A Semantical and Operational Account of Call-by-Value Solvability

Alberto Carraro[1,2] and Giulio Guerrieri[2]

[1] DAIS, Università Ca' Foscari Venezia
alberto.carraro@unive.it
[2] PPS, Univ Paris Diderot, Sorbonne Paris Cité
giulio.guerrieri@pps.univ-paris-diderot.fr

Abstract. In Plotkin's call-by-value lambda-calculus, solvable terms are characterized syntactically by means of call-by-name reductions and there is no neat semantical characterization of such terms. Preserving confluence, we extend Plotkin's original reduction without adding extra syntactical constructors, and we get a call-by-value operational characterization of solvable terms. Moreover, we give a semantical characterization of solvable terms in a relational model, based on Linear Logic, satisfying the Taylor expansion formula. As a technical tool, we also use a resource-sensitive calculus (with tests) in which the elements of the model are definable.

Keywords: (resource) call-by-value lambda-calculus, tests, potential valuability, solvability, relational semantics, weak and stratified reductions.

1 Introduction

In the theory of ordinary (i.e. untyped call-by-name) λ-calculus, the notion of solvability plays a crucial role. A λ-term M is *solvable* if there is a *head context* $\mathtt{H}$ such that $\mathtt{H}(\!|M|\!) \twoheadrightarrow_\beta \lambda x.x = \mathbf{I}$ (the identity); M is *unsolvable* if it is not solvable. Solvability (see [1]) underlies the fundamental notions of approximants, Böhm-trees and separability; moreover, it is possible to encode partial recursive functions in λ-calculus in such a way that undefinedness is represented by unsolvable λ-terms ([1, Ch. 8]). Enforcing the idea of unsolvable-as-meaningless, it is consistent to equate all unsolvable λ-terms (but not all λ-terms having no β-normal form, [1, Ch. 16]). A fundamental theorem for ordinary λ-calculus (see [2,3]) states that for every λ-term M the following are equivalent: (1) M is solvable; (2) the head reduction of M terminates; (3) the semantics of M in the Scott's model D_∞ is not the least element. Equivalence (1)$\Leftrightarrow$(2) (resp. (1)$\Leftrightarrow$(3)) gives a *semantical* (resp. *syntactical* or *operational*) characterization of solvability in ordinary λ-calculus.

The most common parameter passing policy for programming languages is call-by-value (CBV). Plotkin [4] introduced the λ_v-calculus in order to grasp the CBV paradigm in a pure λ-calculus setting. The λ_v-calculus (without constants)

A. Muscholl (Ed.): FOSSACS 2014, LNCS 8412, pp. 103–118, 2014.

has the same syntax as ordinary λ-calculus but its β_v-reduction rule allows the contraction of a β-redex only if the argument is a λ-value, i.e. a variable or an abstraction. As argued in [5], a good CBV λ-calculus should enjoy an *internal* operational characterization (i.e. by using CBV reduction rules) of CBV-solvability. This is not the case for Plotkin's λ_{v}-calculus and the weakness of β_v-reduction is widely recognized and accepted. Following [6,7], a λ-term M is λ_{v}-*solvable* if there is a head context $\mathtt{H}$ such that $\mathtt{H}(\!|M|\!) \twoheadrightarrow_{\beta_v} \mathbf{I}$. Let $\Delta = \lambda x.xx$: there is no head context sending (via β_v-reduction) $N = (\lambda y.\Delta)(x\mathbf{I})\Delta$ to $\mathbf{I}$, thus N is λ_{v}-unsolvable and hence it should be divergent, whereas it is β_v-normal. An operational characterization of λ_{v}-solvability has been provided in [6,7] but through a *call-by-name* reduction; this result is improved in [8] where the characterization is built upon strong normalization of the (call-by-name) lazy β-reduction.

There are many proposals of alternative CBV λ-calculi (see [9,10,11,12,5]) extending Plotkin's one by using explicit substitutions (constructors of the form `let...in`). In particular, Accattoli and Paolini [5] introduced recently the $\lambda_{\mathtt{vsub}}$-calculus where the reduction rule acts at a distance by extending the notion of β_v-redex (with explicit substitutions). In this setting they give an internal operational characterization of solvability and this characterization lifts to Herbelin and Zimmermann's λ_{CBV}-calculus, another CBV λ-calculus with explicit substitutions introduced in [9] (without rules acting at a distance but with commutation rules for explicit substitutions).

Paolini and Ronchi Della Rocca [6,7] made major contributions to the study of CBV-solvability through denotational semantics. In [6] they showed an intersection type system that characterizes λ_{v}-*potentially valuable*[1] (Thm. 6.4) and λ_{v}-solvable λ-terms (Thm. 6.5). We quote from [6, p. 28]: "The type assignment system presented here is strongly related to the system presented in [13] for reasoning on the denotational semantics of the [Plotkin's] λ_{v}-calculus. [...] The two systems have the same typability power". It is not shown whether this type system is "legal" (see [7, Def. 10.1.5]), which is substantially a sufficient condition to turn the type system into a *filter model* (i.e. a true domain model). In [7, Ch. 12] the same authors exhibit two models, $\mathcal{V}$ (§ 12.1) and $\mathcal{VV}$ (§ 12.2), both built from intersection type systems. The model $\mathcal{V}$ comes from a legal type system and it is shown to be isomorphic to the one of [13]. All and only λ_{v}-potentially valuable λ-terms have non trivial interpretation in $\mathcal{V}$, but $\mathcal{V}$ gives only a *partial* semantical characterization of λ_{v}-solvable λ-terms (Thm. 12.1.19). The model $\mathcal{VV}$ characterizes observational equivalence (Thm. 12.2.14) but it is not a filter model. Recently, Ehrhard [14] used a relational model of the λ_{v}-calculus, based on Linear Logic, to show that if the semantics of a λ-term M is not empty, then M is strongly normalizing for the lazy β_v-reduction (which does not reduce under abstractions); the converse is false (the aforesaid λ-term N is a counterexample).

[1] Following [6,7], a λ-term is λ_{v}-potentially valuable if there is a substitution sending it (via $\twoheadrightarrow_{\beta_v}$) into a λ-value. This notion is important for a CBV λ-calculus because if we want to manipulate some subterms, we need first to transform them into λ-values.

The starting points of our work are [6,5,14]. We introduce the $\lambda_{\mathsf{v}}^{\sigma}$-calculus, a CBV λ-calculus having the same syntax as ordinary (and hence Plotkin's CBV) λ-calculus (there are no explicit substitutions) and extending the β_v-reduction by adding two reduction rules, σ_1 and σ_3. For the $\lambda_{\mathsf{v}}^{\sigma}$-calculus we give a semantical and an internal operational characterization of solvability and potential valuability. We use the relational model of [14], which can also be seen as a model of ordinary λ-calculus (unlike the model $\mathcal{V}$ of [7]) and satisfies a version of the Taylor formula (see [14]). We also introduce a resource-sensitive calculus with tests in which the elements of the relational model are definable: this is a promising tool to face the CBV full abstraction problem, along the lines of [15].

Our $\lambda_{\mathsf{v}}^{\sigma}$-calculus springs from Girard's call-by-value "boring" translation $(\cdot)^v$ of λ-calculus into Intuitionistic Multiplicative Exponential Linear Logic (IMELL) proof-nets, identified by $(A \Rightarrow B)^v = !A^v \multimap !B^v$ (see [16]). The images of a σ_1- or σ_3-redex and its contractum under $(\cdot)^v$ are equal modulo some specified "immediate" steps of cut-elimination. Our σ-rules are related to (but partly different from) Regnier's σ-reduction defined in [17,18] for the ordinary λ-calculus. Moreover, σ_1 and σ_3 correspond respectively to the commutation rules let_{app} and (a generalization of) let_{let} in λ_{CBV}-calculus (see [9,5]). In some sense, they can be seen as a finer (and local) decomposition of the reduction rules acting at a distance in $\lambda_{\mathtt{vsub}}$-calculus (it is possible to simulate $\lambda_{\mathtt{vsub}}$- and λ_{CBV}-calculus in our $\lambda_{\mathsf{v}}^{\sigma}$-calculus), but the absence of explicit substitutions in $\lambda_{\mathsf{v}}^{\sigma}$-calculus prevents from lifting the internal operational characterization of CBV-solvability from $\lambda_{\mathtt{vsub}}$- or λ_{CBV}-calculus to our $\lambda_{\mathsf{v}}^{\sigma}$-calculus.

Outline. In §2 we introduce our $\lambda_{\mathsf{v}}^{\sigma}$-calculus. Then, §3, §4 and §5 are devoted to the technical notions which are necessary in order to state our main results: in §3 we present two sub-reductions in the $\lambda_{\mathsf{v}}^{\sigma}$-calculus, called $\mathtt{w}$- and $\mathtt{s}$-reduction; in §4 and §5 we present a resource-sensitive version of the $\lambda_{\mathsf{v}}^{\sigma}$-calculus and the relational model of the (resource) $\lambda_{\mathsf{v}}^{\sigma}$-calculus. In §6 we state and prove our main theorems: the semantical (via the relational model) and syntactical (via $\mathtt{w}$- and $\mathtt{s}$-reductions) characterization of potential valuability and solvability; they say also that weak and strong normalizations coincide for both $\mathtt{w}$- and $\mathtt{s}$-reductions.

2 A CBV Lambda-Calculus with Sigma-Like-Reductions

In this section we introduce $\lambda_{\mathsf{v}}^{\sigma}$, our version of CBV λ-calculus. The syntax of $\lambda_{\mathsf{v}}^{\sigma}$ is the same as the one of ordinary λ-calculus. Given a countable set of *variables* (denoted by $x, y, z, \dots$), the language of $\lambda_{\mathsf{v}}^{\sigma}$ is defined by the following grammar:

$$\begin{array}{lrcll} (\Lambda^{\mathsf{v}}) & V, U & ::= & x \mid \lambda x.M & \lambda\text{-}\mathit{values} \\ (\Lambda) & M, N, L & ::= & V \mid MN & \lambda\text{-}\mathit{terms} \end{array}$$

All λ-terms are considered up to α-conversion. The set of free variables of a λ-term M is denoted by $\mathsf{fv}(M)$. Given pairwise distinct variables $x_1, \dots, x_n$, we denote by $M\{V_1/x_1, \dots, V_n/x_n\}$ the λ-term obtained by the *capture-avoiding* *simultaneous substitution* of each free occurrence of x_i in the λ-term M by the λ-value V_i (for $1 \leq i \leq n$). Notice that, for all λ-values $V, V_1, \ldots, V_n$ and pairwise distinct variables $x_1, \ldots, x_n$, $V\{V_1/x_1, \ldots, V_n/x_n\}$ is a λ-value.

Contexts (with exactly one hole) are defined as usual via the grammar:

$$\mathtt{C} ::= (\!|\cdot|\!) \mid \lambda x.\mathtt{C} \mid \mathtt{C}M \mid M\mathtt{C}.$$

We use $\mathtt{C}(\!|M|\!)$ for the λ-term obtained by the capture-allowing substitution of the λ-term M for $(\!|\cdot|\!)$ in the context $\mathtt{C}$.

Definition 1. *We define the following binary relations from Λ to Λ:*

$$\begin{array}{lll} (\lambda x.M)V & \mapsto_{\beta_v} M\{V/x\} & \textit{with } V \in \Lambda^{\mathsf{v}} \\ (\lambda x.M)NL & \mapsto_{\sigma_1} (\lambda x.ML)N & \textit{with } x \notin \mathsf{fv}(L) \\ V((\lambda x.L)N) & \mapsto_{\sigma_3} (\lambda x.VL)N & \textit{with } x \notin \mathsf{fv}(V) \textit{ and } V \in \Lambda^{\mathsf{v}} \end{array}$$

*For $R \in \{\beta_v, \sigma_1, \sigma_3\}$, if $M \mapsto_R M'$ then M is called R-*redex.

We set $\mapsto_\sigma = \mapsto_{\sigma_1} \cup \mapsto_{\sigma_3}$ and $\mapsto_{\mathsf{v}} = \mapsto_{\beta_v} \cup \mapsto_\sigma$.

The side conditions on $\mapsto_\sigma$ in Def. 1 can be always fulfilled by α-renaming.

Notation. Let $\mapsto_R \subseteq \Lambda \times \Lambda$. We use $\to_R$ (called *R-reduction*) for the closure of $\mapsto_R$ under all contexts; we denote by $\twoheadrightarrow_R$ (resp. $\to_R^+$) the reflexive-transitive (resp. transitive) closure of $\to_R$. Let M be a λ-term: M is *R-normal* if there is no λ-term N such that $M \to_R N$; M is *R-normalizable* if there is a R-normal λ-term N such that $M \twoheadrightarrow_R N$; M is *strongly R-normalizing* if there is no sequence $(N_i)_{i\in\mathbf{N}}$ such that $M = N_0$ and $N_i \to_R N_{i+1}$ for every $i \in \mathbf{N}$.

Notice that, for any λ-value V, if $V \to_{\mathsf{v}} M$, then M is a λ-value.

The $\lambda_{\mathsf{v}}^{\sigma}$*-calculus* is the set Λ of λ-terms endowed with the v-reduction $\to_{\mathsf{v}}$. The set Λ endowed with $\to_{\beta_v}$ is Plotkin's CBV λ-calculus ([4]) without constants.

Informally, σ-rules unblock β_v-redexes which are hidden by the "hyper-sequential structure" of λ-terms. This approach is alternative to the one in [5] where hidden β_v-redexes are reduced thanks to a rule acting at a distance.

Example. $N = (\lambda y.\Delta)(x\mathbf{I})\Delta \to_{\sigma_1} (\lambda y.\Delta\Delta)(x\mathbf{I}) \to_{\beta_v} (\lambda y.\Delta\Delta)(x\mathbf{I}) \to_{\beta_v} \ldots$ is the only possible v-reduction path from N: N is not v-normalizable but β_v-normal.

2.1 Confluence of Our CBV Lambda-Calculus

Our goal here is to prove that the v-reduction is confluent.

Proposition 2. *The reduction $\to_\sigma$ is strongly normalizing.*

Proof. First, we define two sizes $\mathsf{s}(M)$ and $\#M$ by induction on the λ-term M:

$$\begin{array}{ll} \mathsf{s}(x) = 2; & \#x = 1; \\ \mathsf{s}(\lambda x.M) = \mathsf{s}(M) + 1; & \#\lambda x.M = \#M + \mathsf{s}(M); \\ \mathsf{s}(MN) = \mathsf{s}(M) + \mathsf{s}(N). & \#MN = \#M + \#N + 2\mathsf{s}(M)\mathsf{s}(N) - 1. \end{array}$$

It is sufficient to show that if $N \to_\sigma N'$ then $\mathsf{s}(N) = \mathsf{s}(N')$ and $\#N > \#N'$. □

Proposition 3. *The reduction $\to_\sigma$ is (not strongly) confluent.*

Proof. By Newman's Lemma and Prop. 2, it is sufficient to show that $\to_\sigma$ is locally confluent. The proof of local confluence is by induction on M. The λ-term $\Xi = (\lambda x.x')\big((\lambda y.y'\mathbf{I})(z\mathbf{I})\big)(z'\mathbf{I})$ is an objection to strong confluence of $\to_\sigma$. □

Lemma 4 (Hindley–Rosen, [1, p. 64]). *Let $\to_1, \to_2 \subseteq X^2$ (for any set X). If they are both confluent and they commute, i.e. if $t \twoheadrightarrow_1 u_1$ and $t \twoheadrightarrow_2 u_2$ then there exists s such that $u_1 \twoheadrightarrow_2 s$ and $u_2 \twoheadrightarrow_1 s$, then $\to_1 \cup \to_2$ is confluent.*

Lemma 5. *Let $M, M' \in \Lambda$, $V, V', V_1, \ldots, V_m \in \Lambda^{\mathrm{v}}$ and $R \in \{\beta_v, \sigma, \mathsf{v}\}$.*

(i) *If $V \to_R V'$ then $M\{V/x\} \twoheadrightarrow_R M\{V'/x\}$.*
(ii) *If $M \to_R M'$ then $M\{V_1/x_1, \ldots, V_m/x_m\} \to_R M'\{V_1/x_1, \ldots, V_m/x_m\}$.*

Lemma 6. *The reductions $\to_{\beta_v}$ and $\to_\sigma$ commute.*

Proof. It suffices to prove that if $M \to_\sigma N_1$ and $M \to_{\beta_v} N_2$ then there is L s.t. $N_2 \twoheadrightarrow_\sigma L$ and $N_1 \to_{\beta_v} L$. The proof of this statement is by induction on M. □

By Lemmas 4 and 6, Prop. 3 and confluence of $\to_{\beta_v}$ (see [4]), we conclude:

Theorem 7. *The reduction $\to_{\mathsf{v}}$ is (not strongly) confluent.*

The λ-term Ξ (see proof of Prop. 3) is an objection to strong confluence of $\to_{\mathsf{v}}$.
If in the definition of $\mapsto_{\sigma_3}$ (Def. 1) we replace the λ-value V with any λ-term M then $\to_\sigma$ and $\to_{\mathsf{v}}$ are not (locally) confluent: consider $(\lambda x.x')(z\mathbf{I})\big((\lambda y.y')(z'\mathbf{I})\big)$.

3 Weak and Stratified CBV Reductions

In this section we introduce two sub-reductions of $\to_{\mathsf{v}}$: *weak* (or $\mathtt{w}$-)*reduction* and *stratified* (or $\mathtt{s}$-)*reduction*. We will show in §6 that they give an operational characterization of potential valuability and solvability: they are the "CBV counterpart" of head reduction for ordinary λ-calculus. Whereas head reduction is strictly deterministic (any λ-term has at most one head redex), a λ-term might have several (overlapping) $\mathtt{w}$- or $\mathtt{s}$-redexes. Anyway, both $\mathtt{w}$- and $\mathtt{s}$-reductions are confluent (Prop. 10) and for them weak and strong normalization coincide (Thm. 24 and 25). We have gathered our definition of $\mathtt{w}$- and $\mathtt{s}$-reductions from [5].

Definition 8. Weak *and* stratified contexts *(denoted respectively by* $\mathtt{W}$ *and* $\mathtt{S}$*) are contexts defined via the grammar:*

$$\mathtt{W} ::= (\!|\cdot|\!) \mid \mathtt{W}M \mid M\mathtt{W} \mid (\lambda x.\mathtt{W})M \qquad \mathtt{S} ::= \mathtt{W} \mid \lambda x.\mathtt{S} \mid \mathtt{S}M$$

Notation. Let $\mapsto_R \subseteq \Lambda \times \Lambda$: we use $\to_{\mathtt{w}[R]}$ (resp. $\to_{\mathtt{s}[R]}$) for the closure under weak (resp. stratified) contexts of $\mapsto_R$. We set $\mathtt{w} = \mathtt{w}[\mathsf{v}]$ and $\mathtt{s} = \mathtt{s}[\mathsf{v}]$; for instance, $\to_{\mathtt{w}} = \to_{\mathtt{w}[\mathsf{v}]}$ (called $\mathtt{w}$-*reduction*) and $\to_{\mathtt{s}} = \to_{\mathtt{s}[\mathsf{v}]}$ (called $\mathtt{s}$-*reduction*).

Note that $\to_{\mathtt{w}} \subsetneq \to_{\mathtt{s}} \subsetneq \to_{\mathsf{v}}$. In weak contexts, if the hole is under an abstraction then this abstraction is the left-hand side of an application. Stratified contexts never contain the hole under an abstraction which is in the right-hand side of some application, unless the abstraction is the left-hand side of an application.

Example. Let $\Omega = \Delta\Delta$: one has $\Omega \to_{\mathtt{w}} \Omega \to_{\mathtt{w}} \dots$, $\lambda y.\Omega \to_{\mathtt{s}} \lambda y.\Omega \to_{\mathtt{s}} \dots$, and $x(\lambda y.\Omega) \to_{\mathsf{v}} x(\lambda y.\Omega) \to_{\mathsf{v}} \dots$, whereas $\lambda y.\Omega$ (resp. $x(\lambda y.\Omega)$) is w-(resp. s-)normal.

We will now prove that the w- and s-reductions are confluent.

Lemma 9. (i) *The reductions* $\to_{\mathtt{w}[\beta_v]}$ *and* $\to_{\mathtt{s}[\beta_v]}$ *are strongly confluent.*
(ii) *The reductions* $\to_{\mathtt{w}[\sigma]}$ *and* $\to_{\mathtt{s}[\sigma]}$ *are confluent.*
(iii) *The reductions* $\to_{\mathtt{w}[\beta_v]}$ *and* $\to_{\mathtt{w}[\sigma]}$ *(resp.* $\to_{\mathtt{s}[\beta_v]}$ *and* $\to_{\mathtt{s}[\sigma]}$*) commute.*

By Lemmas 4 and 9 we can conclude:

Proposition 10. *The reductions* $\to_{\mathtt{w}}$ *and* $\to_{\mathtt{s}}$ *are (not strongly) confluent.*

The λ-term Ξ (see p. 107) is an objection to strong confluence of $\to_{\mathtt{w}}$ and $\to_{\mathtt{s}}$.

3.1 Characterization of w- and s-Normal Forms

Our goal here is to characterize w- and s-normal forms. Having no explicit substitutions, our characterization appears more concise than the one in [5].

Definition 11. *We define the subsets* $\mathsf{a}_{\mathtt{nf}}$, $\mathsf{s}_{\mathtt{nf}}$ *and* $\mathsf{w}_{\mathtt{nf}}$ *of* Λ *as follows:*

$$
\begin{array}{lrcl}
(\mathsf{a}_{\mathtt{nf}}) & A_{\mathtt{nf}} & ::= & xV \mid xA_{\mathtt{nf}} \mid A_{\mathtt{nf}}W_{\mathtt{nf}} \\
(\mathsf{w}_{\mathtt{nf}}) & W_{\mathtt{nf}} & ::= & V \mid (\lambda x.W_{\mathtt{nf}})A_{\mathtt{nf}} \mid A_{\mathtt{nf}} \\
(\mathsf{s}_{\mathtt{nf}}) & S_{\mathtt{nf}} & ::= & x \mid \lambda x.S_{\mathtt{nf}} \mid (\lambda x.S_{\mathtt{nf}})A_{\mathtt{nf}} \mid A_{\mathtt{nf}}
\end{array}
$$

A β-redex is a λ-term of shape $(\lambda x.M)L$. Notice that $\mathsf{a}_{\mathtt{nf}} \subsetneq \mathsf{s}_{\mathtt{nf}} \subsetneq \mathsf{w}_{\mathtt{nf}}$ and if $N \in \mathsf{a}_{\mathtt{nf}}$ then N has a free "head variable" and it is neither a value nor a β-redex.

Proposition 12. *Let* M *be a* λ*-term.*

(i) M *is* w*-normal iff* $M \in \mathsf{w}_{\mathtt{nf}}$.
(ii) M *is* s*-normal iff* $M \in \mathsf{s}_{\mathtt{nf}}$.
(iii) M *is* w*-(resp.* s*-)normal and is neither a value nor a* β*-redex iff* $M \in \mathsf{a}_{\mathtt{nf}}$.

4 A Resource CBV Lambda-Calculus

We now introduce the *resource* $\lambda_{\mathsf{v}}^{\sigma}$*-calculus*, a valuable tool to prove some parts of our main results. It is an extension of the resource CBV λ-calculus introduced in [14, §5.2]. Its syntax is defined by the following grammar (the same as in [14]):

$$
\begin{array}{lrcll}
(\mathsf{r}\Lambda^{\mathtt{v}}) & u, v & ::= & x \mid \lambda x.t & \textit{resource values} \\
(\mathsf{r}\Lambda^{\mathtt{t}}) & s, t & ::= & st \mid [v_1, \dots, v_k] \quad (k \geq 0) & \textit{resource terms} \\
(\mathsf{r}\Lambda) & e, f & ::= & v \mid t & \textit{expressions}
\end{array}
$$

A resource term like $[v_1, \dots, v_k]$ is a multiset of resource values (called *bag*).

The resource-version of the β_v-rule makes use of *linear substitution*, which requires to enrich the syntax of the calculus with finite sets of resource terms.

Notation. Since the set $\mathcal{P}_{\mathrm{f}}(A)$ of all finite subsets of a set A is the free module $\mathbf{2}\langle A\rangle$ generated by A over the boolean semiring $\{0,1\}$ with $1+1=1$, we will use algebraic notations for operations on its elements (+ for set unions, 0 for the empty set), as done in [15,14].

We denote by $\mathsf{deg}_x(e)$ the number of free occurrences of the variable x in the expression e. Given $e \in \mathsf{r}\Lambda$, $v_1, \dots, v_k \in \mathsf{r}\Lambda^{\mathtt{v}}$ and an enumeration of the free occurrences of variable x in e, if $\mathsf{deg}_x(e) = k$ then by $\sum_{f\in\mathfrak{S}_k} e\{v_{f(1)}/x^1, \dots, v_{f(k)}/x^k\}$ we mean the sum of all expressions obtained by substituting $v_{f(i)}$ for the i-th free occurrence of x in e, as f varies over all elements of the set $\mathfrak{S}_k$ of permutations of $\{1,\dots,k\}$. Finally, the linear substitution of $[v_1,\dots,v_k]$ for x in e is

$$e\langle[v_1,\dots,v_k]/x\rangle = \begin{cases}\sum_{f\in\mathfrak{S}_k} e\{v_{f(1)}/x^1,\dots,v_{f(k)}/x^k\} & \text{if } \mathsf{deg}_x(e)=k\\ 0 & \text{otherwise}\end{cases}$$

Notice that, for $\mathtt{n} \in \{\mathtt{v},\mathtt{t}\}$, if $e \in \mathsf{r}\Lambda^{\mathtt{n}}$ then $e\langle[v_1,\dots,v_k]/x\rangle \in \mathbf{2}\langle\mathsf{r}\Lambda^{\mathtt{n}}\rangle$.

Resource contexts (with exactly one hole) are defined via the grammar:

$$\mathtt{R} ::= (\!|\cdot|\!) \mid \mathtt{R}t \mid t\mathtt{R} \mid [\lambda x.\mathtt{R}, v_1,\dots,v_k] \quad (k \geq 0)$$

Let $\mathtt{R}$ be a resource context. We use $\mathtt{R}(\!|t|\!)$ for the resource term obtained by the capture-allowing substitution of the resource term t for the hole $(\!|\cdot|\!)$ in $\mathtt{R}$. If $\mathbb{T} = \sum_{i=1}^n t_i$ (with $t_1,\dots,t_n \in \mathsf{r}\Lambda^{\mathtt{t}}$), then $\mathtt{R}(\!|\mathbb{T}|\!) = \sum_{i=1}^n \mathtt{R}(\!|t_i|\!) \in \mathbf{2}\langle\mathsf{r}\Lambda^{\mathtt{t}}\rangle$ (see also [14, §5.2] and [15, §2.1]). For example, $\mathtt{R}(\!|0|\!) = 0$ and $[\lambda x.[x](\!|[y][z]+[z][y]|\!), y] = [\lambda x.[x]([y][z]), y] + [\lambda x.[x]([z][y]), y]$.

Definition 13. *We define the following binary relations from* $\mathsf{r}\Lambda^{\mathtt{t}}$ *to* $\mathbf{2}\langle\mathsf{r}\Lambda^{\mathtt{t}}\rangle$*:*

$$[\lambda x.t][v_1,\dots,v_k] \mapsto_{\beta_v} t\langle[v_1,\dots,v_k]/x\rangle \qquad [\lambda x.t]ss' \mapsto_{\sigma_1} [\lambda x.ts']s \quad \textit{if } x \notin \mathsf{fv}(s')$$
$$[v_1,\dots,v_n]t \mapsto_0 0 \quad \textit{if } n \neq 1 \qquad [v]([\lambda x.t]s) \mapsto_{\sigma_3} [\lambda x.[v]t]s \quad \textit{if } x \notin \mathsf{fv}(v)$$

We set $\mapsto_{\mathsf{v}} = \mapsto_{\beta_v} \cup \mapsto_{\sigma_1} \cup \mapsto_{\sigma_3} \cup \mapsto_0$.

According to the convention of §2, $\to_{\mathsf{v}} \subseteq \mathsf{r}\Lambda^{\mathtt{t}} \times \mathbf{2}\langle\mathsf{r}\Lambda^{\mathtt{t}}\rangle$ is the reduction obtained by resource-contextual closure of $\mapsto_{\mathsf{v}}$.

The *resource* $\lambda_{\mathsf{v}}^{\sigma}$*-calculus* consists of the language $\mathsf{r}\Lambda^{\mathtt{t}}$ and the reduction $\to_{\mathsf{v}}$: it is the resource CBV λ-calculus of [14] plus the σ_1- and σ_3-rules.

As a technical simplification, we extend $\to_{\mathsf{v}}$ to a binary relation on $\mathbf{2}\langle\mathsf{r}\Lambda^{\mathtt{t}}\rangle$ by linearity, i.e. $(\sum_{i=1}^n t_i) + \mathbb{S} \to_{\mathsf{v}} (\sum_{i=1}^n \mathbb{T}_i) + \mathbb{S}$ iff $t_i \to_{\mathsf{v}} \mathbb{T}_i$ for every $i = 1,\dots,n$ ($n \geq 1$). With this extension we can concisely state the following theorem:

Theorem 14. *Reduction* $\to_{\mathsf{v}}$ *on* $\mathbf{2}\langle\mathsf{r}\Lambda^{\mathtt{t}}\rangle$ *is strongly normalizing and confluent.*

We omit the proof of Thm. 14. Strong normalization is evident (see [14] for a proof for the resource-contextual closure of $\mapsto_{\beta_v} \cup \mapsto_0$). The proof of local confluence for the resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus is analogous to the one for v-reduction on λ-terms (see §2). Finally, confluence is obtained by Newman's Lemma.

5 A Relational Model of (Resource) CBV Lambda-Calculus

In this section we present a relational model for both the $\lambda_{\mathsf{v}}^{\sigma}$-calculus and the resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus. This model is to be found in the category **Rel** of sets and relations (i.e. $\mathbf{Rel}(X,Y) = \mathcal{P}(X \times Y)$). In **Rel** identities are diagonal relations and composition of morphisms is the standard composition of relations. This category has a symmetric monoidal structure given by $\mathbf{1} = \{1\}$ (arbitrary singleton set) and $X \otimes Y = X \times Y$. This symmetric monoidal category is closed, with $X \multimap Y = X \times Y$, and $*$-autonomous with dualizing object $\bot = \mathbf{1}$. Category **Rel** is cartesian, with $X \,\&\, Y = (\{1\} \times X) \cup (\{2\} \times Y)$, and has an exponential functor ! defined by $!X = \mathcal{M}_{\mathrm{f}}(X)$ (the set of finite multisets on X) and $!f = \{([\alpha_1, \ldots, \alpha_n], [\beta_1, \ldots, \beta_n]) : n \geq 0,\ (\alpha_i, \beta_i) \in f\ \forall 1 \leq i \leq n\}$ for $f \in \mathbf{Rel}(X,Y)$.

All this structure makes **Rel** a new-Seely category and hence a categorical model of Linear Logic (LL). For more details we refer the reader to [19,14].

The model. We build inductively a family of sets $(U_n)_{n \in \mathbf{N}}$ given by $U_0 = \emptyset$ and $U_{n+1} = \mathcal{M}_{\mathrm{f}}(U_n) \times \mathcal{M}_{\mathrm{f}}(U_n)$. Finally, we set $U = \bigcup_{n \in \mathbf{N}} U_n$. Notice that $U_n \subsetneq U_{n+1}$ for all $n \in \mathbf{N}$, and $U = \mathcal{M}_{\mathrm{f}}(U) \times \mathcal{M}_{\mathrm{f}}(U) = !U \multimap !U$.

5.1 Interpreting the CBV Lambda-Calculus

Using the fact that **Rel** has the structure of a LL model, we can give a concrete interpretation of λ-terms as morphisms from $\mathcal{M}_{\mathrm{f}}(U)^n$ to $\mathcal{M}_{\mathrm{f}}(U)$ in **Rel** (where $\mathcal{M}_{\mathrm{f}}(U)^n$ is the n-fold set-theoretic power of $\mathcal{M}_{\mathrm{f}}(U)$). This semantics can also be described by type judgements (see [14]). With $a \uplus b$ we indicate the union of the multisets a and b (accounting for repetitions); if $\vec{a}$ and $\vec{b}$ are two finite sequences (of the same length) of multisets, $\vec{a} \uplus \vec{b}$ is their component-wise union.

Definition 15. *For every λ-term M and repetition-free list $\vec{x} \supseteq \mathsf{fv}(M)$, we define, by induction on M, its* interpretation $[\![M]\!]_{\vec{x}} \subseteq \mathcal{M}_{\mathrm{f}}(U)^n \times \mathcal{M}_{\mathrm{f}}(U)$ *(where n is the length of $\vec{x}$), as follows:*

$$[\![x_i]\!]_{\vec{x}} = \{(\vec{a}, a_i) : a_i \in \mathcal{M}_{\mathrm{f}}(U),\ a_j = [\,]\ \textit{for all}\ 1 \leq j \leq n\ \textit{with}\ j \neq i\}$$
$$[\![\lambda y.N]\!]_{\vec{x}} = \{(\textstyle\biguplus_{i=1}^{k} \vec{a}_i, \biguplus_{i=1}^{k} [(b_i, c_i)]) : k \geq 0,\ \forall i = 1, \ldots, k.\ ((\vec{a}_i, b_i), c_i) \in [\![N]\!]_{\vec{x},y}\}$$
$$[\![MN]\!]_{\vec{x}} = \{(\vec{a}_0 \uplus \vec{a}_1, c) : \exists\, b \in \mathcal{M}_{\mathrm{f}}(U).\ (\vec{a}_0, [(b,c)]) \in [\![M]\!]_{\vec{x}},\ (\vec{a}_1, b) \in [\![N]\!]_{\vec{x}}\}\,.$$

Notation. Hereafter, whenever we write $[\![M]\!]_{\vec{x}}$ we suppose that $\vec{x}$ is a repetition-free list of variables containing $\mathsf{fv}(M)$. Moreover, we will sometimes silently use the fact that $[\![M]\!]_{\vec{x},y} = \{((\vec{a}, [\,]), b) : (\vec{a}, b) \in [\![M]\!]_{\vec{x}}\}$ whenever $y \notin \vec{x}$.

Theorem 16 (soundness). *Let $M, N \in \Lambda$. If $M \to_{\mathsf{v}} N$, then $[\![M]\!]_{\vec{x}} = [\![N]\!]_{\vec{x}}$.*

5.2 Interpreting the Resource CBV Lambda-Calculus

In addition to the structure mentioned above, **Rel** is additive, and more precisely its hom-sets are enriched over the category of complete lattices, with set-theoretic union as join operation. The category **Rel** is a *weak differential* LL *model* (see [14]). Using this structure we can give the concrete interpretation of expressions as morphisms from $\mathcal{M}_f(U)^n$ to $\mathcal{M}_f(U)$ in **Rel**.

Definition 17. *For every expression e and repetition-free list $\vec{x} \supseteq \mathsf{fv}(e)$, we define, by induction on e, its* interpretation $[\![e]\!]_{\vec{x}} \subseteq \mathcal{M}_f(U)^n \times \mathcal{M}_f(U)$ *(where n is the length of $\vec{x}$), as follows:*

$$\begin{aligned}
[\![x_i]\!]_{\vec{x}} &= \{(\vec{a}, [\alpha]) : \alpha \in U,\ a_i = [\alpha],\ a_j = [\,] \text{ for all } 1 \le j \le n \text{ with } j \neq i\} \\
[\![\lambda z.t]\!]_{\vec{x}} &= \{(\vec{a}, [(b,c)]) : ((\vec{a}, b), c) \in [\![t]\!]_{\vec{y},z}\} \\
[\![st]\!]_{\vec{x}} &= \{(\vec{a}_0 \uplus \vec{a}_1, c) : \exists b \in \mathcal{M}_f(U).\ (\vec{a}_0, [(b,c)]) \in [\![s]\!]_{\vec{x}},\ (\vec{a}_1, b) \in [\![t]\!]_{\vec{x}}\} \\
[\![[v_1, \dots, v_k]]\!]_{\vec{x}} &= \{(\biguplus_{i=1}^k \vec{a}_i, \biguplus_{i=1}^k b_i) : k \ge 0,\ \forall i = 1, \dots, k.\ (\vec{a}_i, b_i) \in [\![v_i]\!]_{\vec{x}}\}\,.
\end{aligned}$$

Finally, sums of expressions are interpreted by setting $[\![\sum_{i=1}^n e_i]\!]_{\vec{x}} = \bigcup_{i=1}^n [\![e_i]\!]_{\vec{x}}$.

Notation. As for λ-terms, whenever we write $[\![e]\!]_{\vec{x}}$ we suppose that $\vec{x}$ is a repetition-free list of variables containing $\mathsf{fv}(e)$, and similarly for the sums. Note that $[\![[\,]]\!]_{\vec{x}} = \{([\,]^n, [\,])\} \subseteq \mathcal{M}_f(U)^n \times \mathcal{M}_f(U)$, where $[\,]^n = (\underbrace{[\,], \dots, [\,]}_{n \text{ times}})$.

Theorem 18 (soundness). *Let $\mathbb{S}, \mathbb{T} \in \mathbf{2}\langle \mathsf{r}\Lambda^{\mathsf{t}} \rangle$. If $\mathbb{S} \to_{\mathsf{v}} \mathbb{T}$, then $[\![\mathbb{S}]\!]_{\vec{x}} = [\![\mathbb{T}]\!]_{\vec{x}}$.*

The following notion of CBV Taylor expansion has been introduced in [14].

Definition 19 ([14], Taylor expansion). *Given a λ-term M, we inductively define a set $\mathcal{T}(M)$ of resource terms, called the* Taylor expansion *of M, as follows:*

$$\begin{aligned}
\mathcal{T}(x) &= \{[x^n] : n \ge 0\} \quad \text{where } [x^n] = [\overbrace{x, \dots, x}^{n \text{ times}}] \\
\mathcal{T}(\lambda x.M) &= \{[\lambda x.t_1, \dots, \lambda x.t_n] : n \ge 0,\ \forall i.\ t_i \in \mathcal{T}(M)\} \\
\mathcal{T}(MN) &= \{st : s \in \mathcal{T}(M),\ t \in \mathcal{T}(N)\}\,.
\end{aligned}$$

Theorem 20 ([14]). *Let M be a λ-term. Then $[\![M]\!]_{\vec{x}} = \bigcup_{t \in \mathcal{T}(M)} [\![t]\!]_{\vec{x}}$.*

Thm. 20 shows the semantical connection between λ-terms and their Taylor expansion. In the next section (§6) it will be applied in Thm. 39.1, which is in turn a fundamental part of one of our main results Thm. 24.

Definition 21. *For every expression e we define by induction the set* $\mathsf{strat}(e)$ *of multisets of resource values that occur in e in* stratified position, *as follows:*

$$\mathsf{strat}(x) = \emptyset; \qquad \mathsf{strat}([v_1, \dots, v_n]) = \{[v_1, \dots, v_n]\} \cup \textstyle\bigcup_{i=1}^n \mathsf{strat}(v_i) \ (n \ge 0);$$
$$\mathsf{strat}(st) = \mathsf{strat}(s); \qquad \mathsf{strat}(\lambda x.t) = \mathsf{strat}(t)\,.$$

We set $\mathsf{Strat} = \{t \in \mathsf{r}\Lambda^{\mathsf{t}} : [\,] \notin \mathsf{strat}(t)\}$, *whose elements are called* stratified resource terms.

A stratified resource term t does not contain any $[\,]$ in stratified position, i.e. every occurrence of $[\,]$ in t is a subterm of some subterm of t in argument position. For instance: $[x][\,]$, $[x]([\lambda z.[\,]][\,]) \in \mathsf{Strat}$ but $[\,]$, $[\,][z]$, $[\lambda z.[\,][x,y]] \notin \mathsf{Strat}$.

Stratified resource terms are not closed under v-reduction. For example, the stratified resource term $[\lambda x.[x]][\lambda y.[\,]]$ v-reduces to the non-stratified $[\lambda y.[\,]]$.

Definition 22 (stratified Taylor expansion). *Given a* λ*-term* M*, we define its* stratified Taylor expansion $\mathcal{T}_{\mathsf{s}}(M) = \{t \in \mathcal{T}(M) : \textit{if } t \twoheadrightarrow_{\mathsf{v}} \mathbb{T}\textit{, then } \mathbb{T} \subseteq \mathsf{Strat}\}$.

Example. The λ-term $M = (\lambda xy.x)\Omega$ is neither $\mathtt{w}$- nor $\mathtt{s}$-normalizable and every resource term in $\mathcal{T}(M)$ v-reduces to 0. Instead the non-$\mathtt{s}$-normalizable (but $\mathtt{w}$-normal) λ-term $N = (\lambda xy.\Omega)(zz')$ has infinitely many resource terms in $\mathcal{T}(N)$ that do not v-reduce to 0, like $t = [\lambda x.[\,]]([z][z'])$ for example. However $t \notin \mathcal{T}_{\mathsf{s}}(N)$ and $\mathcal{T}_{\mathsf{s}}(N)$ contains only resource terms that v-reduce to 0, because all resource terms in $\mathcal{T}(N)$ not v-reducing to 0 contain at least one $[\,]$ in stratified position.

The semantical connection between λ-terms and their stratified Taylor expansion is illustrated in one of our main results, Thm. 25. In particular, Thm. 39.2 is the step in which it is proved that the interpretation of $\mathcal{T}_{\mathsf{s}}(M)$ actually witnesses the strong $\mathtt{s}$-normalization of M. Intuitively, if $t \in \mathcal{T}_{\mathsf{s}}(M)$ then the v-normal form of t is a sum $\sum_{i=1}^{n} t_i$ $(n \geq 0)$ of stratified resource terms, each of which does not contain $[\,]$ in stratified position: a subterm $[\,]$ inside a t_i does not "hide" a non-$\mathtt{s}$-normalizable λ-term N such that $M = \mathtt{S}(\!|N|\!)$. So, by Lemma 38.ii one can prove that if $t \neq 0$ then M is strongly $\mathtt{s}$-normalizing.

6 The Main Theorems

In this section we will present our main results: the semantical and internal operational characterization of *potential valuability* (Thm. 24) and *solvability* (Thm. 25) for the $\lambda_{\mathsf{v}}^{\sigma}$-calculus. See §1 for a overview of these notions.

Definition 23 (Potential valuability, solvability). *Let* M *be a* λ*-term:*

– M *is* potentially valuable *if there exist variables* $x_1, \ldots, x_m$ *and* λ*-values* $V, V_1, \ldots, V_m$ *(with* $m \geq 0$*) such that* $M\{V_1/x_1, \ldots, V_m/x_m\} \twoheadrightarrow_{\mathsf{v}} V$*;*
– M *is* solvable *if there exist variables* $x_1, \ldots, x_m$ *and* λ*-terms* $N_1, \ldots, N_n$ *(for some* $n, m \geq 0$*) such that* $(\lambda x_1 \ldots x_m.M)N_1 \cdots N_n \twoheadrightarrow_{\mathsf{v}} \mathbf{I}$.

We state now the two main theorems. In particular, Thm. 24 says that $\mathtt{w}$-normalizability (i.e. potential valuability) plays a role analogous to that of head-normalizability for many call-by-name models, like Scott's D_∞.

Theorem 24. *Let* M *be a* λ*-term with* $\vec{x} \supseteq \mathsf{fv}(M)$*. The following are equivalent:*

(i) M *is* $\mathtt{w}$*-normalizable;*
(ii) M *is potentially valuable;*
(iii) $[\![M]\!]_{\vec{x}} \neq \emptyset$*;*
(iv) M *is strongly* $\mathtt{w}$*-normalizing.*

Theorem 25. *Let* M *be a* λ*-term with* $\vec{x} \supseteq \mathsf{fv}(M)$*. The following are equivalent:*

(i) M *is* $\mathtt{s}$*-normalizable;*
(ii) M *is solvable;*
(iii) $\bigcup_{t \in \mathcal{T}_{\mathsf{s}}(M)} [\![t]\!]_{\vec{x}} \neq \emptyset$*;*
(iv) M *is strongly* $\mathtt{s}$*-normalizing.*

An immediate corollary of Thm. 24 and 25 is that every solvable (i.e. $\mathtt{s}$-normalizable) λ-term is also potentially valuable (i.e. $\mathtt{w}$-normalizable).

The proofs of Thm. 24 and 25 are divided into parts, which are detailed separately in the next subsections, due to the different techniques used for each one of them. The splitting of the two proofs follows the same pattern. The implications (i) $\Rightarrow$ (ii) of both theorems are proved in §6.1 by purely syntactical means. The implication (ii) $\Rightarrow$ (iii) of Thm. 24 is shown in §6.2 using the resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus of §4; for this implication of Thm. 25 we use an extension of the resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus presented in §6.3. The implication (iii) $\Rightarrow$ (iv) of both theorems is proved in §6.4 using simulations of $\mathtt{w}$- and $\mathtt{s}$-reductions in $\lambda_{\mathsf{v}}^{\sigma}$-calculus by the v-reduction of the resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus. Finally, (iv) $\Rightarrow$ (i) are trivial in both cases.

6.1 From Weak and Stratified Normalization to Solvability and Potential Valuability

Our goal here is to prove the implication (i) $\Rightarrow$ (ii) of Thm. 24 and 25. Our approach is largely inspired by [6,7,5].

For every $n \in \mathbf{N}$, we set $\mathsf{o}^n = \lambda x_n \dots x_0.x_0$. Notice that $\mathsf{o}^0 = \mathbf{I}$ and o^n is a closed value for any $n \in \mathbf{N}$. Moreover, $\mathsf{o}^n V \mapsto_{\beta_v} \mathsf{o}^{n-1}$ for any $n > 0$ and $V \in \Lambda^{\mathsf{v}}$.

Lemma 26. *Let* $M \in \mathtt{w_{nf}}$ *with* $\mathsf{fv}(M) \subseteq \{x_1, \dots, x_m\}$ *and let* $j \in \mathbf{N}$*. Then there exists* $h > 0$ *such that for all* $n_1, \dots, n_m \geq j + h$ *there exists a* λ*-term* N *such that* $M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \lambda x.N$ *and* $\lambda x.N$ *is closed.*

Lemma 27. *Let* $M \in \mathtt{s_{nf}}$ *with* $\mathsf{fv}(M) \subseteq \{x_1, \dots, x_m\}$ *and let* $j \in \mathbf{N}$*. Then there exist* $h, k \in \mathbf{N}$ *such that for all* $n_1, \dots, n_{m+k} \geq j + h$ *there exists* $n \geq j$ *such that* $M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\}\mathsf{o}^{n_{m+1}} \dots \mathsf{o}^{n_{m+k}} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^n$*.*

Theorem 28. *Let* M *be a* λ*-term.*

1. [(i)$\Rightarrow$(ii) of Thm. 24] *If* M *is* $\mathtt{w}$*-normalizable then* M *is potentially valuable.*
2. [(i)$\Rightarrow$(ii) of Thm. 25] *If* M *is* $\mathtt{s}$*-normalizable then* M *is solvable.*

Proof. For point 1 (resp. 2), hypothesis means that there is a $\mathtt{w}$-(resp. $\mathtt{s}$-)normal form M' such that $M \twoheadrightarrow_{\mathtt{w}} M'$ (resp. $M \twoheadrightarrow_{\mathtt{s}} M'$), moreover $M' \in \mathtt{w_{nf}}$ (resp. $M' \in \mathtt{s_{nf}}$) by Prop. 12. Let $\mathsf{fv}(M) = \{x_1, \dots, x_m\}$ and thus $\mathsf{fv}(M') \subseteq \{x_1, \dots, x_m\}$.

1. By Lemma 26 (taking $j = 0$) there exists $h > 0$ such that: $M'\{\mathsf{o}^h/x_1, \dots, \mathsf{o}^h/x_m\} \twoheadrightarrow_{\mathsf{v}} \lambda x.N$, for some closed λ-value $\lambda x.N$. One has $M\{\mathsf{o}^h/x_1, \dots, \mathsf{o}^h/x_m\} \twoheadrightarrow_{\mathsf{v}} M'\{\mathsf{o}^h/x_1, \dots, \mathsf{o}^h/x_m\}$ by Lemma 5.ii, so that M is potentially valuable because $\lambda x.N$ is a closed λ-value.
2. By Lemma 27 (taking $j = 0$), there exist $h, k, n \in \mathbf{N}$ such that: $(M'\{\mathsf{o}^h/x_1, \dots, \mathsf{o}^h/x_m\})\mathsf{o}^h \dots \mathsf{o}^h \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^n$ (o^h is applied k times). We conclude that M is solvable because if we set $\mathtt{H} = (\lambda x_1 \dots x_m.(\!|\cdot|\!))\, \underbrace{\mathsf{o}^h \dots \mathsf{o}^h}_{m+k \text{ times}}\, \underbrace{\mathbf{I} \dots \mathbf{I}}_{n \text{ times}}$, then
$$\mathtt{H}(\!|M|\!) \twoheadrightarrow_{\mathsf{v}} \mathtt{H}(\!|M'|\!) \twoheadrightarrow_{\mathsf{v}} (M'\{\mathsf{o}^h/x_1, \dots, \mathsf{o}^h/x_m\})\mathsf{o}^h \dots \mathsf{o}^h\mathbf{I} \dots \mathbf{I} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^n\mathbf{I} \dots \mathbf{I} \twoheadrightarrow_{\mathsf{v}} \mathbf{I}.$$ □

6.2 From Potential Valuability to Non-emptyness

The following theorem proves the implication (ii) ⇒ (iii) of Thm. 24.

Theorem 29. *Let M be a λ-term with $\vec{x} \supseteq \mathsf{fv}(M)$. If M is potentially valuable, then $[\![M]\!]_{\vec{x}} \neq \emptyset$.*

Proof. If M is potentially valuable (see Def. 23) there exist variables $x_1, \ldots, x_m$ and λ-values $V, V_1, \ldots, V_m$ (for some $m \geq 0$) s.t. $M\{V_1/x_1, \ldots, V_m/x_m\} \twoheadrightarrow_{\mathsf{v}} V$. Since variables are λ-values, we can suppose without loss of generality that $\vec{x} = (x_1, \ldots, x_m) \supseteq \mathsf{fv}(M)$. Let $\vec{y} = \mathsf{fv}(V) \cup \bigcup_{i=1}^{m} \mathsf{fv}(V_i)$. One can prove by induction on M that

$$[\![M\{V_1/x_1, \ldots, V_m/x_m\}]\!]_{\vec{y}} = \big\{(\biguplus_{i=1}^{m} \vec{a}_i, c) : \ \exists\, b_1, \ldots, b_m \in \mathcal{M}_{\mathrm{f}}(U) : \\ ((b_1, \ldots, b_m), c) \in [\![M]\!]_{\vec{x}},\ (\vec{a}_i, b_i) \in [\![V_i]\!]_{\vec{y}} \text{ for all } 1 \leq i \leq m\big\}.$$

Since $[\![V]\!]_{\vec{y}} \neq \emptyset$ (this can be proved by simple inspection), by Thm. 16 we obtain that $[\![M\{V_1/x_1, \ldots, V_m/x_m\}]\!]_{\vec{y}} \neq \emptyset$ also holds, so that $[\![M]\!]_{\vec{x}} \neq \emptyset$. □

6.3 From Solvability to Non-emptyness of Stratified Taylor Expansion

The implication (ii) ⇒ (iii) of Thm. 25 seems much more difficult to prove. To accomplish this task we introduce the *resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus with tests*, a CBV version of the resource calculus with tests defined in [15]. In this syntax all elements of the relational model are definable (see Def. 34).

The language extends that of resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus (see §4, p. 108) as follows:

$(\mathsf{r}\Lambda^{\mathsf{v}})$	$u, v ::= x \mid \lambda x.t$		*resource values*
$(\mathsf{r}\Lambda^{\mathsf{t}})$	$s, t ::= t * p \mid st \mid [v_1, \ldots, v_k]$	$(k \geq 0)$	*resource terms*
$(\mathsf{r}\Lambda^{\tau})$	$p, q ::= \tau[t_1, \ldots, t_k]$ $(k \geq 0)$		*tests*

Note the overloaded use of $\mathsf{r}\Lambda^{\mathsf{v}}$ and $\mathsf{r}\Lambda^{\mathsf{t}}$, which now (and until Lemma 36) indicate larger sets than those introduced in §4. We will use this extension to prove Lemma 36 (whose statement concerns only resource terms without tests).

Tests are – formally – multisets of resource terms, the "τ" being a tag for distinguishing them from bags of values. Intuitively, they are constructions which can produce either *success*, represented by $\tau[\,]$, or *failure*, represented by 0.

Notation. We set $\varepsilon = \tau[\,]$ and $\tau[t_1, \ldots, t_k] \parallel \tau[t_{k+1}, \ldots, t_n] = \tau[t_1, \ldots, t_n]$ $(k \leq n)$.

The test $p \parallel q$ represents the (must-)parallel composition of p and q (i.e., $p \parallel q$ succeeds iff both p and q succeed). The composition is parallel in the sense that the order of evaluation is inessential (remember that they are multisets). The binary operator $*$ allows to build a resource term out of a resource term and a test: intuitively, the resource term $t * p$ may be thought of as something that outputs the result of t only if p succeeds. Dually, the "cork construction" $\tau[t]$ may be thought of as a check that tests whether or not t v-reduces to $[\,]$.

Resource, *test-resource* and *test-test contexts* (with exactly one hole), denoted resp. by R, Q and P, are defined by mutual induction via the grammar ($k \geq 0$):

$$\mathtt{R} ::= (\!|\cdot|\!) \mid \mathtt{R}t \mid t\mathtt{R} \mid t * \mathtt{Q} \mid [\lambda x.\mathtt{R}, v_1, \ldots, v_k] \quad (\textit{resource contexts});$$
$$\mathtt{Q} ::= \tau[\mathtt{R}, t_1, \ldots, t_k] \ (\textit{test-resource c.}); \qquad \mathtt{P} ::= (\!|\cdot|\!) \parallel \tau[t_1, \ldots, t_k] \ (\textit{test-test c.}).$$

Let $t, t_1, \ldots, t_n \in \mathsf{r}\Lambda^{\mathsf{t}}$ (resp. $p, p_1, \ldots, p_n \in \mathsf{r}\Lambda^{\tau}$). We use $\mathtt{Q}(\!|t|\!)$ (resp. $\mathtt{P}(\!|p|\!)$) for the test obtained by the capture-allowing substitution of t (resp. p) for the hole $(\!|\cdot|\!)$ in Q (resp. P); similarly for $\mathtt{R}(\!|t|\!)$ (see p. 109). As usual, $\mathtt{R}(\!|\sum_i t_i|\!) = \sum_i \mathtt{R}(\!|t_i|\!)$, $\mathtt{Q}(\!|\sum_i t_i|\!) = \sum_i \mathtt{Q}(\!|t_i|\!)$ and $\mathtt{P}(\!|\sum_i p_i|\!) = \sum_i \mathtt{P}(\!|p_i|\!)$. E.g., $t * 0 = t * \mathtt{Q}(\!|0|\!) = \mathtt{R}(\!|0|\!) = 0$.

Definition 30. *The operational semantics of the resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus with tests extends the set of rules listed in Def. 13 with the following ones:*

$$t(s * p) \mapsto_{\tau_1} ts * p \qquad \tau[t * p] \mapsto_{\tau_4} \tau[t] \parallel p$$
$$(t * p)s \mapsto_{\tau_2} ts * p \qquad \tau[[v_1, \ldots, v_n]] \mapsto_{\tau_5} \begin{cases} \varepsilon & \textit{if } n = 0 \\ 0 & \textit{otherwise} \end{cases}$$
$$(t * p) * q \mapsto_{\tau_3} t * (p \parallel q)$$

We set $\mapsto_{\mathsf{v}\tau} = \mapsto_{\mathsf{v}} \cup (\bigcup_{i=1}^{5} \mapsto_{\tau_i}) \subseteq (\mathsf{r}\Lambda^{\mathsf{t}} \times \mathbf{2}\langle \mathsf{r}\Lambda^{\mathsf{t}}\rangle) \cup (\mathsf{r}\Lambda^{\tau} \times \mathbf{2}\langle \mathsf{r}\Lambda^{\tau}\rangle)$. Then, according to the convention of §2, $\to_{\mathsf{v}\tau} \subseteq \mathsf{r}\Lambda^{\tau} \times \mathbf{2}\langle \mathsf{r}\Lambda^{\tau}\rangle$ is the reduction obtained by test-contextual closure[4] of $\mapsto_{\mathsf{v}\tau}$. The *resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus with tests* consists of the language $\mathsf{r}\Lambda^{\tau}$ and the reduction $\to_{\mathsf{v}\tau}$.

As a technical simplification, we extend $\to_{\mathsf{v}\tau}$ to a binary relation on $\mathbf{2}\langle \mathsf{r}\Lambda^{\tau}\rangle$ by linearity, i.e., $(\sum_{i=1}^{n} q_i) + \mathbb{P} \to_{\mathsf{v}\tau} (\sum_{i=1}^{n} \mathbb{Q}_i) + \mathbb{P}$ iff $q_i \to_{\mathsf{v}\tau} \mathbb{Q}_i$ for every $i = 1, \ldots, n$ ($n \geq 1$). With this extension we can concisely state the following theorem:

Theorem 31. *Reduction $\to_{\mathsf{v}\tau}$ on $\mathbf{2}\langle \mathsf{r}\Lambda^{\tau}\rangle$ is strongly normalizing and confluent.*

Definition 32. *For every test p and repetition-free list $\vec{x} \supseteq \mathsf{fv}(p)$, we define the* interpretation $[\![p]\!]_{\vec{x}} \subseteq \mathcal{M}_{\mathrm{f}}(U)^n \times \mathbf{1}$ *of p, where n is the length of $\vec{x}$, by mutual induction with Def. 17 as follows:*

$$[\![\varepsilon]\!]_{\vec{x}} = \{([\,]^n, 1)\} \qquad [\![p \parallel q]\!]_{\vec{x}} = \{(\vec{a} \uplus \vec{b}, 1) : (\vec{a}, 1) \in [\![p]\!]_{\vec{x}},\ (\vec{b}, 1) \in [\![q]\!]_{\vec{x}}\}$$
$$[\![\tau[t]]\!]_{\vec{x}} = \{(\vec{a}, 1) : (\vec{a}, [\,]) \in [\![t]\!]_{\vec{x}}\} \quad [\![t * p]\!]_{\vec{x}} = \{(\vec{a} \uplus \vec{b}, c) : (\vec{a}, c) \in [\![t]\!]_{\vec{x}},\ (\vec{b}, 1) \in [\![p]\!]_{\vec{x}}\}.$$

Finally, sums of tests are interpreted by setting $[\![\sum_{i=1}^{n} p_i]\!]_{\vec{x}} = \bigcup_{i=1}^{n} [\![p_i]\!]_{\vec{x}}$.

Theorem 33 (soundness). *Let $\mathbb{P}, \mathbb{Q} \in \mathbf{2}\langle \mathsf{r}\Lambda^{\tau}\rangle$. If $\mathbb{P} \to_{\mathsf{v}\tau} \mathbb{Q}$, then $[\![\mathbb{P}]\!]_{\vec{x}} = [\![\mathbb{Q}]\!]_{\vec{x}}$.*

A key tool to connect the semantics with the vτ-reduction is the following transformation of elements of $\mathcal{M}_{\mathrm{f}}(U)$ into resource terms and test contexts. The role of this transformation is made clear in Lemma 35, used to prove Lemma 36.

[4] This means that, for every $p \in \mathsf{r}\Lambda^{\tau}$ and $p' \in \mathbf{2}\langle \mathsf{r}\Lambda^{\tau}\rangle$, if $p \to_{\mathsf{v}\tau} p'$ then either there exist a test-test context P, $q \in \mathsf{r}\Lambda^{\tau}$ and $q' \in \mathbf{2}\langle \mathsf{r}\Lambda^{\tau}\rangle$ such that $p = \mathtt{P}(\!|q|\!)$, $p' = \mathtt{P}(\!|q'|\!)$ and $q \mapsto_{\tau_i} q'$ with $i \in \{4, 5\}$; or there exist a test-resource context Q, $t \in \mathsf{r}\Lambda^{\mathsf{t}}$ and $t' \in \mathbf{2}\langle \mathsf{r}\Lambda^{\mathsf{t}}\rangle$ such that $p = \mathtt{Q}(\!|t|\!)$, $p' = \mathtt{Q}(\!|t'|\!)$ and $t \mapsto_{\mathsf{v}\tau'} t'$ with $\mapsto_{\mathsf{v}\tau'} = \mapsto_{\mathsf{v}} \cup (\bigcup_{i=1}^{3} \mapsto_{\tau_i})$.

Definition 34. *Let* $c = [(a_1, b_1), \dots, (a_n, b_n)] \in \mathcal{M}_\mathrm{f}(U)$ $(n \geq 0)$*. We define:*

- *the closed resource term* $c^- = [\lambda y_1.{b_1}^- * {a_1}^+ (\![y_1^{m_1}]\!), \dots, \lambda y_n.{b_n}^- * {a_n}^+ (\![y_n^{m_n}]\!)]$*, where* m_i *is the cardinality of the multiset* a_i *(for* $i = 1, \dots, n$*);*
- *the test-resource context* $c^+ = \tau[[\lambda x.[\,] * \|_{i=1}^n \tau[[\lambda y.[\,] * {b_i}^+ (\![y^{k_i}]\!)]([x]{a_i}^-)]](\!\cdot\!)]$*, where* k_i *is the cardinality if the multiset* b_i *(for* $i = 1, \dots, n$*).*

Notation. For any $a \in \mathcal{M}_\mathrm{f}(U)$, $\#a$ indicates its cardinality. For $\vec{a} = (a_1, \dots, a_n) \in \mathcal{M}_\mathrm{f}(U)^n$ and $t \in \mathsf{r}\Lambda^\mathsf{t}$, we write $t\langle \vec{a}^- / \vec{x}\rangle$ as a shorthand for $t\langle {a_1}^-/x_1\rangle \cdots \langle {a_n}^-/x_n\rangle$.

Lemma 35. *Let* $(\vec{a}, b) \in \mathcal{M}_\mathrm{f}(U)^n \times \mathcal{M}_\mathrm{f}(U)$*,* $k = \#b$ *and* $t \in \mathsf{r}\Lambda^\mathsf{t}$ *with* $\vec{x} \supseteq \mathsf{fv}(t)$*. Then* $(\vec{a}, b) \in [\![t]\!]_{\vec{x}}$ *iff* $\tau[[\lambda y.[\,] * b^+ (\![y^k]\!)](t\langle \vec{a}^-/\vec{x}\rangle)] \twoheadrightarrow_{\mathsf{v}\tau} \varepsilon$*.*

Lemma 36. *Let* s *and* t *be* v*-normal resource terms without tests (i.e., generated by the grammar on §4, p. 108). If* $s \in \mathsf{Strat}$ *and* $t \notin \mathsf{Strat}$*, then* $[\![s]\!]_{\vec{x}} \cap [\![t]\!]_{\vec{x}} = \emptyset$*.*

Proof. Let $(\vec{a}, b) \in \mathcal{M}_\mathrm{f}(U)^n \times \mathcal{M}_\mathrm{f}(U)$ and $\mathtt{Q}(\!\cdot\!) = \tau[[\lambda y.[\,] * b^+ (\![y^k]\!)]((\!\cdot\!)\langle \vec{a}^-/\vec{x}\rangle)]$, with $k = \#b$. One can prove by induction on the v-normal resource terms (without tests) that: either $\mathtt{Q}(\!t\!) \twoheadrightarrow_{\mathsf{v}\tau} \varepsilon$ and $\mathtt{Q}(\!s\!) \twoheadrightarrow_{\mathsf{v}\tau} 0$; or $\mathtt{Q}(\!s\!) \twoheadrightarrow_{\mathsf{v}\tau} \varepsilon$ and $\mathtt{Q}(\!t\!) \twoheadrightarrow_{\mathsf{v}\tau} 0$; or $\mathtt{Q}(\!s\!) \twoheadrightarrow_{\mathsf{v}\tau} 0$ and $\mathtt{Q}(\!t\!) \twoheadrightarrow_{\mathsf{v}\tau} 0$. Hence, by Lemma 35, $(\vec{a}, b) \notin [\![s]\!]_{\vec{x}} \cap [\![t]\!]_{\vec{x}}$. □

Hereafter, when we will mention resource terms, we will refer to the ones without test (i.e., generated by the grammar on §4, p. 108).

The following theorem proves the implication (ii) ⇒ (iii) of Thm. 25.

Theorem 37. *Let* M *be a* λ*-term and let* $\vec{x} \supseteq \mathsf{fv}(M)$*. If* M *is solvable, then* $\bigcup_{t \in \mathcal{T}_\mathsf{s}(M)} [\![t]\!]_{\vec{x}} \neq \emptyset$*.*

Proof. If M is solvable then there exists a context $\mathtt{C} = (\lambda x_1 \dots x_m.(\!\cdot\!))N_1 \cdots N_n$ (for some $n, m \geq 0$) such that $\mathtt{C}(\!M\!) \twoheadrightarrow_\mathsf{v} \mathbf{I}$. By Thm. 16 and 20, $\bigcup_{t \in \mathcal{T}(\mathtt{C}(\!M\!))} [\![t]\!]_{\vec{x}} = [\![\mathtt{C}(\!M\!)]\!]_{\vec{x}} = [\![\mathbf{I}]\!]_{\vec{x}} = \bigcup_{t \in \mathcal{T}(\mathbf{I})} [\![t]\!]_{\vec{x}}$. Using Lemma 36 we infer that $\bigcup_{t \in \mathcal{T}_\mathsf{s}(\mathtt{C}(\!M\!))} [\![t]\!]_{\vec{x}} = \bigcup_{t \in \mathcal{T}_\mathsf{s}(\mathbf{I})} [\![t]\!]_{\vec{x}}$. Therefore $\bigcup_{t \in \mathcal{T}_\mathsf{s}(\mathtt{C}(\!M\!))} [\![t]\!]_{\vec{x}} \neq \emptyset$ because it is easy to check that $\bigcup_{t \in \mathcal{T}_\mathsf{s}(\mathbf{I})} [\![t]\!]_{\vec{x}} \neq \emptyset$. By Thm. 18 and 14, $\bigcup_{t \in \mathcal{T}_\mathsf{s}(\mathtt{C}(\!M\!))} [\![t]\!]_{\vec{x}} \neq \emptyset$ implies that there is a resource term in $\mathcal{T}_\mathsf{s}(\mathtt{C}(\!M\!))$ that v-reduces to a non-zero v-normal form. Now all resource terms in $\mathcal{T}_\mathsf{s}(\mathtt{C}(\!M\!))$ are of the shape $\mathtt{R}(\!s\!)$ for some $s \in \mathcal{T}_\mathsf{s}(M)$ (because the hole of $\mathtt{C}$ is in stratified position), so that if all resource terms in $\mathcal{T}_\mathsf{s}(M)$ v-reduced to 0, then all resource terms in $\mathcal{T}_\mathsf{s}(\mathtt{C}(\!M\!))$ would v-reduce to 0. Thus, there is $t \in \mathcal{T}_\mathsf{s}(M)$ that v-reduces to a v-normal form $\mathbb{T} \neq 0$. It is easy to prove that $[\![t']\!]_{\vec{x}} \neq \emptyset$ for every v-normal form t', hence $[\![t]\!]_{\vec{x}} = [\![\mathbb{T}]\!]_{\vec{x}} \neq \emptyset$ by Thm. 18. □

6.4 From Non-emptyness to Strong Normalization

Our goal here is to prove the implication (iii) ⇒ (iv) of Thm. 24 and 25.

Lemma 38. *Let M, M' be λ-terms.*

(i) *If $M \to_{\mathtt{w}} M'$ and $t \in \mathcal{T}(M)$, then there exists $\mathbb{T} \subseteq \mathcal{T}(M')$ such that $t \to_{\mathsf{v}} \mathbb{T}$.*
(ii) *If $M \to_{\mathtt{s}} M'$ and $s \in \mathcal{T}_{\mathsf{s}}(M)$, then there exists $\mathbb{S} \subseteq \mathcal{T}_{\mathsf{s}}(M')$ such that $s \to_{\mathsf{v}}^{+} \mathbb{S}$.*

Lemma 38.i is false if we replace the hypothesis $M \to_{\mathtt{w}} M'$ with $M \to_{\mathtt{s}} M'$. For instance, take $M = \lambda x.\Omega$: then $[\,] \in \mathcal{T}(M)$ and $M \to_{\mathtt{s}} M$, but $[\,]$ is v-normal.

Theorem 39. *Let M be a λ-term and let $\vec{x} \supseteq \mathsf{fv}(M)$.*

1. [(iii)⇒(iv) of Thm. 24] *If $[\![M]\!]_{\vec{x}} \neq \emptyset$ then M is strongly $\mathtt{w}$-normalizing.*
2. [(iii)⇒(iv) of Thm. 25] *If $\bigcup_{t \in \mathcal{T}_{\mathsf{s}}(M)} [\![t]\!]_{\vec{x}} \neq \emptyset$, then M is strongly $\mathtt{s}$-normalizing.*

Proof. Let $(\vec{a}, b) \in [\![M]\!]_{\vec{x}}$ (resp. $(\vec{a}, b) \in \bigcup_{t \in \mathcal{T}_{\mathsf{s}}(M)} [\![t]\!]_{\vec{x}}$). By Thm. 20 (resp. Then) there exists $t \in \mathcal{T}(M)$ (resp. $t \in \mathcal{T}_{\mathsf{s}}(M)$) such that $(\vec{a}, b) \in [\![t]\!]_{\vec{x}}$. If $M \to_{\mathtt{w}} M'$ (resp. $M \to_{\mathtt{s}} M'$), then by Lemma 38.i (resp. Lemma 38.ii) there exists $\mathbb{T} \subseteq \mathcal{T}(M')$ (resp. $\mathbb{T} \subseteq \mathcal{T}_{\mathsf{s}}(M')$) such that $t \to_{\mathsf{v}}^{+} \mathbb{T}$. According to Thm. 18, $(\vec{a}, b) \in [\![\mathbb{T}]\!]_{\vec{x}}$, hence $\mathbb{T} \neq \emptyset$ and so there exists $t' \in \mathbb{T}$ such that $(\vec{a}, b) \in [\![t']\!]_{\vec{x}}$. Therefore, if there was an infinite reduction $M \to_{\mathtt{w}} M_1 \to_{\mathtt{w}} M_2 \to_{\mathtt{w}} \dots$ (resp. $M \to_{\mathtt{s}} M_1 \to_{\mathtt{s}} M_2 \to_{\mathtt{s}} \dots$) then there would also be a an infinite reduction $t \to_{\mathsf{v}}^{+} \mathbb{T}_1 \to_{\mathsf{v}}^{+} \mathbb{T}_2 \to_{\mathsf{v}}^{+} \dots$, which is impossible by Thm. 14. □

Conclusions and Future Work

Our approach, that exploits the validity of the Taylor formula for a resource CBV λ-calculus, makes use of purely combinatorial proofs, rather than more standard approaches based on reducibility or some specific machines. The interesting feature of this approach is that it can be used for many different calculi always using a similar relational model and a suitable resource calculus.

We think that using the ordinary syntax of λ-calculus with our reduction will allow to develop a reasonable theory of CBV Böhm trees, never defined before (Paolini's separability result in [20] for λ_{v}-calculus does not use Böhm trees), together with connections between equivalence of Böhm trees and observational equivalence. A future challenge is that of finding other fully abstract denotational models, in view of Paolini and Ronchi Della Rocca's proof of absence of fully abstract filter models (see [7, Thm. 12.1.25]) built from legal type systems.

Another direction is relating two equivalence relations on λ-terms, the one generated by our σ-rules and the one induced by Girard's CBV "boring" translation $(\cdot)^v$ of λ-calculus into IMELL proof-nets (along the lines of [17,18,21]).

References

1. Barendregt, H.P.: The Lambda Calculus: Its Syntax and Semantics. North-Holland, Amsterdam (1984)
2. Barendregt, H.: Solvability in lambda-calculi. J. Symb. Logic 75(3), 191–231 (1975)

3. Wadsworth, C.: The Relation Between Computational and Denotational Properties for Scott's D_∞-Models of the λ-Calculus. SIAM J. Comput. 5(3), 488–521 (1976)
4. Plotkin, G.: Call-by-Name, Call-by-Value and the λ-Calculus. Theor. Comput. Sci. 1(2), 125–159 (1975)
5. Accattoli, B., Paolini, L.: Call-by-Value Solvability, Revisited. In: Schrijvers, T., Thiemann, P. (eds.) FLOPS 2012. LNCS, vol. 7294, pp. 4–16. Springer, Heidelberg (2012)
6. Paolini, L., Ronchi Della Rocca, S.: Call-by-value Solvability. ITA 33(6), 507–534 (1999)
7. Paolini, L., Ronchi Della Rocca, S.: The Parametric λ-calculus: a Metamodel for Computation. Texts in Theoretical Computer Science. Springer, Berlin (2004)
8. Paolini, L., Pimentel, E., Ronchi Della Rocca, S.: Strong Normalization from an unusual point of view. Theor. Comput. Sci. 412(20), 1903–1915 (2011)
9. Herbelin, H., Zimmermann, S.: An operational account of Call-by-Value Minimal and Classical λ-calculus in "Natural Deduction" form. In: Curien, P.-L. (ed.) TLCA 2009. LNCS, vol. 5608, pp. 142–156. Springer, Heidelberg (2009)
10. Hofmann, M.: Sound and Complete Axiomatisations of Call-by-Value Control Operators. Mathematical Structures in Computer Science 5(4), 461–482 (1995)
11. Moggi, E.: Computational Lambda-Calculus and Monads. In: LICS, pp. 14–23 (1989)
12. Dyckhoff, R., Lengrand, S.: Call-by-Value lambda-calculus and LJQ. J. Log. Comput. 17(6), 1109–1134 (2007)
13. Egidi, L., Honsell, F., Ronchi Della Rocca, S.: Operational, denotational and logical descriptions: a case study. Fundamenta Informaticæ 16(2), 149–169 (1992)
14. Ehrhard, T.: Collapsing non-idempotent intersection types. In: CSL, pp. 259–273 (2012)
15. Bucciarelli, A., Carraro, A., Ehrhard, T., Manzonetto, G.: Full Abstraction for the Resource Lambda Calculus with Tests, through Taylor Expansion. Logical Methods in Computer Science 8(4) (2012)
16. Girard, J.: Linear Logic. Theor. Comput. Sci. 50(1), 1–102 (1987)
17. Regnier, L.: Lambda calcul et réseaux. PhD thesis, Université Paris 7 (1992)
18. Regnier, L.: Une équivalence sur les lambda-termes. Theor. Comput. Sci. 126(2), 281–292 (1994)
19. Melliès, P.: Categorical semantics of Linear Logic. In: Interactive Models of Computation and Program Behaviour. Panoramas et Synthèses, vol. 27, pp. 1–196. Société Mathématique de France (2009)
20. Paolini, L.: Call-by-Value Separability and Computability. In: Restivo, A., Ronchi Della Rocca, S., Roversi, L. (eds.) ICTCS 2001. LNCS, vol. 2202, pp. 74–89. Springer, Heidelberg (2001)
21. Accattoli, B., Kesner, D.: The Permutative λ-Calculus. In: Bjørner, N., Voronkov, A. (eds.) LPAR-18. LNCS, vol. 7180, pp. 23–36. Springer, Heidelberg (2012)