Reconstruction and Enumeration of *hv*-Convex Polyominoes with Given Horizontal Projection*

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Abstract. Enumeration and reconstruction of certain types of polyominoes, according to several parameters, are frequently studied problems in combinatorial image processing. Polyominoes with fixed projections play an important role in discrete tomography. In this paper, we provide a linear-time algorithm for reconstructing *hv*-convex polyominoes with minimal number of columns satisfying a given horizontal projection. The method can be easily modified to get solutions with any given number of columns. We also describe a direct formula for calculating the number of solutions with any number of columns, and a recursive formula for fixed number of columns.

Keywords: discrete tomography, reconstruction, enumeration, polyomino, *hv*-convexity.

1 Introduction

Projections of binary images are fundamental shape descriptors that are widely used in tasks of pattern recognition and image processing (see, e.g., [1, 10, 11], and the references given there). In binary tomography [8, 9], projections are used to reconstruct binary images from them. Several theoretical results are known, regarding the efficient reconstruction and the number of solutions, using just the horizontal and vertical projections. From theoretical point of view, *hv*-convex polyominoes form an extensively studied class of binary images. Although, we know quite a lot about the reconstruction complexity and the number of solutions in this class when the horizontal and vertical projections are available [2, 3, 5], surprisingly, those problems have not yet been investigated if only one projection is given. In this paper, we fill this gap by describing a linear-time reconstruction

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algorithm and providing formulas for the number of solutions with minimal and with any given number of columns.

The paper is structured as follows. In Section 2 we give some preliminaries. In Section 3, we provide a linear-time algorithm for reconstructing hv-convex polyominoes from the horizontal projection. Section 4 describes formulas for enumerating hv-convex polyominoes with given horizontal projection, for arbitrary, and also for fixed number of columns. The conclusions are given in Section 5.

2 Preliminaries

A *binary image* is a digital image where each pixel is either black or white. Binary images having m rows and n columns can be represented by binary matrices of size $m \times n$, where the value in the position of the matrix is 1 (respectively, 0) if the corresponding pixel in the image is black (respectively, white).

The *horizontal projection* of a binary image F is a vector representing the number of black pixels in each row of F. Using the matrix representation, it is the vector $\mathcal{H}(F) = (h_1, \ldots, h_m)$, where

$$h_i = \sum_{j=1}^{n} f_{ij} \quad (i = 1, \ldots, m) .$$

The vertical projection of the image can be defined analogously. Throughout the paper, without loss of generality, we assume that each projection component of the binary image is positive.

Two positions $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ in a binary image are said to be *4-adjacent* if $|p_1 - q_1| + |p_2 - q_2| = 1$. The positions P and Q are *4-connected* if there is a sequence of distinct black pixels $P_0 = P, \ldots, P_k = Q$ in the binary image such that P_l is 4-adjacent to P_{l-1}, respectively, for each $l = 1, \ldots, k$. A binary image F is *4-connected* if any two points in F are 4-connected. The 4-connected binary images are also called *polyominoes* [7]. The binary image F is horizontally and vertically convex, or shortly *hv-convex* if the black pixels are consecutive in each row and column of the image (see the polyomino T in Fig. 1). *Upper stack polyominoes* are special hv-convex polyominoes which contain the two bottom corners of their minimal bounding rectangles. Similarly, *lower stack polyominoes* are hv-convex polyominoes that contain the two top corners of their minimal bounding rectangles. Finally, *parallelogram polyominoes* are hv-convex polyominoes that contain both their top left and bottom right, or both their top right and bottom left corners of their minimal bounding rectangles. Any hv-convex polyomino can be constructed (not necessarily uniquely) from an upper stack, a parallelogram and a lower stack polyomino. Figure 1 shows examples for the special types of polyominoes, and such a construction.

3 Reconstruction from the Horizontal Projection

Let $H = (h_1, \ldots, h_m) \in \mathbb{N}^m$ be a vector of size m. We first give an algorithm, called *GreedyRec* which constructs an F binary image with m rows and the

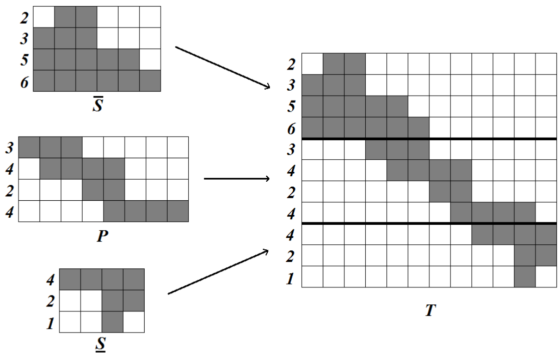


Fig. 1. An hv-convex polyomino T composed of an upper stack $\overline{S}$, a parallelogram P, and a lower stack $\underline{S}$ polyomino

minimal possible number of columns. Due to h-convexity, the 1s are consecutive in each row of the binary image to reconstruct. We will refer to them as the i-th *strip* of the image ($i = 1, \ldots, m$). The sketch of the algorithm is the following (Fig. 3a shows an example result of the algorithm).

1. The first strip must be aligned to the left.
2. The position of the i-th strip of F depends on the position of the $(i-1)$-th strip ($i = 2, \ldots, m$):
   (a) if $h_i = h_{i-1}$, then the i-th strip is just below the $(i-1)$-th strip (see Fig. 2a),
   (b) if $h_i < h_{i-1}$, then the i-th strip is aligned to the right of the $(i-1)$-th strip (see Fig. 2b),
   (c) if $h_i > h_{i-1}$, then the i-th strip is aligned to the left of the $(i-1)$-th strip (see Fig. 2c).

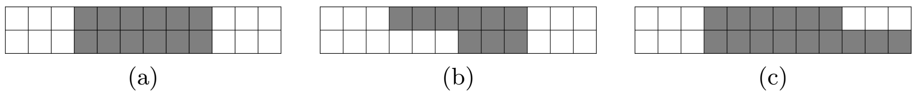
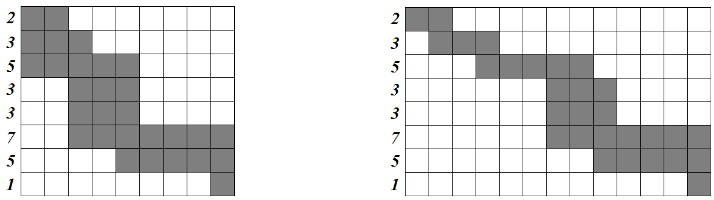


Fig. 2. Steps of *GreedyRec* with the $(i-1)$-th and the i-th rows. Cases: (a) $h_i = h_{i-1}$, (b) $h_i < h_{i-1}$, and (c) $h_i > h_{i-1}$

Theorem 1. *GreedyRec constructs an hv-convex polyomino satisfying the horizontal projection with minimal number of columns, in O(m) time.*

Proof. It is clear that the resulted image is an hv-convex polyomino with the required horizontal projection. We prove by induction that no solution exists with less number of columns.



(a) (b)

Fig. 3. (a) The minimum-size output of *GreedyRec* for $H = (2, 3, 5, 3, 3, 7, 5, 1)$ with 9 columns, and (b) another solution with 13 columns

Let $n_o^{(k)}$ be the number of columns in a minimal-column solution of the problem (i.e., an hv-convex polyomino satisfying the projections with minimal number of columns), considering only the first k components of the input $(h_1, \ldots, h_k)$ $(k \le m)$. Similarly, let $n_g^{(k)}$ be the number of columns in the result of *GreedyRec* for the first k components of the input. For $k = 1$, $n_g^{(1)} = n_o^{(1)} = h_1$, so *GreedyRec* is optimal. For $k > 1$ assume that $n_g^{(k-1)} = n_o^{(k-1)}$.

If $h_k \le h_{k-1}$, then $n_g^{(k)} = n_g^{(k-1)}$ (Cases 2(a) and 2(b) of *GreedyRec*), therefore the number of columns does not change. Since $n_o^{(k)} \ge n_o^{(k-1)}$, therefore $n_g^{(k)} = n_o^{(k)}$, and *GreedyRec* is still optimal.

If $h_k > h_{k-1}$, then $n_g^{(k)} = n_g^{(k-1)} + h_k - h_{k-1}$ (Case 2(c) of *GreedyRec*). Assume to the contrary that an arbitrary optimal algorithm provides a better result, hence $n_o^{(k)} < n_o^{(k-1)} + h_k - h_{k-1}$.

For a further analysis, let us call a column k-*simple* if its $(k-1)$-th element is 0 and its k-th element is 1. The number of k-simple columns is at least $h_k - h_{k-1}$, and due to vertical convexity, in a k-simple column there can be no 1s above the k-th row. Therefore, the first $k-1$ number of strips must fit into $n_o^{(k)} - (h_k - h_{k-1})$ number of non-k-simple columns at most. Due to h-convexity and connectivity, non-k-simple-columns must be successive. Therefore, the first $k-1$ number of strips fit into a matrix with a column number of $n_o^{(k)} - (h_k - h_{k-1}) < n_o^{(k-1)} + h_k - h_{k-1} - (h_k - h_{k-1}) = n_o^{(k-1)}$, which is a contradiction to the minimality of $n_o^{(k-1)}$. Hence, *GreedyRec* is still optimal.

The complexity of the algorithm is straightforward, if the polyomino is represented by the first positions of its strips. □

One can easily modify the output of *GreedyRec* to expand it to have a predefined number of columns (if possible) by moving the k-th, $(k+1)$-st, …, m-th strips further to the right, if the previous strip allows it (i.e., when the image remains hv-convex and 4-connected). The smallest possible number of columns (provided by *GreedyRec*) is $N_{\min} = N_m$, where

$$N_i = \begin{cases} h_i & \text{if } i = 1\ , \\ N_{i-1} & \text{if } h_i \le h_{i-1}\ , \\ N_{i-1} + h_i - h_{i-1} & \text{if } h_i > h_{i-1}\ . \end{cases} \tag{1}$$

This formula can be easily derived from the steps of the algorithm *GreedyRec*. The biggest possible number of columns is

$$N_{\max} = \sum_{i=1}^{m} h_i - m + 1 \ , \tag{2}$$

where every strip is connected with the previous and the next strips through only one element. The modified *GreedyRec* can construct any solution between $N_{\min}$ and $N_{\max}$ in linear time. An example result of the modified algorithm is given in Fig. 3b.

4 Enumerating *hv*-Convex Polyominoes with Fixed Horizontal Projection

Enumeration of polyominoes according to several parameters (e.g., area, perimeter, size of the bounding rectangle, etc.) is an extensively studied field of combinatorial geometry. Regarding the number of *hv*-convex polyominoes satisfying two projections, in [2–5] several results have been published. In [6] a method was proposed to determine the number of *hv*-convex polyominoes that fit into discrete rectangle of given size. In this section, we provide formulas to enumerate *hv*-convex polyominoes satisfying the given horizontal projection.

4.1 Arbitrary Number of Columns

We first give a formula to calculate the number of *hv*-convex polyominoes with a given horizontal projection $H = (h_1, \ldots, h_m)$, if there is no restriction on the number of colums of the resulted image.

Given an *hv*-convex polyomino, the smallest integer k for which $f_{k1} = 1$ is called the *smallest left anchor position*. Similarly, the *greatest right anchor position* is the greatest integer l for which $f_{ln} = 1$. Furthermore, let K denote the greatest integer for which $h_1 \leq h_2 \leq \cdots \leq h_K$. Similarly, let L be the smallest integer for which $h_L \geq h_{L+1} \geq \cdots \geq h_m$. Figure 4 illustrates these definitions.

First, assume that $K < L$. Then, $K < k, l < L$ cannot hold, due to v-convexity. Also note that for every $k < l$ solution, a vertically mirrored image is also a solution with $l < k$, and vice versa. For this reason, we only count the cases with $k < l$ (i.e., $1 \leq k \leq K$ and $L \leq l \leq m$), and multiply the result by 2.

Let $\overline{S}_k(H)$ denote the number of upper stack polyominoes having the horizontal projection $(h_1, \ldots, h_k)$. Similarly, let $\underline{S}_l(H)$ denote the number of lower stack polyominoes having the horizontal projection $(h_l, \ldots, h_m)$. Furthermore, let $P_{k,l}(H)$ denote the number of parallelogram polyominoes with the horizontal projection $(h_k, \ldots, h_l)$, having the smallest left anchor position k and the greatest right anchor position l.

Lemma 1. *$\overline{S}_1(H) = 1$, and $\overline{S}_k(H) = \prod_{i=2}^{k}(h_i - h_{i-1} + 1)$ $(k \geq 2)$. $\underline{S}_m(H) = 1$, and $\underline{S}_l(H) = \prod_{i=l}^{m-1}(h_i - h_{i+1} + 1)$ $(l < m)$.*

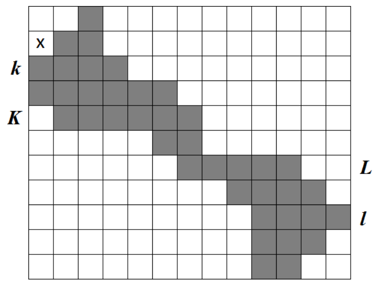


Fig. 4. An hv-convex polyomino with $H = (1, 2, 4, 6, 6, 2, 5, 4, 4, 3, 2)$, where $K = 5$, and $L = 7$. The smallest left anchor position is $k = 3$, the greatest right anchor position is $l = 9$. The $(k-1)$-th strip can be placed on the top of the k-th strip in 2 different ways, and cannot occupy the position marked by $\times$, since the k-th strip must be the leftmost strip

Proof. The formula $\overline{S}_1(H) = 1$ is trivial. If $k \geq 2$, then the $(k-1)$-th strip can be placed on the top of the k-th strip in $h_k - h_{k-1} + 1$ different ways. Similarly, the $(k-2)$-th strip can be placed on the top of the $(k-1)$-th strip, in $h_{k-1} - h_{k-2} + 1$ different ways. And so on. Finally, the first strip can be placed in $h_2 - h_1 + 1$ ways on the top of the second strip. The formula for the lower stack polyominoes can be proven analogously. □

Lemma 2. $P_{k,l}(H) = \prod_{i=k}^{l-1} \min\{h_i, h_{i+1}\}$.

Proof. The k-th strip is fixed (it is in the leftmost position), and we can place the $(k+1)$-th strip under the k-th strip in $\min\{h_k, h_{k+1}\}$ ways. The $(k+2)$-th strip can be placed under the $(k+1)$-th strip in $\min\{h_{k+1}, h_{k+2}\}$ ways. And so on. Finally the l-th strip can be placed under the $(l-1)$-th strip in $\min\{h_{l-1}, h_l\}$ ways. □

In the rest of the paper, we will use the convention that empty (non-defined) factors of a product will be always 1.

Theorem 2. *Let $H \in \mathbb{N}^m$. If $K < L$ then the number of hv-convex polyominoes with the horizontal projection H is*

$$P_{K<L}(H) = 2 \cdot \sum_{k=1}^{K} \sum_{l=L}^{m} \left(\overline{S}_{k-1}(H) \cdot (h_k - h_{k-1}) \cdot P_{k,l}(H) \cdot (h_l - h_{l+1}) \cdot \underline{S}_{l+1}(H) \right) . \tag{3}$$

If $K \geq L$, then the number of solutions is

$$P_{K \geq L}(H) = P_{K<L}(H) - \overline{S}_L(H) \cdot \underline{S}_K(H) . \tag{4}$$

Proof. We observe that an hv-convex polyomino with the smallest left anchor position k and the greatest right anchor position l can be uniquely decomposed into a (possibly empty) upper stack polyomino consisting of the first $k-1$ rows, a (possibly empty) lower stack polyomino of consisting of the last rows from $l+1$ to m, and a parallelogram polyomino consisiting of the k-th, $k+1$-th, ..., l-th rows. If k is the smallest left anchor position, then the $(k-1)$-th strip (the bottom strip of the upper stack polyomino) cannot reach the leftmost position (see the position marked by $\times$, in Fig. 4), therefore the upper stack can be connected to the parallelogram in $(h_k - h_{k-1})$ ways. With a similar argument, the lower stack can be connected to the bottom row of the parallelogram in $(h_l - h_{l+1})$ ways. Thus, using lemmas 1 and 2, for fixed k and l the number of possible solutions is $\overline{S}_{k-1}(H) \cdot (h_k - h_{k-1}) \cdot P_{k,l}(H) \cdot (h_l - h_{l+1}) \cdot \underline{S}_{l+1}(H)$. Including also the mirrored cases we get (3).

If $K \geq L$, then the same formula as in (3) can be applied. However, in this case, it counts some of the solutions twice through symmetry (where the parallelogram poliominoes are rectangular). Note that the longest strips in H are $h_L = h_{L+1} = \cdots = h_K$, and (3) counts all the cases twice when these strips are right under each other. Regarding that the L-th strip is the bottom of the upper stack polyomino, and the K-th strip is the uppermost row of the lower stack polyomino, the number of cases counted twice is $\overline{S}_L(H) \cdot \underline{S}_K(H)$, using Lemma 1. □

4.2 Fixed Number of Columns

Now, we give a recursive formula to calculate the number $P_n(H)$ of hv-convex polyominoes having the horizontal projection $H = (h_1, \ldots, h_m)$, when the number of columns is fixed to n. First, assume again that $K < L$. Let $r \geq 1$ and $P(p_1, \ldots, p_r, n)$ denote the number of parallelogram polyominoes with n columns, having the horizontal projection $(p_1, \ldots, p_r)$.

Lemma 3. *$P(p_1, n) = 1$ if $p_1 = n$. $P(p_1, n) = 0$ if $p_1 \neq n$. Furthermore, for $r > 1$ we have the following recursion*

$$P(p_1, \ldots, p_r, n) = \begin{cases} \sum_{i=1}^{p_1} P(p_2, \ldots, p_r, n-i+1) & \text{if } p_1 \leq p_2 , \\ \sum_{i=1}^{p_2} P(p_2, \ldots, p_r, n-(p_1-p_2)-i+1) & \text{if } p_1 > p_2 . \end{cases}$$

Proof. If $r = 1$, then either the strip itself of length p_1 occupies n number of columns (and should be counted as a solution) or not. If $r > 1$ and $p_1 \leq p_2$, then we count recursively every possible solution where the second strip is shifted to the right under the first strip, and the number of remaining columns decreases proportionately. If $r > 1$ and $p_1 > p_2$, then additionally, we have to substract the difference from the number of required columns, since the second strip must be shifted with at least $p_1 - p_2$ positions to the right, relatively to the first position of the first strip. □

Therefore, including the possible stack polyominoes and the mirrored cases, the number of solutions for a fixed n is

$$P_n(H) = 2 \cdot \sum_{k=1}^{K} \sum_{l=L}^{m} \left(\overline{S}_{k-1}(H) \cdot (h_k - h_{k-1}) \cdot P(h_k, \ldots, h_l, n) \cdot (h_l - h_{l+1}) \cdot \underline{S}_{l+1}(H) \right) ,$$

where $P(h_k, \ldots, h_l, n) = 0$ if $k > l$.

If $K \geq L$ then we have to substract some of the solutions in the same way as in (4). Note that this concerns only $P_{N_{\min}}(H)$ (where n is minimal), since for every other case a mirrored solution is truly a different solution.

$P_n(H)$ also provides a different formula for calculating the number of solutions, if the size of the polyomino can be arbitrary, namely

$$\sum_{n=N_{\min}}^{N_{\max}} P_n(H) ,$$

where $N_{\min}$ and $N_{\max}$ is given by (1) and (2), respectively.

5 Conclusion

In this paper, we showed how to reconstruct *hv*-convex polyominoes from a given horizontal projection with minimal number of columns in linear time. This algorithm can easily be extended to give a solution with any required number of columns, if such a solution exists. We also gave formulas for counting all possible solutions, one for any number of columns, and another one for fixed number of columns. The results can be used in various fields of pattern recognition, image processing, and especially, in binary tomography.

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