q-node having $t_\psi = s_\eta$ and $s_\psi = s_\mu$ as poles. Finally denote by $\eta_1, \eta_2, \ldots, \eta_k$ the remaining children of μ. Recall that $s_{\eta_1} = s_{\eta_i} = s_{\eta_k}$ and that $t_{\eta_1} = t_{\eta_i} = t_{\eta_k}$. If $k = 1$, we first rotate Γ_{η_1} so that the segment $\overline{s_{\eta_1} t_{\eta_1}}$ uses the blue slope $b_{2\Delta}$. If $k > 1$, we combine the drawings $\Gamma_{\eta_1}, \Gamma_{\eta_2}, \ldots, \Gamma_{\eta_k}$ with the same technique described for P-nodes (recall that indeed they were children of a P-node before the creation of the S^*-node), and, again, we rotate the resulting drawing so that the base of its bounding triangle uses the blue slope $b_{2\Delta}$. Then we attach Γ_η to Γ_{η_1} (after Γ_η has been horizontally flipped). Also, we scale Γ_η so that its tail can be redrawn by using the red slope $r_{2\Delta}^+$ and such that $t_\eta = t_\mu$ coincides with $t_{\eta_1} = t_{\eta_k}$. Finally, we redraw the edge associated with ψ, starting from the point representing $t_\psi = s_\eta$, using the red slope r_2^- and stretch it enough that $s_\psi = s_\mu$ and t_μ are horizontally aligned. See also Figure 2(d) for an illustration. Invariants **I1.** and **I2.** hold by construction. Consider now Invariant **I3.**. By construction Γ_μ is contained in a triangle τ_μ such that s_μ and t_μ are placed at the corners of its base. For the sake of description, in what follow we still denote by Γ_η the drawing of G_η minus the tail of η (i.e., minus an edge), and as τ_η the surrounding triangle of Γ_η. To prove the second part of Invariant **I3.**, we should prove that the line ℓ passing through s_μ with slope $b_3 = 2\alpha$ does not cross the drawing of Γ_η, i.e., is such that Γ_η is placed in the half-plane $\mathscr{H}$ defined by ℓ and containing the segment $\overline{s_\mu t_\mu}$. Denote by δx the horizontal distance between the point where s_μ is drawn and the leftmost endpoint of τ_η. Also, denote by h_η the height of τ_η. Our condition is satisfied if the following inequality holds $\tan(2\alpha)\delta x \geq \tan(\alpha)\delta x + h_\eta$. Let w_η be the length of the base of τ_η, in the worst case (the case that maximizes h_η), we have that $h_\eta = \frac{w_\eta}{2}\frac{1}{\tan(\alpha)}$, which means that the degree of the two vertices placed as endpoints of the base of τ_η is Δ. Moreover, it is possible to see that $w_\eta = \frac{\tan(\alpha)\delta x - \tan(\alpha-\varepsilon)\delta x}{\tan(\alpha-\varepsilon)}$. Substituting w_η in h_η and h_η in the above inequality we have: $\tan(2\alpha) \geq \tan(\alpha) + \frac{\tan(\alpha)-\tan(\alpha-\varepsilon)}{2\tan(\alpha-\varepsilon)\tan(\alpha)}$. With some manipulation we get: $\tan(\alpha-\varepsilon) \geq \frac{\tan(\alpha)}{2\tan(2\alpha)\tan(\alpha)-2\tan(\alpha)\tan(\alpha)+1}$. Now, since the tangent function is strictly increasing in $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have: $\varepsilon \leq \alpha - \arctan\left(\frac{\tan(\alpha)}{2\tan(2\alpha)\tan(\alpha)-2\tan(\alpha)\tan(\alpha)+1}\right)$. Since the value of ε has been chosen equal to the right-hand side of the above inequality, the inequality holds. Hence, $\beta_\mu < 2\alpha = (\Delta(s_\mu)+1)\alpha$ (since $\Delta(s_\mu) = 1$). With a symmetric argument one can prove that the line ℓ' passing through t_μ with slope $b_{2\Delta-1} = \frac{(\Delta-1)\pi}{\Delta}$ does not cross the drawing of Γ_η. Since $\Delta(t_\mu) = \Delta(t_{\eta_k}) + 1$, and $\gamma_\mu = \gamma_{\eta_k} + \alpha < (\Delta(t_{\eta_k}) + 1)\alpha + \alpha = (\Delta(t_{\eta_k}) + 2)\alpha = (\Delta(t_\mu) + 1)\alpha$, Invariant **I3.** holds. □

Lemma 4. *Let μ be an R-node different from ξ. Then G_μ admits a straight-line drawing Γ_μ that respects Invariants **I1.**, **I2.** and **I3**.*

Proof. Refer to Figure 2(e). Recall that, by Property 2, (i) the skeleton $\sigma(\mu)$ is isomorphic to K_4 and it has one crossing; (ii) the children of μ are all *OS*; (iii) two children of μ are Q-nodes whose associated edges cross each other in G_μ. Hence, denote by η_1, η_2, η_3 the three children of μ whose associated virtual edges lie on the boundary of the outer face of $\sigma(\mu)$ with $s_\mu = s_{\eta_1}$, $t_{\eta_1} = s_{\eta_2}$, $t_{\eta_2} = s_{\eta_3}$, and $t_{\eta_3} = t_\mu$. Also, denote by η_4 and η_5 the two children of μ that are Q-nodes whose associated edges cross each other in G_μ, and so that the poles of η_4 coincides with t_{η_1} and t_{η_3}, while the poles of η_5 coincides with t_{η_2} and s_{η_1}. We rotate Γ_{η_1} in such a way that the segment $\overline{s_{\nu_1} t_{\nu_1}}$ uses the blue slope b_2. Similarly, we rotate Γ_{η_3} in such a way that the segment $\overline{s_{\eta_3} t_{\eta_3}}$ uses the blue slope $b_{2\Delta}$. Furthermore, we scale one of the two drawings so that t_{η_1} and s_{η_3} are horizontally aligned. Moreover, we redraw the edge associated with η_4 by using the red slope $r^+_{2\Delta}$ and we redraw the edge associated with η_5 by using the red slope r^-_2. Observe that, attaching η_4 and η_5 to η_1 and η_3, the length of the segment $\overline{t_{\eta_1} s_{\eta_3}}$ is determined. Thus, we attach Γ_{η_2} so that s_{η_2} coincides with t_{η_1} and that t_{η_2} coincides with s_{η_3}.

It is easy to see that Invariant **I1.** and **I2.** are respected by construction. Concerning Invariant **I3.**, again by construction Γ_μ is contained in a triangle τ_μ such that s_μ and t_μ are placed at the corners of its base. Moreover, with the same argument used in the proof of Lemma 3, one can show that $\beta_\mu = \beta_{\eta_1} + \alpha$ and that $\gamma_\mu = \gamma_{\eta_3} + \alpha$. Since $\Delta(s_\mu) = \Delta(\eta_1) + 1$ and $\Delta(t_\mu) = \Delta(\eta_3) + 1$, Invariant **I3.** holds. □

Lemma 5. *Let ρ be the root of T and let ξ be its unique child. Graph $G = G_\rho \cup G_\xi$ admits a straight-line drawing Γ that respects Invariants **I1.**, **I2.** and **I3**.*

Proof sketch: It is possible to prove that at least one edge (s,t) of the outer face of G is not crossed. If we root T at the Q-node associated with (s,t), the root's child ξ is *OS* and a drawing of $G_\rho \cup G_\xi$ can be computed as in Lemmas 1, 2, 3, and 4. □

Lemma 6. *Let G be a biconnected outer* 1*-plane graph with n vertices and with maximum degree Δ. G admits an outer* 1*-planar straight-line drawing that maintains the given outer* 1*-planar embedding, and that uses at most* 6Δ *slopes. Also, this drawing can be computed in $O(n)$ time.*

Proof sketch: By Lemmas 1, 2, 3, 4, and 5, G has an outer 1-planar straight-line drawing that maintains the embedding, with at most 6Δ slopes. □

A simply connected outerplane graph can be augmented (in linear time) into a biconnected outerplane graph by adding edges so that the maximum degree is increased by at most two. This technique can be directly applied also to outer 1-plane graphs.

Theorem 1. *Let G be an outer* 1*-plane graph with n vertices and with maximum degree Δ. G admits an outer* 1*-planar straight-line drawing that maintains the given outer* 1*-planar embedding, and that uses at most* $6\Delta + 12$ *slopes. Also, this drawing can be computed in $O(n)$ time.*

4 The Planar Slope Number

In this section we describe an algorithm, called BP-DRAWER, that computes a planar drawing of an outer 1-planar graph G, using at most $4\Delta^2 - 4\Delta$ slopes. This result is then extended to simply connected graphs with a number of slopes equal to $4\Delta^2 + 12\Delta + 8$.

A Universal Set of Slopes. We start by defining a universal set of slopes that are used by algorithm BP-DRAWER. Let $\theta = \frac{\pi}{4\Delta}$ and observe that $0 < \theta \leq \frac{\pi}{12}$ when $\Delta \geq 3$. We call *green slopes* the set of slopes defined as $g_i = (i-1)\theta$, for $i = 1, 2, \ldots, 4\Delta$. For each green slope g_i, we define $\Delta - 1$ *yellow slopes* as $y_{i,j} = g_i + \arctan\left(\frac{\tan(g_{4\Delta})\tan(g_3)}{\tan(g_j)}\right)$ with $j = 3\Delta, \ldots, 4\Delta - 2$. The reason of this choice will be clarified in the proof of Lemma 10. The union of the green and yellow slopes defines the universal set of slopes $\mathscr{T}_\Delta$. It is possible to see that $g_i < y_{i,j} < g_{i+1}$, for each $1 \leq i < 4\Delta$ and $3\Delta \leq j \leq 4\Delta - 2$.

Algorithm Overview. Algorithm BP-DRAWER takes as input a biconnected outer 1-plane graph G with maximum degree Δ and returns a planar straight-line drawing Γ of G that uses only slopes in $\mathscr{T}_\Delta$. As in Section 3 we construct the *SPQR*-tree T of G rooted at a Q-node associated with an edge that is not crossed and belongs to the boundary of the outer face of G in the outer 1-planar embedding of G. Then we draw G by visiting T bottom-up. At each internal node μ of T we compute a drawing Γ_μ of G_μ by combining the already computed drawings of the pertinent graphs of the children of μ. For each drawing Γ_μ we maintain the following three invariants: **Ia.** Γ_μ is planar. **Ib.** Γ_μ uses only slopes in $\mathscr{T}_\Delta$. **Ic.** Γ_μ is contained in a triangle τ_μ such that s_μ and t_μ are placed at the corners of its base. Also, $\beta_\mu < (\Delta(s_\mu) - 1)\theta$ and $\gamma_\mu < (\Delta(t_\mu) - 1)\theta$, where β_μ and γ_μ are the internal angles of τ_μ at s_μ and t_μ, respectively.

As in Section 3 the root ρ of T and its unique child ξ will be handled in a special way. Also, in order to construct Γ_μ we may shift, scale and rotate the drawings of the pertinent graphs of the children of μ. We observe that if we rotate τ_μ by an angle $c \cdot \theta$, with c integer, the resulting drawing maintains invariant *Ib*. Namely each green slope g_i, for $i = 1, 2, \ldots, 4\Delta$, used in τ_μ will be transformed in another green slope $g_{i+c} = g_i + c \cdot \theta = (i - 1 + c)\theta$, where $i + c$ is considered modulo 4Δ. Similarly, any yellow slope $y_{i,j}$ will be transformed into another yellow slope $y_{i+c,j}$.

Before describing how the drawing of the pertinent graph of each node μ is obtained by combining the drawing of the pertinent graphs of its children, we observe that the structural properties described in Properties 2, 3, or 4 hold, depending on the type of μ. However, since we want to produce a planar drawing, our algorithm embeds each pertinent graph in a planar way. One of the consequence of this fact is that we no longer need to introduce S^*-nodes; namely, the P-nodes that are *AOS* in the outer 1-planar embedding must be embedded in a planar way and therefore they do not need to be handled in a special way anymore. On the other hand, we need to distinguish between R-nodes whose poles are adjacent in G and R-nodes whose poles are not adjacent in G. For this reason we introduce R^*-nodes. Let μ be an R-node; if the poles s_μ and t_μ of μ are adjacent in G, then the parent ν of μ is a P-node that has (at least) another child η that is a Q-node (the edge associated with η is (s_μ, t_μ)). We replace μ and η in T with a new node φ, that, for the sake of description, is called an *R^*-node*. Also, the children of μ become children of φ. If μ and η were the only two children of ν, then we also

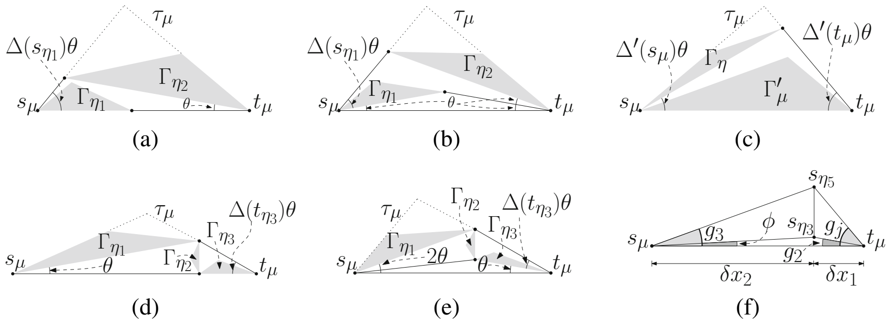

Fig. 3. The planar drawing of the pertinent graph of: (a) a *P*-node with two children such that none of them is a *Q*-node; (b) a *P*-node with three children, one of which is a *Q*-node; (c) a *P*-node that is *AOS* in the outer 1-planar embedding of *G*; (d) an *R*-node; (e) an R^*-node. (f) Illustration for the proof of Lemma 10.

replace v with φ. The pertinent graph of φ is $G_\varphi = G_\mu \cup G_\eta$, and the reference edge of φ is (s_μ, t_μ). We now explain how the different types of node are handled.

The proof of next lemmas are omitted. An illustration of how Γ_μ is constructed is shown in Figures 2(a) and 3.

Lemma 7. *Let μ be an S-node different from ξ. Then G_μ admits a straight-line drawing Γ_μ that respects Invariants* ***Ia., Ib.*** *and* ***Ic****.*

Lemma 8. *Let μ be a P-node different from ξ. Then G_μ admits a straight-line drawing Γ_μ that respects Invariants* ***Ia., Ib.*** *and* ***Ic****.*

Lemma 9. *Let μ be an R-node different from ξ. Then G_μ admits a straight-line drawing Γ_μ that respects Invariants* ***Ia., Ib.*** *and* ***Ic****.*

Lemma 10. *Let μ be an R^*-node different from ξ. Then G_μ admits a straight-line drawing Γ_μ that respects Invariants* ***Ia., Ib.*** *and* ***Ic****.*

Proof. Since μ is an R^*-node, it is obtained by merging an R-node μ' and a Q-node representing the edge $(s_{\mu'}, t_{\mu'})$. By Property 2, the skeleton $\sigma(\mu')$ of μ' is isomorphic to K_4 and two children of μ' are Q-nodes. The two edges corresponding to these Q-nodes do not share an end vertex and each one of them is incident to a distinct pole of μ. Let $\eta_1, \eta_2, \eta_3, \eta_4$, and η_5 be the children of μ'; we assume that η_4 and η_5 are the two Q-nodes. Also, μ has a sixth child η_6 that is a Q-node corresponding to the edge (s_μ, t_μ). We assume that $s_\mu = s_{\eta_1} = s_{\eta_4}$, $t_\mu = t_{\eta_3} = t_{\eta_5}$, $t_{\eta_1} = t_{\eta_2} = s_{\eta_5}$, and $t_{\eta_4} = s_{\eta_2} = s_{\eta_3}$. We construct a drawing of G_μ as follows (see Figure 3(e)). We rotate Γ_{η_3} so that the segment $\overline{s_{\eta_3} t_{\eta_3}}$ uses the green slope $g_{4\Delta}$, and draw the edge associated with η_5 as a segment whose slope is the green slope $(4\Delta - \Delta(t_{\eta_3}))\theta$ and whose length is such that s_{η_5} is vertically aligned with s_{η_3}. We rotate Γ_{η_2} so that the segment $\overline{s_{\eta_2} t_{\eta_2}}$ uses the green slope $g_{2\Delta+1} = \frac{\pi}{2}$. We then attach Γ_{η_2}, Γ_{η_3}, and Γ_{η_5} (possibly scaling some of them). We draw the edge corresponding to η_6 with the horizontal slope g_1 and stretch it so that

$s_{\eta_6} = s_\mu$ belongs to the line with slope g_2 passing through s_{η_5}. We now rotate Γ_{η_1} so that the segment $\overline{s_{\eta_1}t_{\eta_1}}$ uses the green slope g_2 and attach it to Γ_{η_5} and Γ_{η_6}. Finally, the edge corresponding to η_4 is drawn as the segment $\overline{s_\mu s_{\eta_3}}$. Invariant **Ia.** holds because the drawings Γ_{η_1}, Γ_{η_2}, Γ_{η_3}, Γ_{η_4}, Γ_{η_5}, and Γ_{η_6} do not intersect each other except at common endpoints. About this, let τ be the triangle defined by the three vertices s_μ, s_{η_3}, and s_{η_5}; it is easy to see that Γ_{η_2} is completely contained inside τ except for the segment $\overline{s_{\eta_3}s_{\eta_5}}$ that Γ_{η_2} shares with τ. Namely the angle inside τ at s_{η_3} is $\frac{\pi}{2} + \theta$, while the angle inside τ at s_{η_5} is at least $\frac{\pi}{4}$ (because the angle inside τ at s_μ is θ and $2\theta < \frac{\pi}{4}$). Since $\beta_{\eta_2} < \frac{\pi}{4}$ and $\gamma_{\eta_2} < \frac{\pi}{4}$, the triangle τ_{η_2} is completely inside τ except for the vertical side shared by the two triangles. Concerning Invariant **Ib.**, we observe that Γ_{η_1}, Γ_{η_2}, Γ_{η_3}, Γ_{η_4}, and Γ_{η_5} are rotated by an angle that is a multiple of θ and therefore **Ib.** holds by construction for each of them. We now show that the slope ϕ of the edge corresponding to η_4 is in fact either a green slope or a yellow one (refer to Figure 3(f)). Let δx_1 be the horizontal distance between s_{η_3} and t_μ and let δx_2 be the horizontal distance between s_μ and s_{η_3}. By simple trigonometry we have $\delta x_1 \tan(g_{4\Delta}) = \delta x_2 \tan(\phi)$ and $\delta x_1 \tan(g_j) = \delta x_2 \tan(g_3)$, where g_j is the slope of the segment representing the edge corresponding to η_5 (and therefore $j = 4\Delta - \Delta(t_{\eta_3})$). From the two previous equations we obtain $\tan(\phi) = \frac{\tan(g_{4\Delta})\tan(g_3)}{\tan(g_j)}$. Notice that $1 \leq \Delta(t_{\eta_3}) \leq \Delta$ and therefore $3\Delta \leq j \leq 4\Delta - 1$. If $j = 4\Delta - 1$, then $\tan(g_3) = -\tan(g_j)$ and $\tan(\phi) = -\tan(g_{4\Delta}) = \tan(g_2)$, hence $\phi = g_2$, i.e., ϕ is a green slope. Otherwise $\phi = \arctan\left(\frac{\tan(g_{4\Delta})\tan(g_3)}{\tan(g_j)}\right) =$ and therefore ϕ is the yellow slope $y_{1,j}$ (recall that $g_1 = 0$). Concerning Invariant **Ic.**, we have that $\Delta(s_\mu) = \Delta(s_{\eta_1}) + 2$ and $\Delta(t_\mu) = \Delta(t_{\eta_3}) + 2$. Moreover, $\beta_\mu = \beta_{\eta_1} + 2\theta \leq (\Delta(s_{\eta_1}) - 1)\theta + 2\theta = (\Delta(s_\mu) - 1)\theta$. Finally, $\gamma_\mu = \gamma_{\eta_3} + 2\theta \leq (\Delta(t_{\eta_3}) - 1)\theta + 2\theta = (\Delta(t_\mu) - 1)\theta$. □

Lemma 11. *Let ρ be the root of T and let ξ be its unique child. Graph $G = G_\rho \cup G_\xi$ admits a straight-line drawing Γ that respects Invariants **Ia.**, **Ib.** and **Ic**.*

By Lemmas 7, 8, 9, 10, and 11, we can prove the following lemma.

Lemma 12. *Let G be a biconnected outer 1-plane graph with n vertices and with maximum degree Δ. G admits a planar straight-line drawing that uses at most $4\Delta^2 - 4\Delta$ slopes. Also, this drawing can be computed in $O(n)$ time.*

The result above can be extended to simply connected outer 1-planar graph with the same technique described in Section 3. We obtain the following theorem.

Theorem 2. *Let G be an outer 1-plane graph with n vertices and with maximum degree Δ. G admits a planar straight-line drawing that uses at most $4\Delta^2 + 12\Delta + 8$ slopes. Also, this drawing can be computed in $O(n)$ time.*

5 Open Problems

An interesting open problem motivated by our result of Section 3 is whether the 1-*planar slope number* of 1-planar straight-line drawable graphs (not all 1-planar graphs admit a 1-planar straight-line drawing [12]), is bounded in Δ or not. A second problem is whether the quadratic upper bound of Section 4 is tight or not. Finally, it could be interesting to further explore trade-offs between slopes and crossings, e.g., can we draw planar partial 3-trees with $o(\Delta^5)$ slopes and a constant number of crossings per edge?

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