Directional Convexity Measure for Binary Tomography*

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Abstract. There is an increasing demand for a new measure of convexity for discrete sets for various applications. For example, the well-known measures for h-, v-, and hv-convexity of discrete sets in binary tomography pose rigorous criteria to be satisfied. Currently, there is no commonly accepted, unified view on what type of discrete sets should be considered nearly hv-convex, or to what extent a given discrete set can be considered convex, in case it does not satisfy the strict conditions. We propose a novel directional convexity measure for discrete sets based on various properties of the configuration of 0s and 1s in the set. It can be supported by proper theory, is easy to compute, and according to our experiments, it behaves intuitively. We expect it to become a useful alternative to other convexity measures in situations where the classical definitions cannot be used.

Keywords: Binary Tomography, Discrete Geometry, Convexity Measure.

1 Introduction

Convexity is a crucial geometrical feature of a discrete set, e.g. in binary tomography[6], where the aim is to reconstruct binary images from their projections. Several reconstruction methods utilize preliminary information — such as horizontal (h), vertical (v) or both horizontal and vertical (hv) convexity — about the set to be reconstructed [2,4,5,7]. However, definitions for convexity are strict, in the sense that a change in a single pixel of the corresponding binary image could cause the set to lose its horizontal and/or vertical convexity,

* L.G. Nyúl was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. P. Balázs was supported by the OTKA PD100950 grant of the National Scientific Research Fund. The research of P. Balázs and T.S. Tasi was partially supported by the European Union and the State of Hungary co-financed by the European Social Fund under the grant agreement TÁMOP 4.2.4.A/2-11-1-2012-0001 ('National Excellence Program') and under the grant agreement TÁMOP-4.2.2.A-11/1/KONV-2012-0073 ('Telemedicine-focused research activities on the field of Mathematics, Informatics and Medical sciences').

J. Ruiz-Shulcloper and G. Sanniti di Baja (Eds.): CIARP 2013, Part II, LNCS 8259, pp. 9–16, 2013.

thus the previous methods cannot be applied anymore. Instead of providing a binary property to determine whether a set is convex or not, we prefer to assign each discrete set a degree of convexity. One could view this as *fuzzifying* the set of convex shapes by determining a membership value for each shape. Such a measure of convexity describes the image better than a binary value, and it should be more robust to noisy data. Therefore, it can be used to give a more detailed feature of the image, and even guide the reconstruction, in case such task is performed.

Various continuous and several discrete convexity measures have been introduced in image processing over the years, most of them belonging to a few, well-defined categories. Area based measures have been popular for quite some time [3,11,12], as well as boundary-based ones, like [13]. Other methods use simplification of the contour [8] to derive a shape hierarchy, or even use a probabilistic approach [9,10] to solve the problem. Our proposed method falls into the latter category, but takes a different approach. Instead of an approximation based on random inner points, or on certain pixels on the boundary, it treats all points equivalently.

The structure of the present paper is the following. In Section 2 we present the preliminaries for our problem and some basic features of images in binary tomography. Section 3 introduces our new convexity measure. In Section 4 we present few experimental results, and finally Section 5 is for the conclusion.

2 Preliminaries

2.1 Definitions and Notation

Let us consider the two-dimensional integer lattice $\mathbb{Z}^2$ on the plane. Any finite subset of this lattice is called a *discrete set*. A discrete set cannot only be represented by its elements, but also by a binary matrix or a binary image.

Adjacency in binary tomography is defined as follows: two positions $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ in a discrete set are *4-adjacent* if $|p_1 - q_1| + |p_2 - q_2| = 1$. A discrete set F is called *4-connected* if for two arbitrary positions $P, Q \in F$, there exists a sequence of distinct positions $P = P_0, P_1, \ldots, P_l = Q$ in the set F, so that P_i is 4-adjacent to P_{i-1}, respectively, for each $i = 1, \ldots, l$. Sometimes we also call 4-connected discrete sets *polyominoes*. A discrete set F is *h-convex* if no individual row contains any intervening 0s in the sequence of 1s, *v-convex* if no individual column contains any intervening 0s in the sequence of 1s, and *hv-convex* if both conditions are met.

Run-length encoding (RLE) is a simple, yet useful form of discrete data representation, mainly used for data compression purposes. Instead of storing data as is, each sequence in which the same data value appears in consecutive elements is stored as a single data value and a counter. This, of course, is effective only if there are relatively few, but preferably long runs of identical values in the data. This representation becomes highly beneficial when calculating convexity of rows or columns of discrete sets. A *token* or *run* is a maximal sequence in which the same data value occurs in consecutive data elements. The *length* of a given token is the number of occurences of the same data value in that particular token. A *1-token* is a token of 1s and a *0-token* is a token of 0s.

Consider, for example the following row of length 15: 001111100011111. We can represent this row in a more visible way, $0^2 1^5 0^3 1^5$, where the superscripts represent the length of each token (counters). The *span* of the data in a single row (column) is the distance between the outermost 1s in that row (column), while the *length* of the row (column) is the total number of bits present in that row (column). Obviously, the length is never smaller than the span. In our example the span of the row is 13, and its length is 15.

2.2 Basic Properties of a Convexity Measure

From a one-dimensional convexity measure we expect the following desirable, basic properties, several of which were also considered in [9,10]:

– It should give a value for all discrete sets in the interval [0, 1].
– For the least convex sets in the defined sense, it should give 0.
– It should give 1 if and only if the set is convex in the defined sense.
– It should be invariant under appropriate geometrical transformations.

In discrete geometry, thus in binary tomography the latter should be treated carefully. The measure is usually expected to be translation invariant, and it could be invariant with respect to uniform scaling. However, it is not expected to be rotation invariant, since elements of binary images are arranged in rows and columns, thus rotation is inherently problematic.

3 Introducing a New Convexity Measure

3.1 Preliminary Experiments

We carried out various early statistical experiments on discrete sets, mostly on the benchmark sets of [1] with added noise and distortions. We tried several combinations of the following features that, we believed, could truly describe the convexity of a sequence of binary digits: number of bit changes in the sequence, the span of the data, the length of the sequence, the number of 1-tokens in the sequence, the number of 0-tokens between the outermost 1-tokens in the sequence, etc. Although some results were promising, we concluded that each of these ad hoc measures unjustifiably favored some patterns as the most or least convex sets.

3.2 A Measure for Directional Convexity

To provide a measure that is objective and universally applicable, a theoretical approach is needed. One should consider the definition of convex shapes in the continuous domain. A planar shape $\mathbf{C}$ is said to be convex if for arbitrary points $A, B \in \mathbf{C}$, all points of the line segment $\overline{AB}$ belong to $\mathbf{C}$. One could determine the convexity of $\mathbf{C}$ by taking all possible pairs of points in $\mathbf{C}$ and measuring the proportion of points in the formed line segments that are not in $\mathbf{C}$ (the outer points). Of course, when $\mathbf{C}$ is convex, no line segment connecting points of $\mathbf{C}$ will contain outer points. Unfortunately, even if the shape is discretized, thus there is only a finite number of points, it is computationally too expensive to calculate all contributions of outer points in all possible line segments in $\mathbf{C}$. To overcome this problem possible solutions so far include random sampling of inner points, randomly choosing points on the boundary only [10,13], or using probabilistic measures to estimate the convexity of the shape [9].

Instead of limiting our calculations to only a random selection of points, we consider all pairs of points in the set and the line segments connecting them, but only in a few number of predefined directions. This approach suits binary tomography better, since there are certain fundamental directions for describing and examining binary images, such as horizontal and vertical. Although throughout this paper we measure convexity in these two directions only, one could select any particular direction.

3.3 Calculating Directional Convexity

We compute *directional* convexity of a row (or column) in the following way. We split the row (column) into a sequence of 1-tokens and 0-tokens (see Section 2). From the construction above trivially follows that leading and trailing 0-tokens do not contribute to the measure, thus hereafter we shall omit them. The rest of the row (column) can be encoded as $1^{k_1}0^{l_1}1^{k_2}0^{l_2}\dots 1^{k_n}$, where n is the number of 1-tokens and $k_1, l_1, k_2, l_2 \dots, k_n > 0$. Trivially, taking two 1s from the same 1-token, the line segment connecting them will not contain any 0s and will not contribute to the convexity measure. Now, let us take two arbitrary 1s from different 1-tokens, say the ith and jth, such that $i < j$. The contribution of 0s (outer points) in the line segment that connects them is

$$\sum_{t=i}^{j-1} l_t \,. \tag{1}$$

For two different 1-tokens (ith and jth), we can form $k_i k_j$ possible pairs of 1s, by picking one from each. The contribution of this particular 1-token pair is

$$k_i k_j \sum_{t=i}^{j-1} l_t \,. \tag{2}$$

Finally, to get the contributions for the entire row (column) one has to sum up (2) for all possible $\binom{n}{2}$ combinations of 1-token pairs:

$$\varphi = \sum_{1 \le i < j \le n} k_i k_j \sum_{t=i}^{j-1} l_t \,. \tag{3}$$

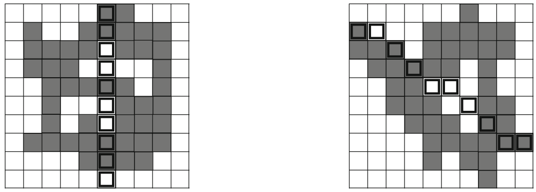

Fig. 1. A 4-connected discrete set with the highlighted column 1100100110 (left). For that particular column $\varphi = 2 \cdot 1 \cdot 2 + 2 \cdot 2 \cdot 4 + 1 \cdot 2 \cdot 2 = 24$. Another 4-connected discrete set (right), with a line segment in a different direction. The highlighted line segment to be used for the calculation in this case is 1011000111, with $\varphi = 32$.

The higher φ is, the less convex the row (column) is. Therefore, φ actually indicates the *directional non-convexity* of the row (column), rather than the directional convexity. Later, we shall describe a way to define a directional convexity measure based on φ.

Figure 1 shows an example discrete set on the left, represented by its binary image and the calculation of φ of its highlighted column. On the right, it shows a different discrete set and a discrete line segment that connects two of its inner points, emphasizing on the fact that the same calculations can be made for arbitrary discrete directions.

3.4 Normalizing Directional Non-convexity

To obtain a normalized measure, it is required to know what is the maximum value a single row can produce for (3), i.e. which is the least convex row according to our measure. Initially we ran simulated annealing to find rows with such property, but eventually it became clear that such rows have the form of $1^{K/3}0^{K/3}1^{K/3}$, where K is the length of the row and $K \equiv 0 \pmod 3$. (In case $K \equiv 1 \pmod 3$ is true, then $1^{\lfloor K/3 \rfloor}0^{\lceil K/3 \rceil}1^{\lfloor K/3 \rfloor}$ is the correct form, while if $K \equiv 2 \pmod 3$ is true, then $1^{\lceil K/3 \rceil}0^{\lfloor K/3 \rfloor}1^{\lceil K/3 \rceil}$ is the correct form.) The following two lemmas form the basis of this fact.

Lemma 1. *Let a row be given such that* $1^{k_1}0^{l_1}1^{k_2}0^{l_2}\ldots 1^{k_n}$, *with* $k_1 = l_1 = \cdots = k_n = \frac{K}{2n-1}$. *Then the non-convexity of the row is maximal if* $n = 2$.

Proof. There are $n-1$ pairs of 1-tokens having exactly one 0-token between them, $n-2$ pairs of 1-tokens having exactly two 0-tokens between them, and so on. Finally, there is one pair of 1-tokens having $n-1$ 0-tokens between them. For 1-tokens with exactly i 0-tokens between them the non-convexity sum is

$$(n-i)\left(\frac{K}{2n-1}\right)^2 \frac{iK}{2n-1} \,. \tag{4}$$

Thus, the total non-convexity sum for the row is

$$\begin{aligned}\varphi_n &= \sum_{i=1}^{n-1}(n-i)\Big(\frac{K}{2n-1}\Big)^2\frac{iK}{2n-1} = \\ &= \frac{K^3}{(2n-1)^3}\sum_{i=1}^{n-1} i(n-i) = \frac{K^3}{(2n-1)^3}\left(n\sum_{i=1}^{n-1} i - \sum_{i=1}^{n-1} i^2\right) = \\ &= \frac{K^3}{(2n-1)^3}\left(\frac{n^2(n-1)}{2} - \frac{(n-1)n(2n-1)}{6}\right) = \frac{K^3 n(n-1)(n+1)}{6(2n-1)^3} .\end{aligned} \tag{5}$$

Similarly,

$$\varphi_{n+1} = \frac{K^3(n+1)n(n+2)}{6(2n+1)^3} . \tag{6}$$

Then, for an arbitrary $n \geq 2$

$$\varphi_n - \varphi_{n+1} = \frac{K^3 n(n+1)}{6}\left(\frac{(n-1)(2n+1)^3 - (n+2)(2n-1)^3}{(2n-1)^3(2n+1)^3}\right) > 0 \tag{7}$$

since $(n-1)(2n+1)^3 - (n+2)(2n-1)^3 = 12n^2 - 16n + 1 > 0$. Thus, φ_n is maximal if and only if $n = 2$. □

Lemma 2. *Let a row be given such that* $1^a0^b1^c$ *with* $a, b, c > 0$ *and* $K = a+b+c$. *Then the non-convexity of the row is maximal if* $a = b = c$.

Proof. From the definition it follows that $\varphi(1^a0^b1^c) = abc = ab(K-a-b)$. For the maximality of this expression the derivatives must be equal to 0, i.e.,

$$bK - 2ba - b^2 = 0 \quad \text{and} \quad aK - a^2 - 2ba = 0 . \tag{8}$$

Knowing that $a, b > 0$ we get that

$$K - 2a - b = 0 \quad \text{and} \quad K - a - 2b = 0 , \tag{9}$$

thus $2K - 3a - 3b = 0$ and therefore $\frac{2}{3}K = a + b$. Substituting $b = \frac{2}{3}K - a$ into (8) the lemma follows. □

Hence, the maximum value of non-convexity of a row (column) is $(K/3)^3$, where K denotes the length of the row (column). Using this, the normalized non-convexity of a row (column) is

$$\hat{\varphi} = \frac{\varphi}{(K/3)^3} . \tag{10}$$

3.5 Directional Convexity of a Two-Dimensional Discrete Set

The directional non-convexity of a two-dimensional discrete set can be defined as the mean of the normalized non-convexity value of all rows (columns). For

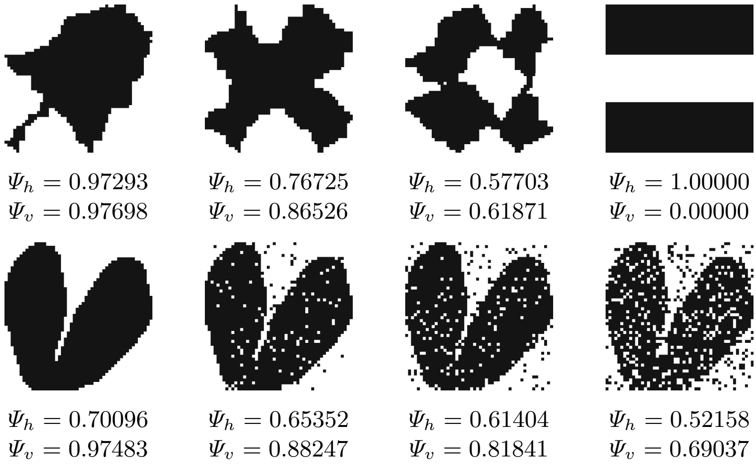

Fig. 2. Example binary images of size 50×50, with horizontal (Ψ_h) and vertical (Ψ_v) convexity shown. Bottom row: same image without, and with 5%, 10%, and 20% noise.

example, for an entire binary matrix consisting of m rows, the directional non-convexity is

$$\Phi_h = \frac{\sum_{r=1}^{m} \hat{\varphi}_r}{m} \quad , \tag{11}$$

where $\hat{\varphi}_r$ is the normalized non-convexity of the rth row. From Φ_h one can simply derive a directional convexity measure for a discrete set by any monotonic continuous mapping from $[0, 1]$ to $[1, 0]$, e.g. $\Psi_h = 1 - \Phi_h$ can be considered such a measure. Analogously we can define the vertical convexity Ψ_v.

4 Experimental Results

The proposed method for measuring directional convexity of binary images along a defined direction has been tested thoroughly. Most of the images used were binary images derived from the 4-connected convex discrete sets of [1] by performing various operations resulting in a wide variety of non-convex images. Such operations included adding salt and pepper noise, applying morphological, topological, and set operations. Figure 2 shows a few display examples, along with the corresponding convexity measures.

We found that our method performs particularly well on noisy images, contrary to other methods that we tested that use the convex hull of the object or the span of each row or column. Another advantage is that the transition of the measure from convex to concave images is much smoother compared to other methods. With several other methods we experienced huge declines in function value caused by only a small distortion or noise in the image, which, we firmly believe, is unacceptable. Our model does not include any artificially favoured structure in the image to be considered the least convex.

5 Summary and Conclusion

In this paper we propose a new method to measure directional convexity of discrete sets. So far in all experiments we used horizontal and vertical directions, conventional in binary tomography, but the method works with any predefined direction as well. We are already working on the generalization to combine several directions to build a more global convexity measure for multidimensional discrete sets. We also have preliminary results on extending this work to measure convexity of not neccessarily binary discrete images, which may open ways to explore the connections between representations of binary, discrete, fuzzy, and continuous properties of shapes in images.

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