Nonlinear Delay Evolution Inclusions on Graphs

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Abstract. We prove a necessary and a sufficient condition for a time-dependent closed set to be viable with respect to a delay evolution inclusion. An application to a null controllability problem is also included.

Keywords: Delay differential inclusion · m-dissipative operator · Viability · Null controllability problem

1 Introduction

Let X be a real Banach space, $I = [a, b) \subseteq \mathbb{R}$ and let $A : D(A) \subseteq X \rightsquigarrow X$ be the infinitesimal generator of a nonlinear semigroup of nonexpansive mappings $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \geq 0\}$. Let $\sigma \geq 0$ and let $C_\sigma = C([-\sigma, 0]; X)$ be endowed with the usual sup-norm $\|\varphi\|_\sigma = \sup\{\|\varphi(t)\|;\ t \in [-\sigma, 0]\}$.

If $u \in C([\tau - \sigma, T], X)$ and $t \in [\tau, T]$, we denote by $u_t \in C_\sigma$ the function defined by $u_t(s) = u(t + s)$ for $s \in [-\sigma, 0]$. It should be noticed that for $\sigma = 0$, i.e. when de delay is absent, C_σ reduces to X. Let $K : I \rightsquigarrow X$ and $F : \mathcal{K} \rightsquigarrow X$ be nonempty-valued multi-functions, where $\mathcal{K} = \{(t, \varphi) \in I \times C_\sigma;\ \varphi(0) \in K(t)\}$.

In this paper prove a necessary and a sufficient condition in order that $\mathcal{K}$ be viable with respect to $A + F$. Let $(\tau, \varphi) \in \mathcal{K}$ and let us consider

$$\begin{cases} u'(t) \in Au(t) + F(t, u_t) \\ u_\tau = \varphi. \end{cases} \tag{1}$$

Definition 1. A function $u \in C([\tau - \sigma, T]; X)$ is said to be a C^0*-solution of* (1) *on* $[\tau, T] \subseteq I$, if $(t, u_t) \in \mathcal{K}$ for $t \in [\tau, T]$, $u(t) = \varphi(t - \tau)$ for $t \in [\tau - \sigma, \tau]$ and there exists $f \in L^1(\tau, T; X)$ with $f(t) \in F(t, u_t)$ a.e. for $t \in [\tau, T]$ and such that u is a C^0-solution of the Cauchy problem

$$\begin{cases} u'(t) \in Au(t) + f(t), \quad t \in [\tau, T] \\ u(\tau) = \varphi(0) \end{cases}$$

in the usual sense. See Cârjă, Necula, Vrabie [2], Definition 1.6.2, p. 17.

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We say that the function $u : [\,\tau - \sigma, T) \to X$ is a *C^0-solution of* (1) *on* $[\,\tau - \sigma, T)$, if u is a C^0-solution on $[\,\tau - \sigma, \tilde{T}\,]$ for every $\tilde{T} < T$.

Definition 2. We say that $\mathcal{K}$ is C^0 *-viable* with respect to $A + F$, if for each $(\tau, \varphi) \in \mathcal{K}$, there exists $T > \tau$, such that $[\,\tau, T\,] \subseteq I$ and (1) has at least one C^0-solution $u : [\,\tau - \sigma, T\,] \to X$. If $T = \sup I$, we say that $\mathcal{K}$ is *globally C^0-viable* with respect to $A + F$.

Viability results concerning evolution inclusions without delay, i.e., when $\sigma = 0$, using the concepts of tangent set and quasi-tangent set – introduced and studied by Cârjă, Necula and Vrabie [2–4] and [5] –, were obtained by Necula, Popescu and Vrabie [16,17]. For viability results referring to delay evolution equations and inclusions, we mention the pioneering papers of Pavel and Iacob [18] and Haddad [9]. For related results see Gavioli and Malaguti [8], Lakshmikantham, Leela and Moauro [12], Leela and Moauro [13], Lupulescu and Necula [14]. The semilinear case was very recently considered by Necula and Popescu [15] and the present paper extends to the fully nonlinear case the results there obtained.

The paper is divided into five sections, the second one being concerned with the definitions of the basic concepts used in that follows. In Sect. 3 we state and prove a necessary condition for C^0-viability, while Sect. 4 contains the main result of the paper: a sufficient condition for C^0-viability. In Sect. 5, we include an application to a control problem.

2 Preliminaries

Let $f \in L^1(\tau, T; X)$ and $\xi \in \overline{D(A)}$. We denote by $u(\cdot, \tau, \xi, f) : [\,\tau, T\,] \to \overline{D(A)}$ the unique C^0-solution, i.e. integral solution of the Cauchy problem

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in [\,\tau, T\,] \\ u(\tau) = \xi. \end{cases}$$

Clearly, $u(\cdot, \tau, \xi, 0) = S(\cdot - \tau)\xi$, where $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \geq 0\}$ is the semigroup of nonexpansive mappings generated by A on $\overline{D(A)}$ by the Crandall and Liggett Exponential Formula. See Crandall and Liggett [7].

We assume familiarity with the basic concepts and results in nonlinear evolution equations, delay equations and inclusions and we refer the reader to Barbu [1], Cârjă, Necula and Vrabie [2], Lakshmikantham and Leela [11], Hale [10] and Vrabie [19] for details.

The metric d on $\mathcal{K}$ is defined by $d((\tau, \varphi), (\theta, \psi)) = \max\{|\tau - \theta|, \|\varphi - \psi\|_\sigma\}$, for all $(\tau, \varphi), (\theta, \psi) \in \mathcal{K}$. Furthermore, whenever we use the term *strongly-weakly u.s.c.* multi-function we mean that the domain of the multi-function in question is equipped with the strong topology, while the range is equipped with the weak topology. The term *u.s.c.* refers to the case in which both domain and range are endowed with the strong, i.e. norm, topology.

Thereafter, $D(\xi, r)$ denotes the closed ball with center ξ and radius r.

Definition 3. The multi-function $F : \mathcal{K} \rightsquigarrow X$ is called *locally bounded* if, for each (τ, φ) in $\mathcal{K}$, there exist $\delta > 0$, $\rho > 0$ and $M > 0$ such that for all (t, ψ) in $([\,\tau - \delta, \tau + \delta\,] \times D(\varphi, \rho)) \cap \mathcal{K}$, we have $\|F(t, \psi)\| \leq M$.

Let $(\tau, \varphi) \in \mathcal{K}$, let $\eta \in X$ and let $E \subset X$ be a nonempty, bounded subset, let $h > 0$ and let $\mathcal{F}_E = \left\{f \in L^1_{\mathrm{loc}}(\mathbb{R}; X);\ f(s) \in E \text{ a.e. for } s \in \mathbb{R}\right\}$. We denote by $u(\tau + h, \tau, \varphi(0), \mathcal{F}_E) = \{u(\tau + h, \tau, \varphi(0), f);\ f \in \mathcal{F}_E\}$.

Definition 4. We say that E is *A-right-quasi-tangent* to $\mathcal{K}$ at (τ, φ) if

$$\liminf_{h \downarrow 0} h^{-1} \mathrm{d}\left(u(\tau + h, \tau, \varphi(0), \mathcal{F}_E), K(\tau + h)\right) = 0.$$

We denote by $\mathfrak{QTS}^A_{\mathcal{K}}(\tau, \varphi)$ the set of all A-right-quasi-tangent sets to $\mathcal{K}$ at (τ, φ).

If K is constant, E is right-quasi-tangent to $\mathcal{K}$ at (τ, φ) if and only if it is A-quasi-tangent to K at $\xi = \varphi(0)$ in the sense of Cârjă, Necula, Vrabie [2].

3 Necessary Conditions for Viability

The following lemma was proved in Necula and Popescu [15].

Lemma 1. *Let $f : [\,\tau, T\,] \to X$ be a measurable function and $B, C \subset X$ two nonempty sets such that $f(t) \in B + C$ a.e. for $t \in [\,\tau, T\,]$. Then, for every $\varepsilon > 0$ there exist $b : [\,\tau, T\,] \to B$, $c : [\,\tau, T\,] \to C$ and $r : [\,\tau, T\,] \to S(0, \varepsilon)$, all measurable, such that $f(t) = b(t) + c(t) + r(t)$* a.e. *for $t \in [\,\tau, T\,]$.*

Theorem 1. *If $F : \mathcal{K} \rightsquigarrow X$ is u.s.c. and $\mathcal{K}$ is C^0-viable with respect to $A + F$ then, for all $(\tau, \varphi) \in \mathcal{K}$, $\lim\limits_{h \downarrow 0} h^{-1} \mathrm{d}\left(u(\tau + h, \tau, \varphi(0), \mathcal{F}_{F(\tau,\varphi)}), K(\tau + h)\right) = 0$.*

Proof. Let $(\tau, \varphi) \in \mathcal{K}$ and $u : [\,\tau - \sigma, T\,] \to X$ be a C^0-solution of 1. Hence there exists $f \in L^1(\tau, T; X)$ such that $f(s) \in F(s, u_s)$ a.e. for $s \in [\,\tau, T\,]$ and $u(t) = u(t, \tau, \varphi(0), f)$ for all $t \in [\,\tau, T\,]$. Let $\varepsilon > 0$ be arbitrary but fixed.

Since F is u.s.c. at (τ, φ) and $\lim_{t \to \tau} u_t = u_\tau = \varphi$ in C_σ, we may find $\delta > 0$ such that $f(s) \in F(s, u_s) \subseteq F(\tau, \varphi) + S(0, \varepsilon)$ a.e. for $s \in [\,\tau, \tau + \delta\,]$.

Taking $B = F(\tau, \varphi)$ and $C = S(0, \varepsilon)$, from Lemma 1, we deduce that there exist two integrable functions $g : [\,\tau, \tau + \delta\,] \to F(\tau, \varphi)$ and $r : [\,\tau, \tau + \delta\,] \to S(0, 2\varepsilon)$ such that $f(s) = g(s) + r(s)$ a.e. for $s \in [\,\tau, \tau + \delta\,]$. Since $u(\tau + h) \in K(\tau + h)$, we deduce that, for each $0 < h < \delta$, $\mathrm{d}\left(u(\tau + h, \tau, \varphi(0), \mathcal{F}_{F(\tau,\varphi)}), K(\tau + h)\right)$ $\leq \mathrm{d}\left(u(\tau + h, \tau, \varphi(0), g), u(\tau + h, \tau, \varphi(0), f\right) \leq \displaystyle\int_\tau^{\tau + h} \|g(s) - f(s)\| ds \leq 2\varepsilon h$. So, $\limsup\limits_{h \downarrow 0} h^{-1} \mathrm{d}\left(u(\tau + h, \tau, \varphi(0), \mathcal{F}_{F(\tau,\varphi)}), K(\tau + h)\right) \leq 2\varepsilon$. The proof is complete.

Theorem 2. *If $F : \mathcal{K} \rightsquigarrow X$ is u.s.c. and $\mathcal{K}$ is C^0-viable with respect to $A + F$ then $F(\tau, \varphi) \in \mathfrak{QTS}^A_{\mathcal{K}}(\tau, \varphi)$ for all $(\tau, \varphi) \in \mathcal{K}$.*

4 Sufficient Conditions for Viability

Definition 5. We say that the multi-function $K : I \rightsquigarrow X$ is:

(i) *closed from the left on* I if for any sequence $((t_n, x_n))_{n\geq 1}$ from $I \times X$, with $x_n \in K(t_n)$ and $(t_n)_n$ nondecreasing, $\lim_n t_n = t \in I$ and $\lim_n x_n = x$, we have $x \in K(t)$.

(ii) *locally closed from the left* if for each $(\tau, \xi) \in I \times X$ with $\xi \in K(\tau)$ there exist $T > \tau$ and $\rho > 0$ such that the multi-function $t \rightsquigarrow K(t) \cap D(\xi, \rho)$ is closed from the left on $[\,\tau, T\,]$.

Definition 6. An m-dissipative operator $A : D(A) \subseteq X \rightsquigarrow X$ is *of complete continuous type* if for each sequence $(f_n, u_n)_n$ in $L^1(\tau, T; X) \times C([\,\tau, T\,]; X)$ with u_n a C^0-solution of the problem $u_n'(t) \in Au_n(t) + f_n(t)$ on $[\,\tau, T\,]$ for $n = 1, 2, \ldots$, $\lim_n f_n = f$ weakly in $L^1(\tau, T; X)$ and $\lim_n u_n = u$ strongly in $C([\,\tau, T\,]; X)$, it follows that u is a C^0-solution of the problem $u'(t) \in Au(t) + f(t)$ on $[\,\tau, T\,]$.

If the dual of X is uniformly convex and A generates a compact semigroup, then A is of complete continuous type. See Vrabie [19, Corollary 2.3.1, p. 49].

Theorem 3. *Let K be locally closed from the left and let $F : \mathcal{K} \rightsquigarrow X$ be nonempty, convex and weakly compact valued. If F is strongly-weakly u.s.c., locally bounded and $A : D(A) \rightsquigarrow X$ is of complete continuous type and generates a compact semigroup, then a sufficient condition in order that $\mathcal{K}$ be C^0-viable with respect to $A + F$ is the tangency condition $F(\tau, \varphi) \in \mathfrak{QTS}^A_{\mathcal{K}}(\tau, \varphi)$ for all $(\tau, \varphi) \in \mathcal{K}$. If, in addition, F is u.s.c., then the tangency condition is also necessary in order that $\mathcal{K}$ be C^0-viable with respect to $A + F$.*

The next lemma is inspired from Cârjă and Vrabie [6].

Lemma 2. *Let $K : I \rightsquigarrow X$ be locally closed from the left, $F : \mathcal{K} \rightsquigarrow X$ be locally bounded and let $(\tau, \varphi) \in \mathcal{K}$. Let us assume that the tangency condition is satisfied. Let $\rho > 0$, $T > \tau$ and $M > 0$ be such that:*

(1) *the multi-function $t \rightsquigarrow K(t) \cap D(\varphi(0), \rho)$ is closed from the left on $[\,\tau, T)$;*
(2) *$\|F(t, \psi)\| \leq M$ for all $t \in [\,\tau, T\,]$ and all $\psi \in D_\sigma(\varphi, \rho)$ with $(t, \psi) \in \mathcal{K}$;*
(3) $\displaystyle\sup_{t\in[\,\tau,T\,]} \|S(t-\tau)\varphi(0) - \varphi(0)\| + \sup_{|t-s|\leq T-\tau} \|\varphi(t) - \varphi(s)\| + (T-\tau)(M+1) < \rho.$

Then, for each $\varepsilon \in (0, 1)$, there exist a family $\mathcal{P}_T = \{[\,t_m, s_m); m \in \Gamma\}$ of disjoint intervals, with Γ finite or at most countable, and two functions: $f \in L^1(\tau, T; X)$, and $u \in C([\,\tau - \sigma, T\,]; X)$ such that:

(i) *$\cup[\,t_m, s_m) = [\,\tau, T)$ and $s_m - t_m \leq \varepsilon$, for all $m \in \Gamma$;*
(ii) *$u(t_m) \in K(t_m)$, for all $m \in \Gamma$ and $u(T) \in K(T)$;*
(iii) *$f(s) \in F(t_m, u_{t_m})$ a.e. for $s \in [\,t_m, s_m)$ and $\|f(s)\| \leq M$ a.e. for $s \in [\,\tau, T\,]$;*
(iv) *$u(t) = \varphi(t - \tau)$ for $t \in [\,\tau - \sigma, \tau\,]$ and*
$\|u(t) - u(t, t_m, u(t_m), f)\| \leq (t - t_m)\varepsilon$ for $t \in [\,t_m, T\,]$ and $m \in \Gamma$;
(v) *$\|u_t - \varphi\|_\sigma < \rho$ for all $t \in [\,\tau, T\,]$;*
(vi) *$\|u(t) - u(t_m)\| \leq \varepsilon$ for all $t \in [\,t_m, s_m)$ and all $m \in \Gamma$.*

Proof. Let us observe that, if $(i)\sim(iv)$ are satisfied, then (v) is satisfied too, i.e. $\|u(t+s)-\varphi(s)\|<\rho$ for all $t\in[\tau,T]$ and $s\in[-\sigma,0]$. Indeed, if $t+s\leq\tau$ then

$$\|u(t+s)-\varphi(s)\|=\|\varphi(t+s-\tau)-\varphi(s)\|\leq\sup_{|t_1-t_2|\leq T-\tau}\|\varphi(t_1)-\varphi(t_2)\|<\rho.$$

If $t+s>\tau$ then $|s|<T-\tau$ and from (3), (iii) and (iv), we get

$$\|u(t+s)-\varphi(s)\|\leq\|u(t+s)-u(t+s,\tau,\varphi(0),f)\|$$

$$+\|u(t+s,\tau,\varphi(0),f)-u(t+s,\tau,\varphi(0),0)\|$$

$$+\|u(t+s,\tau,\varphi(0),0)-\varphi(0)\|+\|\varphi(0)-\varphi(s)\|$$

$$\leq(t+s-\tau)\varepsilon+\int_\tau^{t+s}\|f(\theta)\|d\theta+\|S(t+s-\tau)\varphi(0)-\varphi(0)\|+\|\varphi(0)-\varphi(s)\|$$

$$\leq(T-\tau)(1+M)+\|S(t+s-\tau)\varphi(0)-\varphi(0)\|+\|\varphi(0)-\varphi(s)\|<\rho.$$

Let $\varepsilon\in(0,1)$ be arbitrary, but fixed. We will show that there exist $\delta=\delta(\varepsilon)$ in (τ,T) and $\mathcal{P}_\delta$, f, u such that $(i)\sim(vi)$ hold true with δ instead of T.

From the tangency condition, it follows that there exist $h_n\downarrow 0$, $g_n\in\mathcal{F}_{F(\tau,\varphi)}$ and $p_n\in X$, with $\|p_n\|\to 0$ and $u(\tau+h_n,\tau,\varphi(0),g_n)+p_nh_n\in K(\tau+h_n)$ for every $n\in\mathbb{N}$, $n\geq 1$. Let $n_0\in\mathbb{N}$ and $\delta=\tau+h_{n_0}$ be such that $\delta\in(\tau,T)$, $h_{n_0}<\varepsilon$ and $\|p_{n_0}\|<\varepsilon$.

Let $\mathcal{P}_\delta=\{[\tau,\delta)\}$, $f(t)=g_{n_0}(t)$ and $u(t)=u(t,\tau,\varphi(0),g_{n_0})+(t-\tau)p_{n_0}$ for $t\in[\tau,\delta]$. Obviously, $(i)\sim(v)$ are satisfied. Moreover, we may diminish $\delta>\tau$ (increase n_0), if necessary, in order to (vi) be satisfied too.

Let $\mathcal{U}=\{(\mathcal{P}_\delta,f,u);\ \delta\in(\tau,T]$ and $(i)\sim(vi)$ are satisfied with δ instead of $T\}$.

As we already have shown, $\mathcal{U}\neq\emptyset$. On $\mathcal{U}$ we define a partial order by:

$$(\mathcal{P}_{\delta_1},f_1,u_1)\preceq(\mathcal{P}_{\delta_2},f_2,u_2),$$

if $\delta_1\leq\delta_2$, $\mathcal{P}_{\delta_1}\subseteq\mathcal{P}_{\delta_2}$, $f_1(s)=f_2(s)$ a.e. for $s\in[\tau,\delta_1]$ and $u_1(s)=u_2(s)$ for all $s\in[\tau,\delta_1]$. We will prove that each nondecreasing sequence in $\mathcal{U}$ is bounded from above. Let $((\mathcal{P}_{\delta_j},f_j,u_j))_{j\geq 1}$ be a nondecreasing sequence in $\mathcal{U}$ and let $\delta=\sup_{j\geq 1}\delta_j$. If there exists $j_0\in\mathbb{N}$ such that $\delta_{j_0}=\delta$, then $(\mathcal{P}_{\delta_{j_0}},f_{j_0},u_{j_0})$ is an upper bound for the sequence. So, let us assume that $\delta_j<\delta$, for all $j\geq 1$. Obviously, $\delta\in(\tau,T]$. We define $\mathcal{P}_\delta=\cup_{j\geq 1}\mathcal{P}_{\delta_j}$, $f(t)=f_j(t)$ and $u(t)=u_j(t)$ for all $j\geq 1$ and $t\in[\tau,\delta_j)$. Clearly, $f\in L^1(\tau,\delta;X)$ and $u\in C([\tau,\delta);X)$.

Let us observe that, in view of (iv), we have

$$\|u(t)-u(s)\|\leq\|u(t)-u(t,\delta_j,u(\delta_j),f)\|$$

$$+\|u(t,\delta_j,u(\delta_j),f)-u(s,\delta_j,u(\delta_j),f)\|+\|u(s,\delta_j,u(\delta_j),f)-u(s)\|$$

$$\leq(t-\delta_j)\varepsilon+\|u(t,\delta_j,u(\delta_j),f)-u(s,\delta_j,u(\delta_j),f)\|+(s-\delta_j)\varepsilon$$

$$\leq 2(\delta-\delta_j)\varepsilon+\|u(t,\delta_j,u(\delta_j),f)-u(s,\delta_j,u(\delta_j),f)\|$$

for all $j\geq 1$ and all $t,s\in[\delta_j,\delta)$. Since $\lim_j\delta_j=\delta$ and $u(\cdot,\delta_j,u(\delta_j),f)$ is continuous at $t=\delta$, we conclude that u satisfies the Cauchy condition for the existence of the limit at $t = \delta$. So, u can be extended by continuity to the whole interval $[\tau, \delta]$. By observing that $u(\delta) = \lim_{t \uparrow \delta} u(t) = \lim_{j \to \infty} u(\delta_j) = \lim_{j \to \infty} u_j(\delta_j)$, $u_j(\delta_j) \in D(\varphi(0), \rho) \cap K(\delta_j)$ and the latter is closed from the left, we deduce that $u(\delta) \in D(\varphi(0), \rho) \cap K(\delta)$. The rest of conditions in lemma being obviously satisfied, it follows that $(\mathcal{P}_\delta, f, u)$ is an upper bound for the sequence. Consequently, $(\mathcal{U}, \preceq)$ and $\mathcal{N} : (\mathcal{U}, \preceq) \to R$, defined by $\mathcal{N}(\mathcal{P}_\delta, f, u) = \delta$, for each $(\mathcal{P}_\delta, f, u) \in \mathcal{U}$, satisfy the hypotheses of the Brezis-Browder Ordering Principle – see Cârjă, Necula and Vrabie [2, Theorem 2.1.1, p. 30]. Accordingly, there exists an $\mathcal{N}$-maximal element in $\mathcal{U}$. This means that there exists $(\mathcal{P}_{\delta^*}, f^*, u^*) \in \mathcal{U}$ such that, whenever $(\mathcal{P}_{\delta^*}, f^*, u^*) \preceq (\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{u})$, we necessarily have $\mathcal{N}(\mathcal{P}_{\delta^*}, f^*, u^*) = \mathcal{N}(\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{u})$. We will show that $\delta^* = T$. To this aim, let us assume by contradiction that $\delta^* < T$.

Since $(\delta^*, u^*_{\delta^*}) \in \mathcal{K}$, using the tangency condition, we deduce that there exist the sequences $h_n \downarrow 0$, $g_n \in \mathcal{F}_{F(\delta^*, u^*_{\delta^*})}$ and $p_n \in X$, with $\|p_n\| \to 0$, such that $u(\delta^* + h_n, \delta^*, u^*(\delta^*), g_n) + p_n h_n \in K(\delta^* + h_n)$ for all $n \in \mathbb{N}$, $n \geq 1$. Let $n_0 \in \mathbb{N}$ and $\overline{\delta} = \delta^* + h_{n_0}$ with $\overline{\delta} \in (\delta^*, T)$, $h_{n_0} < \varepsilon$ and $\|p_{n_0}\| < \varepsilon$. Let $\mathcal{P}_{\overline{\delta}} = \mathcal{P}_{\delta^*} \cup \{[\delta^*, \overline{\delta}]\}$,

$$\overline{f}(t) = \begin{cases} f^*(t), & t \in [\tau, \delta^*] \\ f_{n_0}(t), & t \in (\delta^*, \overline{\delta}] \end{cases},$$

$$\overline{u}(t) = \begin{cases} u^*(t), & t \in [\tau, \delta^*] \\ u(t, \delta^*, u^*(\delta^*), f_{n_0}) + (t - \delta^*) p_{n_0}, & t \in (\delta^*, \overline{\delta}]. \end{cases}$$

By (v), we have $u^*_{\delta^*} \in S_\sigma(\varphi, \rho)$. So, (2) implies that $\|\overline{f}(s)\| \leq M$ a.e. for $s \in (\tau, \overline{\delta})$.

Clearly $(i)\sim(iii)$ are satisfied. In order to prove (iv) we will consider only the case $t_m \leq \delta^* \leq t$, the other cases being obvious. Using the evolution property, i.e. $u(t, a, \xi, f) = u(t, b, u(b, a, \xi, f), f)$ for $\tau \leq a \leq b \leq t \leq T$, we get

$$\begin{aligned} &\|\overline{u}(t) - u(t, t_m, u^*(t_m), \overline{f})\| \\ &\leq \|u(t, \delta^*, u^*(\delta^*), \overline{f}) - u(t, t_m, u^*(t_m), \overline{f})\| + (t - \delta^*)\varepsilon \\ &= \|u(t, \delta^*, u^*(\delta^*), \overline{f}) - u(t, \delta^*, u(\delta^*, t_m, u^*(t_m), \overline{f}), \overline{f})\| + (t - \delta^*)\varepsilon \\ &\leq \|u^*(\delta^*) - u(\delta^*, t_m, u^*(t_m), \overline{f})\| + (t - \delta^*)\varepsilon \\ &\leq (\delta^* - t_m)\varepsilon + (t - \delta^*)\varepsilon = (t - t_m)\varepsilon, \end{aligned}$$

which proves (iv).

Similarly, we can diminish $\overline{\delta}$ (increase n_0) in order that (vi) be satisfied too.

So, $(\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{u}) \in \mathcal{U}$, $(\mathcal{P}_{\delta^*}, f^*, u^*) \preceq (\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{u})$, but $\delta^* < \overline{\delta}$ which contradicts the maximality of $(\mathcal{P}_{\delta^*}, f^*, u^*)$. Hence $\delta^* = T$, and $\mathcal{P}_{\delta^*}$, f^* and u^* satisfy all the conditions $(i)\sim(vi)$. The proof is complete.

Definition 7. Let $\varepsilon > 0$. An element $(\mathcal{P}_T, f, u)$ satisfying $(i)\sim(vi)$ in Lemma 2, is called an *ε-approximate C^0-solution of* (1).

We can proceed now to the proof of Theorem 3.

Proof. The necessity follows from Theorem 2. As long as the proof of the sufficiency is concerned, let $\rho > 0$, $T > \tau$ and $M > 0$ be as in Lemma 2. Let $\varepsilon_n \in (0,1)$, with $\varepsilon_n \downarrow 0$. Let $((\mathcal{P}_T^n, f_n, u_n))_n$ be a sequence of ε_n-approximate C^0-solutions of (1) given by Lemma 2. If $\mathcal{P}_T^n = \{[\,t_m^n, s_m^n);\ m \in \Gamma_n\}$ with Γ_n finite or at most countable, we denote by $a_n : [\,\tau, T) \to [\,\tau, T)$ the step function, defined by $a_n(s) = t_m^n$ for each $s \in [\,t_m^n, s_m^n)$. Clearly $\lim\limits_n a_n(s) = s$ uniformly for $s \in [\,\tau, T)$, while from (vi), deduce that $\lim\limits_n \|u_n(t) - u_n(a_n(t))\| = 0$, uniformly for $t \in [\,\tau, T)$. From (iv), we get

$$\lim_n (u_n(t) - u(t, \tau, \varphi(0), f_n)) = 0 \tag{2}$$

uniformly for $t \in [\,\tau, T\,]$. Since $\|f_n(t)\| \leq M$ for all $n \in \mathbb{N}$ and a.e. for $t \in [\,\tau, T\,]$ and the semigroup generated by A is compact, by Vrabie [19, Theorem 2.3.3, p. 47], we deduce that the set $\{u(\cdot, \tau, \varphi(0), f_n);\, n \geq 1\}$ is relatively compact in $C([\,\tau, T\,]; X)$. From this remark and (2), we conclude that $(u_n)_n$ has at least one uniformly convergent subsequence to some function u, subsequence denoted again by $(u_n)_n$.

Since $a_n(t) \uparrow t$, $\lim_n u_n(a_n(t)) = u(t)$, uniformly for $t \in [\,\tau, T)$ and the mapping $t \to K(t) \cap D(\varphi(0), \rho)$ is closed from the left, we get that $u(t) \in K(t)$ for all $t \in [\,\tau, T\,]$. But $\lim_n (u_n)_{a_n(t)} = u_t$ in C_σ, uniformly for $t \in [\,\tau, T)$. Hence, the set $C = \overline{\{(a_n(t), (u_n)_{a_n(t)});\, n \geq 1,\ t \in [\,\tau, T)\}}$ is compact and $C \subseteq \mathcal{K}$.

At this point, recalling that F is strongly-weakly u.s.c. and has weakly compact values, by Cârjă, Necula and Vrabie [2, Lemma 2.6.1, p. 47], it follows that $B = \overline{\text{conv}} \left(\bigcup_{n \geq 1} \bigcup_{t \in [\,\tau, T)} F(a_n(t), (u_n)_{a_n(t)}) \right)$ is weakly compact. We notice that $f_n(s) \in B$ for all $n \geq 1$ and a.e. for $s \in [\,\tau, T\,]$. An appeal to Cârjă, Necula and Vrabie [2, Theorem 1.3.8, p. 10] shows that, at least on a subsequence, $\lim\limits_n f_n = f$ weakly in $L^1(\tau, T; X)$. As F is strongly-weakly u.s.c. with closed and convex values while, by Lemma 2, for each $n \geq 1$, we have $f_n(s) \in F(a_n(s), (u_n)_{a_n(s)})$ a.e. for $s \in [\,\tau, T\,]$, from Vrabie [19, Theorem 3.1.2, p. 88], we conclude that $f(s) \in F(s, u_s)$ a.e. for $s \in [\,\tau, T\,]$.

Finally, by (2) and the fact that A is of complete continuous type, we get $u(t) = u(t, \tau, \varphi(0), f)$ for each $t \in [\,\tau, T\,]$ and so, u is a C^0-solution of (1).

Theorem 4. *Let K be closed from the left and let $F : \mathcal{K} \rightsquigarrow X$ be nonempty, convex and weakly compact valued. If there exist $a, b \in C(I)$ such that*

$$\|F(t, \varphi)\| \leq a(t) + b(t)\|\varphi(0)\| \quad \text{for all } t \in I \text{ and all } \varphi \in C_\sigma,$$

F is strongly-weakly u.s.c. and $A : D(A) \rightsquigarrow X$ is of complete continuous type and generates a compact semigroup, then a sufficient condition in order that $\mathcal{K}$ be globally C^0-viable with respect to $A + F$ is the tangency condition in Theorem 3. If, in addition, F is u.s.c., then the tangency condition is also necessary in order that $\mathcal{K}$ be mild-viable with respect to $A + F$.

5 A Sufficient Condition for Null Controllability

Let X be a Banach space, $A : D(A) \subseteq X \rightsquigarrow X$ an m-dissipative operator, $g : \mathbb{R}_+ \times C_\sigma \to X$ a given function and $(\tau, \varphi) \in \mathbb{R}_+ \times C_\sigma$ with $\varphi(0) \in \overline{D(A)}$. The problem is how to find a measurable control $c(\cdot)$ taking values in $D(0,1)$ in order to reach the origin in some time T, by C^0-solutions of the state equation

$$\begin{cases} u'(t) \in Au(t) + g(t, u_t) + c(t) \\ u_\tau = \varphi. \end{cases} \tag{3}$$

With $G : \mathbb{R}_+ \times C_\sigma \rightsquigarrow X$, defined by $G(t,v) = av(0) + g(t,v) + D(0,1)$, the above problem reformulates: find $T > 0$ and a C^0-solution of problem

$$\begin{cases} u'(t) \in (A - aI)u(t) + G(t, u_t) \\ u_\tau = \varphi, \quad u(\tau + T) = 0. \end{cases} \tag{4}$$

Theorem 5 and Corollary 1 below are "delay" versions of Cârjă, Necula and Vrabie [3, Theorem 12.1 and Corollary 12.1].

Theorem 5. *Let X be a reflexive Banach space and let $A : D(A) \subseteq X \rightsquigarrow X$ be such that, for some $a \in \mathbb{R}$, $A - aI$ is an m-dissipative operator of complete continuous type and which is the infinitesimal generator of a compact semigroup of contractions, $\{S(t) : \overline{D(A)} \to \overline{D(A)};\quad t \geq 0\}$. Let $g : \mathbb{R}_+ \times C_\sigma \to X$ be a continuous function such that for some $L > 0$ we have*

$$\|g(t,v)\| \leq L\|v(0)\|, \quad \text{for all } (t,v) \in \mathbb{R}_+ \times C_\sigma. \tag{5}$$

Assume that $0 \in D(A)$ and $0 \in A0$. Then, for each $(\tau, \varphi) \in \mathbb{R}_+ \times C_\sigma$ with $\xi = \varphi(0) \in \overline{D(A)} \setminus \{0\}$, there exists a C^0-solution $u : [\tau, \infty) \to X$ of (4) satisfying

$$\|u(t)\| \leq \|\xi\| - (t - \tau) + (L + a)\int_\tau^t \|u(s)\| ds, \quad \text{for all } t \geq \tau \text{ with } \ u(t) \neq 0. \tag{6}$$

Proof. Let $(\tau, \varphi) \in \mathbb{R}_+ \times C_\sigma$ with $\xi = \varphi(0) \in \overline{D(A)} \setminus \{0\}$. We show that there exist $T \in (0, +\infty)$ and a noncontinuable C^0-solution $(z, u) : [\tau, \tau + T) \to \mathbb{R} \times X$ of the problem

$$\begin{cases} z'(t) = (L + a)\|u(t)\| - 1, & t \in [\tau, \tau + T) \\ u'(t) \in (A - aI)u(t) + G(t, u_t), & t \in [\tau, \tau + T) \\ z_\tau = \|\varphi\| \quad \text{and} \quad u_\tau = \varphi, & \\ \|u(t)\| \leq z(t), & t \in [\tau, \tau + T). \end{cases} \tag{7}$$

On the Banach space $\mathcal{X} = \mathbb{R} \times X$ the operator $\mathcal{A} = (0, A - aI)$ generates a compact semigroup of contractions $\{(1, S(t));\ (1, S(t)) : \mathbb{R} \times \overline{D(A)} \to \mathcal{X}\}$.

We denote by $\mathcal{C}_\sigma = C([-\sigma, 0]; \mathcal{X}) = C([-\sigma, 0]; \mathbb{R}) \times C([-\sigma, 0]; X)$. Let K be the locally closed set $K = \{(x_1, x_2) \in \mathbb{R}_+ \times (\overline{D(A)} \setminus \{0\});\ \|x_2\| \leq x_1\}$, with the associate set $\mathcal{K} = \{(t, \psi) \in \mathbb{R} \times \mathcal{C}_\sigma;\ \psi(0) \in K\}$, i.e.

$$\mathcal{K} = \{(t, \psi_1, \psi_2) \in \mathbb{R} \times C([-\sigma, 0]; \mathbb{R}) \times C([-\sigma, 0]; X);\ \|\psi_2(0)\| \leq \psi_1(0)\}$$

and let the multi-function $\mathcal{F} : \mathcal{K} \rightsquigarrow \mathbb{R} \times X$ be defined by

$\mathcal{F}(t,\psi_1,\psi_2) = ((L+a)\|\psi_2(0)\|-1, a\psi_2(0)+g(t,\psi_2)+D(0,1))$, for $(t,\psi_1,\psi_2)\in\mathcal{K}$.

To show that $\mathcal{F}(\tau,\psi_1,\psi_2)\in\mathfrak{QTS}^{\mathcal{A}}_{\mathcal{K}}(\tau,\psi_1,\psi_2)$, for every $(\tau,\psi_1,\psi_2)\in\mathcal{K}$, we shall prove the stronger condition: there exists $(\eta_1,\eta_2)\in\mathcal{F}(\tau,\psi_1,\psi_2)$ such that

$$\liminf_{h\downarrow 0} h^{-1}\mathrm{d}\left(\mathcal{U}(\tau+h,\tau,(\xi_1,\xi_2),(\eta_1,\eta_2)),K\right)=0, \tag{8}$$

where $(\xi_1,\xi_2)=(\psi_1(0),\psi_2(0))$ and $\mathcal{U}(\cdot,\tau,(\xi_1,\xi_2),(\eta_1,\eta_2))$ is the C^0-solution of the corresponding Cauchy problem for the operator $\mathcal{A}$, i.e.

$$\mathcal{U}(t,\tau,(\xi_1,\xi_2),(\eta_1,\eta_2))=(\xi_1+(t-\tau)\eta_1, u(t,\tau,\xi_2,\eta_2))\in\mathcal{X},$$

$u(\cdot,\tau,\xi_2,\eta_2)$ being the corresponding solution for $A-aI$. To this end, it suffices to prove that there exist $(h_n)_n$ in $\mathbb{R}_+$, with $h_n\downarrow 0$, and (θ_n,p_n) in $\mathbb{R}\times X$, with $(\theta_n,p_n)\to(0,0)$, such that, for every $n\in\mathbb{N}$, we have

$$\|u(\tau+h_n,\tau,\xi_2,\eta_2)+h_np_n\|\le \xi_1+h_n\eta_1+h_n\theta_n. \tag{9}$$

Clearly, $\|u\,(\tau+h,\tau,\xi_2,\eta_2)\|\le\|\xi_2\|+\int_\tau^{\tau+h}[\,u(s,\tau,\xi_2,\eta_2),\eta_2\,]_+ds$ for all $h>0$. The normalized semi-inner product, $(x,y)\mapsto[\,x,y\,]_+=\lim_{h\downarrow 0}h^{-1}(\|x+hy\|-\|x\|)$, is u.s.c. Hence, setting $\ell(s):=u(s,\tau,\xi_2,\eta_2)$, we get

$$\liminf_{h\downarrow 0}h^{-1}\int_\tau^{\tau+h}[\ell(s),\eta_2]_+ds\le\limsup_{h\downarrow 0}h^{-1}\int_\tau^{\tau+h}[\,\ell(s),\eta\,]_+ds\le[\,\xi_2,\eta_2\,]_+.$$

Let $\eta_1=(L+a)\|\psi_2(0)\|-1=(L+a)\|\xi_2\|-1$ and $\eta_2=a\xi_2+g(\tau,\psi_2)-\frac{\xi_2}{\|\xi_2\|}$. Clearly, $\eta_2\in a\xi_2+g(\tau,\psi_2)+D(0,1)$ and so, $(\eta_1,\eta_2)\in\mathcal{F}(\tau,\psi_1,\psi_2)$. From (5), we get $[\xi_2,\eta_2]_+=a\|\xi_2\|+[\xi_2,g(\tau,\psi_2)]_+-1\le(L+a)\|\xi_2\|-1=\eta_1$ and hence $\liminf_{h\downarrow 0}h^{-1}\left(\|u(\tau+h,\tau,\xi_2,\eta_2)\|-\|\xi_2\|\right)\le\eta_1$. Keeping in mind that $\|\xi_2\|=\|\psi_2(0)\|\le\psi_1(0)=\xi_1$ since $(\tau,\psi_1,\psi_2)\in\mathcal{K}$, the last inequality proves (9) with $p_n=0$. Thus we get (8). From Theorem 3, $\mathcal{K}$ is C^0-viable with respect to $\mathcal{A}+\mathcal{F}$. As $(\tau,\|\varphi\|,\varphi)\in\mathcal{K}$, thanks to Brezis-Browder Ordering Principle [2, Theorem 2.1.1, p. 30] –, we obtain further that there exist $T\in(0,+\infty\,]$ and a noncontinuable C^0-solution of $(z,u):[\,\tau,\tau+T)\to\mathbb{R}\times X$ of (7) which satisfies $(z(t),u(t))\in K$ for every $t\in[\,\tau,\tau+T)$. This means that (6) is satisfied for every $t\in[\,\tau,\tau+T)$. Since G has sublinear growth, u, as a solution of (4), can be continued to $\mathbb{R}_+$. So, $u(\tau+T)$ exists, even though the solution (z,u) of (7) is defined merely on $[\,\tau,\tau+T)$ if T is finite. In this case, $u(\tau+T)=0$ since otherwise (z,u) can be continued to the right of T which is a contradiction.

Corollary 1. *Under the hypothesis of Theorem 5, the following properties hold.*

(*i*) *If $L+a\le 0$, for any $(\tau,\varphi)\in\mathbb{R}_+\times C_\sigma$ with $\xi=\varphi(0)\in\overline{D(A)}\setminus\{0\}$, there exist a control $c(\cdot)$ and a C^0-solution of (3) that reaches the origin of X in some time $T\le\|\xi\|$ and satisfies $\|u(t)\|\le\|x\|-(t-\tau)$* for all $\tau\le t\le\tau+T$.

(ii) *If* $L+a>0$*, for every* $(\tau,\varphi)\in\mathbb{R}_+\times C_\sigma$ *with* $\xi=\varphi(0)\in\overline{D(A)}\setminus\{0\}$ *satisfying* $0<\|\xi\|<1/(L+a)$*, there exist a control* $c(\cdot)$ *and a* C^0*-solution of (3) that reaches the origin of* X *in some time* $T\leq(L+a)^{-1}\log\left\{[1-(L+a)\|\xi\|]^{-1}\right\}$ *and* $\|u(t)\|\leq e^{(L+a)(t-\tau)}\left[\|\xi\|-(L+a)^{-1}\right]+(L+a)^{-1}$ *for* $t\in[\tau,\tau+T]$.

References

1. Barbu, V.: Nonlinear Differential Equations of Monotone Type in Banach Spaces. Springer Monographs in Mathematics. Springer, New York (2010)
2. Cârjă, O., Necula, M., Vrabie, I.I.: Viability, Invariance and Applications. North-Holland Mathematics Studies, vol. 207. Elsevier, Amsterdam (2007)
3. Cârjă, O., Necula, M., Vrabie, I.I.: Tangent sets, necessary and sufficient conditions for viability for nonlinear evolution inclusions. Set-Valued Anal. **16**, 701–731 (2008)
4. Cârjă, O., Necula, M., Vrabie, I.I.: Necessary and sufficient conditions for viability for semilinear differential inclusions. Trans. Am. Math. Soc. **361**, 343–390 (2009)
5. Cârjă, O., Necula, M., Vrabie, I.I.: Tangent sets, viability for differential inclusions and applications. Nonlinear Anal. **71**, e979–e990 (2009)
6. Cârjă, O., Vrabie, I.I.: Some new viability results for semilinear differential inclusions. NoDEA Nonlinear Differ. Equ. Appl. **4**, 401–424 (1997)
7. Crandall, M.G., Liggett, T.M.: Generation of semi-groups of nonlinear transformations in general Banach spaces. Am. J. Math. **93**, 265–298 (1971)
8. Gavioli, A., Malaguti, L.: Viable mild solutions of differential inclusions with memory in Banach spaces. Port. Math. **57**, 203–217 (2000)
9. Haddad, G.: Monotone trajectories of differential inclusions and functional-differential inclusions with memory. Isr. J. Math. **39**, 83–100 (1981)
10. Hale, J.: Functional differential equations. Applied Mathematical Sciences, vol. 3. Springer, Heidelberg (1971)
11. Lakshmikantham, V., Leela, S.: Nonlinear Differential Equations in Abstract Spaces. International Series in Nonlinear Mathematics, vol. 2. Pergamon Press, New York (1981)
12. Lakshmikantham, V., Leela, S., Moauro, V.: Existence and uniqueness of mild solutions of delay differential equations on a closed subset of a Banach space. Nonlinear Anal. **2**, 311–327 (1978)
13. Leela, S., Moauro, V.: Existence of mild solutions in a closed set for delay differential equations in Banach spaces. Nonlinear Anal. **2**, 47–58 (1978)
14. Lupulescu, V., Necula, M.: A viability result for nonconvex semilinear functional differential inclusions. Discuss. Math. Differ. Incl. Control Optim. **25**, 109–128 (2005)
15. Necula, M. Popescu, M.: A viability result for differential inclusions on graphs, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat., doi:10.2478/aicu-2013-0016
16. Necula, M., Popescu, M., Vrabie, I.I.: Viability for differential inclusions on graphs. Set-Valued Anal. **16**, 961–981 (2008)
17. Necula, M., Popescu, M., Vrabie, I.I.: Evolution equations on locally closed graphs and applications. Nonlinear Anal. **71**, e2205–e2216 (2009)
18. Pavel, N.H., Iacob, F.: Invariant sets for a class of perturbed differential equations of retarded type. Isr. J. Math. **28**, 254–264 (1977)
19. Vrabie, I.I.: Compactness Methods for Nonlinear Evolutions. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 75, 2nd edn. Longman, New York (1995)