Distribution of Energy Levels in Quantum Systems with Integrable Classical Counterpart. Rigorous Results.

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Let $E_0 \le E_1 \le E_2 \le \ldots$ be the energy levels (eigenvalues) of the Schrödinger operator $H = -\frac{1}{2}\Delta + U(q)$ on a closed d-dimensional Riemannian manifold M^d. Here

$$-\Delta = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial q^i}(\sqrt{g}\, g^{ij}\frac{\partial}{\partial g^j}) \tag{1}$$

is the Laplace-Beltrami operator and to ensure the discreteness of the spectrum of H we assume, in the case of a non-compact M^d, that $\lim_{q\to\infty} U(q) = \infty$. For simplicity we assume also that M^d has no boundary. Otherwise it is neccessary to supply H with Dirichlet (or some other) boundary conditions.

The main problem we are interested in is the statistics of the energy levels $\{E_k, k \ge 0\}$ in large spectral intervals. To be more precise we describe, following works [4], [7], two concrete problems. Let $N(E) = \#\{k \mid E_k \le E\}$ be the counting function of the energy levels. By $\#A$ we denote the number of elements in a set A.

1 Distribution of Level Spacings. Let $\Delta E_k = E_k - E_{k-1}, k \ge 0$, be spacings between neighboring energy levels. We are interested in the limit distribution of ΔE_k in the spectral interval $E_0 \le E_k \le E$, when $E \to \infty$. Let us define the distribution function of the spacings,

$$P(x; E) = \frac{\#\{k \mid E_k \le E,\ \Delta E_k \le x\Delta E\}}{N(E)} \quad ,$$

where

$$\Delta E = \frac{E - E_0}{N(E)} \tag{2}$$

is the mean spacing. The problem is: Does the limit

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$$P(x) = \lim_{E\to\infty} P(x;E) \tag{3}$$

exist and if so, what is $P(x)$?

2 Distribution of the Number of Energy Levels in Intervals of a Prescribed Length. Let $\lambda > 0$ be a fixed constant. Let us define

$$A_n(E) = \{E_0 \le E' \le E \mid \#\{k \mid E' \le E_k \le E' + \lambda\Delta E\} = n\},\ n \ge 0\ , \tag{4}$$

and

$$\pi(n;E) = (E - E_0)^{-1} \int_{A_n(E)} dE' \quad .$$

One can see that $\pi(n;E)$ is the probability that a spectral interval $E' \le E_k \le E' + \lambda\Delta E$, with E' uniformly distributed in $[E_0, E]$, contains exactly n energy levels. The problem is the existence and the calculation of

$$\lim_{E\to\infty} \pi(n;E) = \pi(n)\ ,\ \ n \ge 0\,. \tag{5}$$

A general conjecture is the following. Assume that the flow in the cotangent bundle T^*M^d over M^d generated by the Hamilton equations

$$\dot q = \frac{\partial H(p,q)}{\partial p} \qquad \dot p = -\frac{\partial H(p,q)}{\partial q}$$

with $H(p,q) = \frac{1}{2}p^2 + U(q)$ is a completely integrable system, i.e. there exist d Poisson-commuting independent integrals of the motion $I_1(p,q), \dots, I_d(p,q)$. Then in 'typical case' the limit distribution $P(x)$ in (3) is exponential, i.e.

$$p(x) \equiv -\frac{dP(x)}{dx} = \exp(-x) \tag{6}$$

and the limit distribution $\pi(n)$ in (5) is Poisson, i.e.

$$\pi(n) = \frac{\lambda^n}{n!}\exp(-\lambda) \quad . \tag{7}$$

It is noteworthy, that the limit distributions (6), (7) are the same as if the energy levels E_k were independent random variables, uniformly distributed in the interval $E_0 \le E_k \le E$. Therefore (6), (7) can be interpreted as the absence of an interaction between energy levels.

A proof of (6) was given in the pioneer work [4] of Berry and Tabor, but it was not completely rigorous. The ideas of [4] were used by Sinai in [7] to prove (7) for the Laplace-Beltrami operator on two-dimensional revolution surfaces. In that case the geodesic flow is integrable because of the Clairaut integral. The strategy of Sinai was the following. Let M^2 be a periodic revolution surface in $\mathbf{R}^3$ which is defined in cylindric coordinates (r, φ, z) by the equation $r = f(z)$, where $f(z) > 0$ is a smooth periodic function, $f(z+a) = f(z)$. Let ds^2 be the Euclidean metric, restricted to M^2. Because of the periodicity of $f(z)$ we may consider z as a cyclic coordinate, so that M^2 is a compact Riemannian manifold diffeomorphic to a torus. For the sake of simplicity we will assume that $f(z)$ has only one maximum and one minimum, both non-degenerate, in a period. One can choose a new cyclic coordinate $s = s(z)$ in such a way that the eigenvalue problem $-\Delta u = Eu$ on M^2 is written as

$$-\frac{\partial^2 u}{\partial s^2} - \frac{\partial^2 u}{\partial \varphi^2} = Ef^2(s)u \quad . \tag{8}$$

We look for u in the form $u(s,\varphi) = v(s)e^{in\varphi}$, and for $v(s)$ we get the equation

$$-v''(s) + \left(Ef^2(s) - n^2\right)v(s) = 0 \tag{9}$$

with periodic boundary condition. The integer n is a parameter of the equation. For every n we have, obviously, a sequence of eigenvalues E_{mn}, $m \geq 0$, of (9).

Now the problem is decomposed into two stages. At the first stage we solve (9) in the quasi-classical approximation (QCA) and at the second one we deal with a geometric probability problem. The main difficulty of the first stage is to obtain uniform estimates of the remainder term of the QCA and to show that almost all eigenvalues of (9) can be found in the QCA. This problem was solved in [1]. It is important to stress here an aspect of multiplicity of the eigenvalues. Let us note first that actually every eigenvalue E_{mn} is twice degenerate, when $n \neq 0$, since $E_{m,-n} = E_{mn}$. Besides, an analysis of the QCA shows that for every $n \geq 0$ there are two series of eigenvalues,

$$\Sigma_n^{(1)} = \{E_{mn} \mid E_{mn}f_{min}^2 - n^2 > 0\}, \qquad f_{min} = \min_s f(s),$$
$$\Sigma_n^{(2)} = \{E_{mn} \mid E_{mn}f_{min}^2 - n^2 < 0\},$$

such that the eigenvalues of the first series are asymptotically twice degenerate, namely for all $N > 0$,

$$\mid E_{2m'-1,n} - E_{2m',n} \mid = O\left((m'^2 + n^2)^{-N}\right) \quad ,$$

while the eigenvalues of the second series are strongly separated each from other. The Bohr-Sommerfeld quantization condition is written as

$$\frac{1}{\pi}\int_0^{s(a)} \sqrt{E_{mn}f^2(s) - n^2}ds = 2m', \quad m = 2m' - 1,\ 2m'$$

for the first series of eigenvalues and as

$$\frac{1}{\pi}\int_{s1}^{s2} \sqrt{E_{mn}f^2(s) - n^2}ds = m + \frac{1}{2} \quad ,$$

where s_1, s_2 are the turning points, $E_{mn}f^2(s_1) - n^2 = E_{mn}f^2(s_2) - n^2 = 0$, for the second one. Because of different asymptotic multiplicity of eigenvalues in the two series, it is neccessary to establish the Poisson law (7) for each of them separately. With the help of quasi-classical quantization conditions this problem is reduced to a geometric probability problem. Simplifying it somewhat, we can formulate it in the following way (see [7]; for a full formulation see [1]).

Let γ be a curve in the plane which is defined in polar coordinates (r, α) by the equation $r = G(\alpha), \alpha_0 \leq \alpha \leq \alpha_1$, where $G(\alpha)$ is a positive continuous function. Let γ_R with $R > 0$ be the curve which is defined by the equation $r = RG(\alpha)$, $\alpha_0 \leq \alpha \leq \alpha_1$, so that γ_R is just the homothety of γ with the coefficient R. Let Π_R be the strip in the sector $\alpha_0 \leq \alpha \leq \alpha_1$ between γ_R and $\gamma_{R+\frac{\lambda}{2R}}$, where $\lambda > 0$ is the same as in (4). Let $\pi_k(E)$ be the probability that Π_R contains exactly k integer points (m, n) assuming that R^2 is uniformly distributed in $[0, E]$. The problem is to prove that

$$\lim_{k \to \infty} \pi_k(E) = \frac{\lambda^k}{k!} \exp(-\lambda) \quad . \tag{10}$$

A 'stochastic' version of this problem was solved by Sinai and Major (see [8], [5]). They introduced a class of probability distributions on the space $\mathrm{Lip}^+([\alpha_0, \alpha_1])$ of positive continuous Lipschitz functions on $[\alpha_0, \alpha_1]$, for which they proved the following theorem: For every probability distribution P from this class, (10) is true for P-almost every function $G(\alpha) \in \mathrm{Lip}^+([\alpha_0, \alpha_1])$. In particular, it proves an existence theorem of curves $\gamma = \{r = G(\alpha)\}$, for which (10) is true. In this respect it resembles the famous Cantor's proof of the existence of transcendental numbers. The situation here is however even more surprising: No example of γ for which (10) is true is known now. On the contrary, there exist examples for which (10) is known to be false, e.g. ellipses with rational principal axes (it corresponds to a plain torus with rational periods as M^2) and some others.

The Sinai - Major result combined with the estimates of QCA in [1] supports strongly the following 'stochastic' version of the Poisson conjecture (7): For almost every revolution surface with respect to a probability distribution P, $\lim_{E \to \infty} \pi(n, E) = \pi(n)$ exists and is a mixture of Poisson distributions with multiplicities 2 and 4. This problem remains open.

Let us turn now to linear quantum systems. In that case the Schrödinger operator is written as

$$H = \frac{1}{2} \sum_{j=1}^{d} \left(-\frac{\partial^2}{\partial q_j^2} + \omega_j^2 q_j^2 \right)$$

and its energy levels are $E_m = E_0 + (m, \omega)$, where $m = (m_1, \ldots, m_d)$, $m_1, \ldots, m_d \geq 0$, $\omega = (\omega_1, \ldots, \omega_d)$, $(m, \omega) = m_1\omega_1 + \ldots + m_d\omega_d$, $E_0 = 1/2(\omega_1 + \ldots + \omega_d)$, so we are dealing here with the statistics of numbers of the form (m,ω). Let $d = 2$. Since $m_1\omega_1 + m_2\omega_2 = \omega_1(m_1 + m_2\alpha)$, $\alpha = \omega_2/\omega_1$, it is enough to consider the numbers $\{\lambda_{mn} = m+n\alpha; m,n \geq 0\}$. Let $\Delta\lambda_{mn} = \lambda_{mn} - \lambda_{m'n'}$ be the spacing between λ_{mn} and the previous level, and $P_\alpha(x;E)$ be the distribution of $\Delta\lambda_{mn}/\Delta\lambda$ in the spectral interval $0 \leq \lambda_{mn} \leq E$. Here $\Delta\lambda = E/n(E)$ is the mean spacing. Again the problem is: Does $\lim_{E\to\infty} P_\alpha(x;E) = P_\alpha(x)$ exist and what is $P_\alpha(x)$? This problem was studied in works [4], [6], [2,3] and others. We present here the main result of [2,3].

Let $0 < \alpha < 1$ be an irrational number, represented by a continued fraction $[a_1, a_2, \ldots]$. Let $\frac{p_k}{q_k} = [a_1, a_2, \ldots, a_k], k \geq 1$, be the approximants of α. We are interested in

$$\lim_{k\to\infty} P_\alpha(x;p_k) = P_\alpha(x)$$

and we formulate two results, negative and positive.

Theorem 1 (negative result) *For almost every α, $\lim\limits_{k\to\infty} P_\alpha(x;p_k)$ does not exist.*

Theorem 2 (positive result) *For almost every α, $\lim\limits_{k\to\infty} \frac{1}{k}\sum\limits_{j=1}^{k} P_\alpha(x;p_j) = P(x)$ exists and does not depend on α.*

Actually in [2,3] a more detailed description of $p_\alpha(x;E)$ is given, but we have no place here to discuss it.

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