On the Diversity Order of UW-OFDM

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Abstract. Unique word (UW) OFDM is a multicarrier technique that follows a different approach than standard multicarrier techniques like cyclic prefix (CP) OFDM. It has been reported that UW-OFDM outperforms CP-OFDM in the sense that it performs better in fading channels [1,2], and it has much lower out-of-band radiation [3] compared to CP-OFDM. However, a theoretical analysis of the error rate performance of UW-OFDM was never addressed in the literature. In this paper, we derive analytical expressions for the bit error rate for UW-OFDM, from which we can obtain the diversity order. It turns out that when the code generator matrix, needed to construct the UW-OFDM signal, is full rank, the UW-OFDM system reaches the maximum diversity order. This is in contrast with standard CP-OFDM, where only diversity order one can be reached, unless additional precoding is applied. Further, in the paper, we propose a construction method for the code generator matrix to achieve a (close to) maximum coding gain.

1 System Description

In multicarrier techniques, typically a guard interval is used to avoid intersymbol interference between successively transmitted symbols. In standard multicarrier techniques such as CP-OFDM, this guard interval is added on top of the DFT interval, implying the length of a transmitted symbol is increased. A different approach is used in UW-OFDM: here the guard interval is part of the DFT block. Further, in contrast with CP-OFDM, where the guard interval samples depend on the transmitted data, and are thus a priori unknown to the receiver, the guard interval in UW-OFDM is filled with known samples. The UW-OFDM signal is constructed in two steps. First, the data is modulated on the carriers such that after the N-point DFT, the last N_u time domain samples are zero. In this zero part of the signal, the unique word consisting of N_u known samples will be added. In order to obtain the zeroes in the time domain, we have to introduce redundancy in the frequency domain. Assume that because of the presence of

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guard bands, $N_m \leq N$ carriers are modulated. In that case maximally $N_m - N_u$ data symbols can be transmitted per DFT interval.

Let us assume we transmit $N_d \leq N_m - N_u$ data symbols $\mathbf{x}_d$. The required redundancy in the frequency domain is added by multiplying the data symbols with the $N_m \times N_d$ code generator matrix $\mathbf{G}$. To select which carriers are modulated and which are reserved as guard carriers, we use the $N \times N_m$ carrier selection matrix $\mathbf{B}$, where $N_m \leq N$ is the number of modulated carriers. This carrier selection matrix is a reduced version of the $N \times N$ identity matrix, where the columns corresponding to the unmodulated carriers are deleted. The resulting frequency domain vector is applied to the inverse DFT, resulting in the time domain samples

$$\mathbf{y} = \mathbf{F}_N^H \mathbf{B}\mathbf{G}\mathbf{x}_d = \begin{pmatrix} * \\ \mathbf{0} \end{pmatrix} \tag{1}$$

where $(\mathbf{F}_N)_{k,\ell} = \frac{1}{\sqrt{N}} e^{-j2\pi\frac{k\ell}{N}}$. The last N_u elements of $\mathbf{y}$ must be zero, implying that the matrix $\mathbf{G}$ must belong to the null space of the matrix $\tilde{\mathbf{F}}$, which consists of the N_u bottom rows of $\mathbf{F}_N^H \mathbf{B}$. As $\mathbf{F}_N^H$ is an orthogonal matrix and $\mathbf{B}$ is full rank, also the submatrix $\tilde{\mathbf{F}}$ has full rank, inferring the null space has dimension $N_m - N_u$. Let us define the $N \times (N_m - N_u)$ matrix $\mathbf{U}$ as the matrix containing an orthonormal basis for this null space. Such a basis can easily be found using the singular value decomposition of $\tilde{\mathbf{F}}$. As the columns of the matrix $\mathbf{G}$ belong to this null space, the matrix $\mathbf{G}$ can be written as the following linear combination:

$$\mathbf{G} = \mathbf{U}\mathbf{W} \tag{2}$$

where the $(N_m - N_u) \times N_d$ matrix $\mathbf{W}$ can freely be selected.

2 Theoretical Error Performance

In this section we derive an upper bound on the bit error rate when the UW-OFDM signal is transmitted over a Rayleigh fading channel. The channel is modelled as a tapped delay line with $L+1$ taps: $\mathbf{h} = [h(0) \dots h(L)]^T$, and the channel adds white Gaussian noise with spectral density $N_0/2$ per real dimension. To avoid intersymbol interference, we assume that the guard interval, i.e. the unique word, is longer than the channel length: $N_u \geq L$. Neglecting the presence of the unique word, the received sequence, is applied to a DFT resulting in the samples

$$\mathbf{r} = \mathbf{F}_N \mathbf{H} \mathbf{F}_N^H \mathbf{B}\mathbf{G}\mathbf{x}_d + \mathbf{F}_N \mathbf{w} = \tilde{\mathbf{H}}\mathbf{B}\mathbf{G}\mathbf{x}_d + \mathbf{F}_N \mathbf{w}. \tag{3}$$

where $\tilde{\mathbf{H}}_{k,k'} = \delta_{k,k'} \sum_{\ell=0}^{L} h(\ell) e^{j2\pi\frac{k\ell}{N}}$.

To derive the error rate performance, we use a similar approach as in [4]. Let us define the pairwise error probability (PEP) of the transmitted data vector $\mathbf{x}_d$ and the detected data vector $\mathbf{x}'_d \neq \mathbf{x}_d$, given the channel realization $\mathbf{h}$, by $Pr(\mathbf{x}'_d \neq \mathbf{x}_d|\mathbf{h})$. This PEP can be upper bounded using the Chernoff bound:

$$Pr(\mathbf{x}'_d \neq \mathbf{x}_d|\mathbf{h}) \leq \exp\left(-\frac{d^2(\mathbf{v}, \mathbf{v}')}{4N_0}\right) \tag{4}$$

with $d^2(\mathbf{v}, \mathbf{v}')$ the Euclidean distance between the vectors $\mathbf{v}$ and $\mathbf{v}'$ that depend on the transmitted and detected data sequences: $\mathbf{v} = \tilde{\mathbf{H}}\mathbf{B}\mathbf{G}\mathbf{x}_d$ and $\mathbf{v}' = \tilde{\mathbf{H}}\mathbf{B}\mathbf{G}\mathbf{x}'_d$. Defining $\tilde{\mathbf{H}}\mathbf{B}\mathbf{G}(\mathbf{x}_d - \mathbf{x}'_d) = \tilde{\mathbf{H}}\mathbf{B}\mathbf{G}\mathbf{e} = \mathbf{B}_e\mathbf{h}$, we rewrite the Euclidean distance as $d^2(\mathbf{v}, \mathbf{v}') = \mathbf{h}^H\mathbf{B}_e^H\mathbf{B}_e\mathbf{h}$. Averaging over the random channel, an upper bound on the average PEP is found:

$$Pr(\mathbf{x}_d \neq \mathbf{x}'_d) \leq \prod_{\ell=0}^{L} \frac{1}{1 + \alpha_L \frac{\lambda_{e,\ell}}{4N_0}} \tag{5}$$

where $\lambda_{e,\ell}$ are the eigenvalues of the matrix $\mathbf{B}_e^H\mathbf{B}_e$. We assumed in the derivation of (5) that the channel taps were uncorrelated: $E[\mathbf{h}\mathbf{h}^H] = \alpha_L^2\mathbf{I}_{L+1}$, with $\alpha_L = \frac{1}{L+1}$. The upper bound on the average PEP still depends on the unknown error vector $\mathbf{e} = \mathbf{x}_d - \mathbf{x}'_d$ through the eigenvalues. We assume, without loss of generality, that $\mathbf{e}^H\mathbf{e} = 1$.

Because of the definition of $\mathbf{B}_e^H\mathbf{B}_e$, i.e., $d^2(\mathbf{v}, \mathbf{v}') = \mathbf{h}^H\mathbf{B}_e^H\mathbf{B}_e\mathbf{h} \geq 0$, it follows that the eigenvalues are real-valued and non-negative. Assuming there are r_e non-zero eigenvalues $\lambda_{e,\ell} > 0$, we can further upper bound (5):

$$Pr(\mathbf{x}_d \neq \mathbf{x}'_d) < \left(\frac{1}{4N_0}\right)^{-r_e} \left(\prod_{\ell=1}^{r_e} \alpha_L \lambda_{e,\ell}\right)^{-1}. \tag{6}$$

The first factor determines the diversity order, i.e., the diversity order equals r_e, and the second factor is related to the coding gain γ_e. If there are no zero eigenvalues, i.e. when $\mathbf{B}_e^H\mathbf{B}_e$ is full rank, the diversity order is maximized: $r_e = L+1$. In that case, the coding gain equals $\gamma_e = \alpha_L \left[\det(\mathbf{B}_e^H\mathbf{B}_e)\right]^{\frac{1}{L+1}}$. The coding gain and diversity order still depend on the unknown error vector $\mathbf{e}$. The maximum obtainable diversity order and coding gain, irrespective of the data sequence is obtained by minimizing r_e and γ_e over the data.

$$\begin{aligned} r &= \min_{\mathbf{e} \neq \mathbf{0}} r_e \\ \gamma &= \min_{\mathbf{e} \neq \mathbf{0}} \gamma_e. \end{aligned} \tag{7}$$

It has been shown in [4] that the maximum diversity order can be achieved provided that the minimum Euclidean distance is larger than the channel length: $d_{\min} = \min_{\mathbf{e}} d(\mathbf{v}, \mathbf{v}') \geq L+1$. It turns out that a sufficient condition to reach the maximum diversity order is that $\text{rank}(\mathbf{B}\mathbf{G}) = N_d$, and $N_d \geq L+1$, implying that the matrix $N_m \times N_d$ matrix $\mathbf{G}$ must be full rank. Taking into account the decomposition $\mathbf{G} = \mathbf{U}\mathbf{W}$, where the matrix $\mathbf{U}$ is full rank because it consists of orthogonal basis vectors of the null space, it follows that the matrix $\mathbf{W}$ must be full rank to obtain full diversity. Hence, it turns out to be quite simple to design a UW-OFDM system with full diversity: it can be shown that the standard implementations for UW-OFDM, given in [1] and [2], reach full diversity. This is in contrast with CP-OFDM, which reaches a diversity of one only, unless we apply precoding.

3 Coding Gain

In the following, we restrict our attention to the case where the code generator matrix is full rank, i.e. the system has full diversity. We are interested in the code generator matrix that maximizes the coding gain γ. Comparing the upper bounds (4) and (5), it is obvious that if we find a code generator matrix $\mathbf{G}$ that maximizes the minimum Euclidean distance $d^2(\mathbf{v}, \mathbf{v}')$ irrespective of the error vector, also the coding gain will be (close to) maximum. Hence, let us look closer at the Euclidean distance. Given that $\mathbf{v} = \tilde{\mathbf{H}}\mathbf{B}\mathbf{G}\mathbf{x}_d$ and $\mathbf{v}' = \tilde{\mathbf{H}}\mathbf{B}\mathbf{G}\mathbf{x}'_d$, it follows that

$$d^2(\mathbf{v}, \mathbf{v}') = \mathbf{e}^H\mathbf{G}^H\mathbf{B}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{B}\mathbf{G}\mathbf{e} = \mathbf{e}^H\mathbf{A}_R\mathbf{e} \tag{8}$$

where the matrix $\mathbf{A}_R$ is a positive semi-definite Hermitian $N_d \times N_d$ matrix. In the following, we assume that the error vector $\mathbf{e} \in \mathbb{C}^{N_d \times 1}$, with $\mathbf{e}^H\mathbf{e} = 1$. Using the property that the Rayleigh quotient $(\mathbf{e}^H\mathbf{A}_R\mathbf{e})/(\mathbf{e}^H\mathbf{e})$ is bounded by the minimum and maximum eigenvalue of the matrix $\mathbf{A}_R$ [5], it follows that the minimum of the product $\mathbf{e}^H\mathbf{A}_R\mathbf{e}$ corresponds to the minimum eigenvalue of $\mathbf{A}_R$. If we want the minimum Euclidean distance to be as large as possible, this implies that the minimum eigenvalue of $\mathbf{A}_R$ must be as large as possible.

To maximize the minimum eigenvalue, we use the property of the Gerschgorin circles [5]. For a Hermitian matrix $\mathbf{A}_R$, this property states that the real-valued eigenvalues λ_k are located in the intervals $1 - R_k \leq \lambda_k \leq 1 + R_k$, with

$$R_k = \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_d} |(\mathbf{A}_R)_{k,\ell}|. \tag{9}$$

In the following we assume that $(\mathbf{G}^H\mathbf{B}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{B}\mathbf{G})_{k,k} = 1$. The normalization of the received energy per symbol implies that all data symbols have the same error performance [6,7], which results in the lowest error rate performance if we average over all data symbols. As a consequence, the sum of the eigenvalues λ_k is a constant: $\text{trace}(\mathbf{A}_R) = \sum_{k=1}^{N_d} \lambda_k = N_d$. It is straightforward to show through Lagrange optimization that maximizing the minimum eigenvalue under the constraint that the sum of the eigenvalues is known, corresponds to the case where all eigenvalues are equal, which infers $\mathbf{A}_R$ must be the identity matrix. This implies that we have to select the code generator matrix $\mathbf{G}$ such that

$$\mathbf{A}_R = \mathbf{G}^H\mathbf{B}^H\tilde{\mathbf{H}}\tilde{\mathbf{H}}\mathbf{B}\mathbf{G} = \mathbf{I}_{N_d}. \tag{10}$$

However, the matrix $\mathbf{A}_R$ depends on the channel taps $\mathbf{h}$ through the diagonal matrix $\tilde{\mathbf{H}}$. Hence, unless the channel is known, finding the code generator matrix that results in $\mathbf{A}_R = \mathbf{I}_{N_d}$ is generally not possible. Therefore, we restrict our attention to the case where we know the channel. This case could correspond to the case of a fixed wired link, or a wireless link with slowly varying channel, where the channel is estimated and fed back to the transmitter.

In the following, we propose a systematic construction method for the matrix $\mathbf{G}$ based on the decomposition $\mathbf{G} = \mathbf{U}\mathbf{W}$ (2). Note that the $N_m \times (N - N_u)$

matrix $\mathbf{U}$ is composed using the orthonormal basis vectors of the null space of the matrix $\tilde{\mathbf{F}}$, and forces the last N_u time domain samples in the DFT block to be zero. As soon as the system parameters (N, N_m and N_u) are known, the matrices $\mathbf{U}$ and $\mathbf{B}$ are fixed. Hence, we only need to select the matrix $\mathbf{W}$, which needs to be full rank to have full diversity. Using $\mathbf{G} = \mathbf{UW}$ in (10), we obtain the following restriction on $\mathbf{W}$: $\mathbf{W}^H\mathbf{U}^H\mathbf{B}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{BUW} = \mathbf{I}_{N_d}$. Let us consider the eigenvalue decomposition of the known $(N_m - N_u) \times (N_m - N_u)$ matrix $\mathbf{U}^H\mathbf{B}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{BU} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^H$. Because of the Hermitian nature of the matrix, its eigenvalues are real-valued and its eigenvector matrix $\mathbf{V}$ is a unitary matrix. Further, $\mathbf{U}^H\mathbf{B}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{BU}$ is the Gram matrix of $\tilde{\mathbf{H}}\mathbf{BU}$ implying the matrix is positive semi-definite. Assuming the diagonal matrix $\mathbf{B}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{B}$ is full rank, i.e., the channel at the frequencies of the modulated carriers does not contain spectral nulls, its eigenvalues are strictly non-zero. Without loss of generality, we can decompose $\mathbf{W}$ as $\mathbf{W} = \mathbf{VZ}$, resulting in $\mathbf{W}^H\mathbf{U}^H\mathbf{B}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{BUW} = \mathbf{Z}^H\mathbf{V}^H\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^H\mathbf{VZ} = \mathbf{Z}^H\boldsymbol{\Lambda}\mathbf{Z} = \mathbf{I}_{N_d}$. Further, defining $\boldsymbol{\Lambda} = \boldsymbol{\Gamma}\boldsymbol{\Gamma}$, where the real-valued diagonal matrix $\boldsymbol{\Gamma}$ equals $\boldsymbol{\Gamma}$ =diag($\sqrt{\lambda_k}$), with λ_k the eigenvalues contained in $\boldsymbol{\Lambda}$, we can substitute $\mathbf{Z} = \boldsymbol{\Gamma}^{-1}\mathbf{X}$, resulting in:

$$\mathbf{X}^H\mathbf{X} = \mathbf{I}_{N_d}. \tag{11}$$

In the case of $N_r = N_u$, the matrix $\mathbf{X}$ is a square matrix. It can be verified that in this case the condition (11) can only be fulfilled when $\mathbf{X}$ is a unitary matrix. On the other hand, if $N_r > N_u$, the condition (11) requires that $\mathbf{X}$ is a finite frame [8]. In the following, we restrict our attention to the case where $N_u = N_r$.

4 Transmit Versus Received Power

In the previous section, we have proposed a systematic construction method to generate the code generation matrix that achieves (a close to maximum) coding gain if the channel is known at the transmitter. The degree of freedom that follows from the construction method (i.e., a unitary matrix $\mathbf{X}$ must be selected) gives us a large number of possible code generator matrices. In our solution, we have set a restriction on the received power, but no specifications were given on the transmit power. Let us look closer at the power at the transmitter P_T and the received power P_R:

$$\begin{aligned}
P_R &= E_s\text{trace}(\mathbf{G}^H\mathbf{B}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{BG}) = N_dE_s \\
P_T &= E_s\text{trace}(\mathbf{G}^H\mathbf{B}^H\mathbf{BG}) \\
&= E_s\text{trace}(\mathbf{X}^H\boldsymbol{\Gamma}^{-1}\mathbf{V}^H\mathbf{U}^H\mathbf{B}^H\mathbf{BUV}\boldsymbol{\Gamma}^{-1}\mathbf{X}) \\
&= E_s\text{trace}(\mathbf{U}^H\mathbf{B}^H\mathbf{BUV}\boldsymbol{\Gamma}^{-1}\mathbf{X}\mathbf{X}^H\boldsymbol{\Gamma}^{-1}\mathbf{V}^H) \\
&= E_s\text{trace}(\mathbf{U}^H\mathbf{B}^H\mathbf{BU}(\mathbf{U}^H\mathbf{B}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{BU})^{-1}).
\end{aligned} \tag{12}$$

where in the last line, we used the unitary nature of the matrix $\mathbf{X}$, i.e., $\mathbf{X}\mathbf{X}^H = \mathbf{I}_{N_d}$, and $\mathbf{V}\boldsymbol{\Gamma}^{-1}\boldsymbol{\Gamma}^{-1}\mathbf{V}^H = \mathbf{V}\boldsymbol{\Lambda}^{-1}\mathbf{V}^H = (\mathbf{U}^H\mathbf{B}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{BU})^{-1}$. Note that the transmit power nor the received power depend on the selected unitary matrix $\mathbf{X}$, but only depend on the channel and system parameters, i.e., all

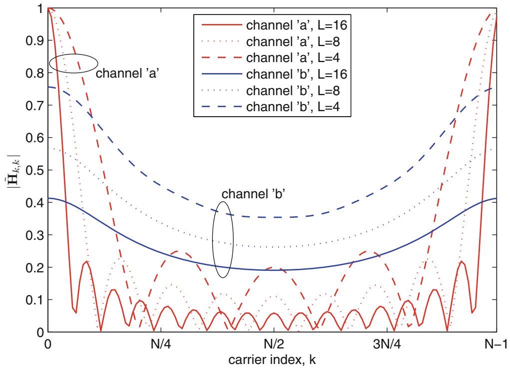


Fig. 1. Frequency response $|\tilde{\mathbf{H}}_{k,k}|$ of the channel for a) $h(\ell) = \nu$ and b) $h(\ell) = \nu \exp(-\ell)$; $\ell = 0, \ldots, L$, ν is a constant to normalize the channel: $\mathbf{h}^H\mathbf{h} = \alpha_L$.

matrices $\mathbf{X}$ lead to the same transmit/received power. In general, it turns out that the transmitted energy per symbol is not normalized. However, because $\mathbf{BG}$ needs to be full rank in order to have full diversity and as we assumed in our construction method that $\tilde{\mathbf{H}}$ does not contain spectral nulls at the modulated carriers, we know that in this case the transmit power is finite.

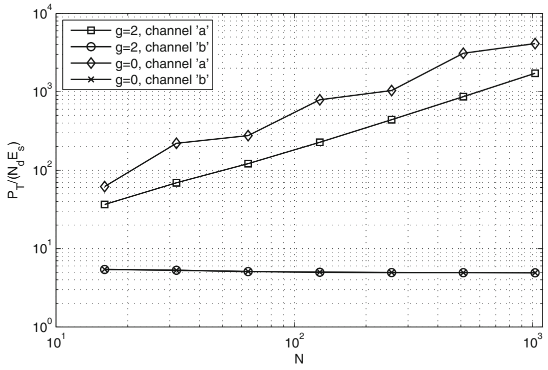
As an illustration, we evaluate the influence of the system parameters and the channel on the transmit power P_T for the following two channels:

$$\text{channel a} : h(\ell) = \nu$$
$$\text{channel b} : h(\ell) = \nu e^{-\ell}$$

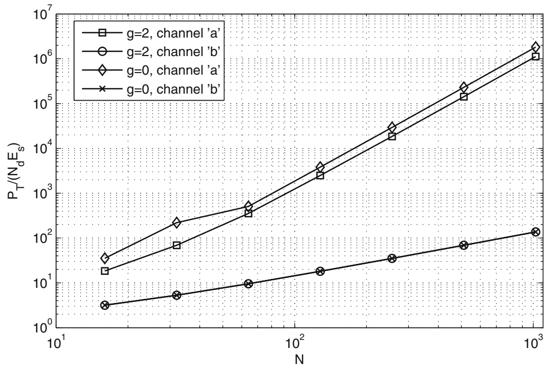
where the channel parameter ν is selected such that $\mathbf{h}^H\mathbf{h} = \frac{1}{L+1}$. Figure 1 shows the frequency response $|\tilde{\mathbf{H}}_{k,k}|$ for these two channels, for different values for L. In this figure, we observe that the frequency response of channel ‘b’ is reasonably flat, whereas channel ‘a’ is more frequency selective. None of the channels show spectral nulls[1], implying their channel matrix $\tilde{\mathbf{H}}$ is full rank. Further, it follows from the figure that the amplitude of the frequency response reduces when L increases.

Figure 2 shows the required transmit power normalized to the received power, i.e., $P_T/(N_d E_s)$, as function of the DFT size N, with and without guard band. The guard band consists of g unmodulated edge carriers at both sides of the frequency band. Hence, the number of modulated carriers equals $N_m = N - 2g$. In Fig. 2(a), the channel length is kept constant, while in Fig. 2(b), the channel length increases proportional to the DFT size. The transmit power for channel

[1] Although channel ‘a’ from Fig. 1 shows small values for the frequency response $|\tilde{\mathbf{H}}_{k,k}|$, they are non-zero. If we plot the y-axis in the figure with a logarithmic scale, it can be observed that the largest and smallest value of $|\tilde{\mathbf{H}}_{k,k}|$ differ approximately a factor of 100. This difference is not large enough to make the matrix $\tilde{\mathbf{H}}$ singular.



(a) $N_r = N_u = L = 4$



(b) $N_r = N_u = L = N/8$

Fig. 2. Transmit power P_T, normalized to the received power $N_d E_s$.

'a' is larger than for channel 'b', and the transmit power is larger for $g = 2$ than for $g = 0$, although the difference for channel 'b' is small. To explain this effect, look at the eigenvalues of the matrix $\mathbf{U}^H \mathbf{B}^H \tilde{\mathbf{H}}^H \tilde{\mathbf{H}} \mathbf{B} \mathbf{U}$ for both channels, with and without guard band, for $N = 32$. For other values of N, similar results are obtained. From Fig. 3, it follows that for both channels, the eigenvalues are larger for $g = 0$ than for $g = 2$, and the difference is larger in channel 'a' than in channel 'b'. This can be explained using Fig. 1. In our channel models, the amplitude of the channel frequency response is larger at the band edges. By not using these carriers, the average channel frequency response decreases, resulting in smaller eigenvalues. This effect is larger in channel 'a' because the channel is more frequency selective than channel 'b'. Further, the eigenvalues for channel 'a' are smaller than for channel 'b', as the frequency response of channel 'a' is for (almost) all carriers much smaller than for channel 'b'. Hence, it is expected that, to obtain the same received power, the required transmit power in channel 'a' will be larger than in channel 'b', and increasing the guard band width will result in an increased transmit power, which is confirmed in Fig. 2.

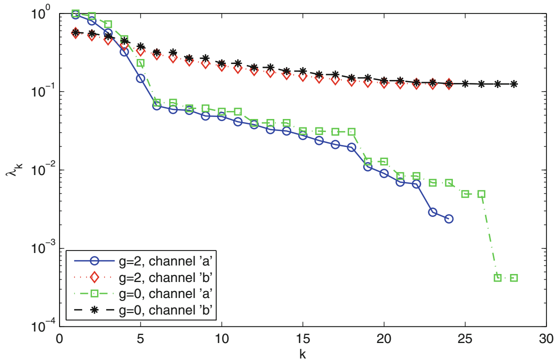


Fig. 3. Eigenvalues of the matrix $\mathbf{U}^H\mathbf{B}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{B}\mathbf{U}$ in decreasing order of magnitude, $N = 32$, $N_r = N_u = L = 4$

Further, in Fig. 2, it also can be observed that when the channel length is kept constant, the required transmit power for channel 'b' is essentially constant as function of the DFT size, whereas for channel 'a' the transmit power linearly increases with the DFT size. When the channel length increases with the DFT size, both the transmit power in channel 'a' and channel 'b' increase, although in channel 'a', the transmit power increases faster. To explain this effect, we consider the special case where all carriers are modulated, i.e., with $g = 0$. In that case, the carrier selection matrix $\mathbf{B}$ reduces to the identity matrix. In addition, the null space matrix $\mathbf{U}$ reduces to $\hat{\mathbf{F}}$, where $\hat{\mathbf{F}}$ corresponds to the first $N - N_u$ rows of $\mathbf{F}_N^H$. Consequently, the transmit power reduces to

$$P_T = E_s \text{trace}(\mathbf{U}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{U})^{-1}. \tag{13}$$

After some straightforward computations, it follows that for the two channels 'a' and 'b', the matrix $\mathbf{U}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{U}$ reduces to a symmetric banded Toeplitz matrix with elements $(\mathbf{U}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{U})_{m,m'} = \frac{1}{L+1}w(m - m')$, with

$$w_a(m) = \begin{cases} 1 - \frac{|m|}{L+1} & |m| \leq L \\ 0 & \text{else} \end{cases} \tag{14}$$

$$w_b(m) = \begin{cases} e^{-|m|}\frac{1-e^{-2(L+1-|m|)}}{1-e^{-2(L+1)}} & |m| \leq L \\ 0 & \text{else} \end{cases} \tag{15}$$

for channel 'a' and 'b', respectively[2]. For channel 'b', it follows from (15) that the matrix $\mathbf{U}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{U}$ is diagonally dominant. Hence, taking into account the Gerschgorin theorem [5], the $N - N_u$ eigenvalues λ_k of the matrix will be of

[2] For general channels, it can be verified that the matrix $\mathbf{U}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{U}$ is a banded matrix having L non-zero side diagonals at both sides of the main diagonal, although in general, the matrix is not symmetric Toeplitz.

the order of the diagonal elements, i.e., $\lambda_k \approx \frac{1}{L+1}$. This can also be observed in Fig. 3. The normalized transmit power $\frac{P_T}{N_d E_s} = \frac{1}{N_d}\text{trace}(\mathbf{U}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{U})^{-1} = \sum_{k=0}^{N-N_u-1}\frac{1}{\lambda_k}$ can therefore be approximated by $\frac{P_T}{N_d E_s} \approx L+1$, with $N_d = N-N_u$. For channel 'a', the matrix $\mathbf{U}^H\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}\mathbf{U}$ is not diagonally dominant. However, in [9], it is shown that square banded Toeplitz matrices are asymptotically equivalent to a circulant matrix for increasing matrix size. Hence, the eigenvalues for channel 'a' can be approximated by the eigenvalues of the asymptotically equivalent circulant matrix, which are obtained by computing its spectrum $f(z)$:

$$\begin{aligned} f(z) &= \sum_{m=-L}^{L}\frac{1}{L+1}w_a(m)e^{jmz} \\ &= \left(\frac{\sin(L+1)\frac{z}{2}}{(L+1)\sin\frac{z}{2}}\right)^2. \end{aligned} \tag{16}$$

The eigenvalues of the matrix are given by $\lambda_k = f(\frac{2\pi k}{N-N_u})$. Hence, the eigenvalue spread will be larger than for channel 'b', which is also observed in Fig. 3. The transmit power is mainly determined by the smallest eigenvalues. These smallest eigenvalues correspond to the values of k for which $\frac{2\pi k}{N-N_u}$ is close to a zero of the function $f(z)$ in the interval $[0,1[$, where the zeros are given by $z_m = \frac{2\pi m}{L+1}$, $m = 1,\ldots,\lfloor\frac{L+1}{2\pi}\rfloor$. To find the smallest eigenvalues, we derive the Taylor series expansion of $f(z)$ at $z = z_m$:

$$f(z) \approx \frac{(z-z_m)^2}{2\sin^2(\frac{\pi m}{L+1})}. \tag{17}$$

After some straightforward computations, we find that the smallest eigenvalues can be upper bounded by $\lambda_{\hat{k}} \leq 2[(N-N_u)\sin\frac{\pi m}{L+1}]^2$, where $\hat{k} = [\frac{\pi m(N-N_u)}{L+1}]_I$ and $[x]_I$ rounds x to the nearest integer. Taking this into account, the normalized transmit power for channel 'a' can be approximated by

$$\frac{P_T}{N_d E_s} \approx \frac{2(N-N_u)^2}{N_d}\sum_{m=1}^{\lfloor\frac{L+1}{2\pi}\rfloor}\sin^2\frac{\pi m}{L+1} \sim (N-N_u)(L+1). \tag{18}$$

Hence, the theoretical approximations for the transmit power for both channels confirm the behaviour of the transmit power in Fig. 2.

5 Conclusions

In this paper, we derived an analytical expression for the theoretical error rate performance for UW-OFDM. From this expression we obtained the conditions to achieve full diversity and a (close to) maximum coding gain: full diversity is reached if the code generator matrix is full rank, and a (close to) maximum coding gain requires that the matrix $\mathbf{A}_R$ must be the identity matrix. Based on these conditions, we proposed a systematic construction method for the code generator matrix. We showed that the transmit power, given the received power is normalized, is independent of the selected code generator matrix, but only depends on the channel and the system parameters.

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