Eigenvector Sign Correction for Spectral Correspondence Matching

Muhammad Haseeb[1,2] and Edwin R. Hancock[1,⋆]

[1] Department of Computer Science, University of York, UK
[2] Department of Computer Science, University of Peshawar, Pakistan

Abstract. In this paper we describe a method to correct the sign of eigenvectors of the proximity matrix for the problem of correspondence matching. The signs of the eigenvectors of a proximity matrix are not unique and play an important role in computing the correspondences between a set of feature points. We use the coefficients of the elementary symmetric polynomials to make the direction of the eigenvectors of the two proximity matrices consistent with each other for the problem correspondence matching. We compare our method to other methods presented in the literature. The empirical results show that using the coefficients of the elementary symmetric polynomials for eigenvectors sign correction is a better choice to solve the problem.

Keywords: Eigenvector Direction Correction, Symmetric Polynomials, Correspondence Matching, Point Pattern Matching.

1 Introduction

Correspondence matching between 2D images is the preprocessing step for a number of computer vision algorithms. The problem of feature correspondence matching is to find a one-to-one correspondence between feature points in a pair of 2D images that represent an object in the image. The images can be taken from a different point of view, at different times. In literature many different methods have been presented to address the problem of correspondence matching. These methods can be broadly categorized into two classes namely the non-spectral methods [3, 4] and the spectral methods [1,2,5,7,9–11]. Spectral methods solve the problem using the eigenvalues and eigenvectors of the adjacency matrix or the Laplacian matrix (degree matrix minus the adjacency matrix) for the point set arrangement. Correspondence matchings are computed by embedding the graphs into a common eigenspace using eigen-decomposition of the point-proximity matrices, where correspondences are computed by closest point matching in this eigenspace. However, arbitrary determination of the signs of the eigenvectors returned by a numerical solver causes error in correct correspondence matchings. This problem needs to be handled and has already been reported in previous works [1, 9].

In this paper we address the problem of eigenvector sign correction for the problem of correspondence matching. We propose a novel method that solves the problem of eigenvector sign flipping by using the coefficients of the symmetric polynomials.

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2 Eigenvector Sign Flipping Problem

Spectral graph based correspondence matching algorithms commence by constructing proximity matrices from the given set of points. The structural information present in the proximity matrices of the point sets are used to establish correspondences between the point sets. The method developed by Shapiro and Brady [9] is one of the earliest works on spectral correspondence matching. Shapiro and Brady proposed an algorithm to match 2D feature points across a pair of images using the eigenvectors of a proximity matrix computed from the intra-image distances between each pair of feature points. As input, the algorithm receives a set of m feature points x_i in image I_1 and n feature points y_j in image I_2. Each image feature point is assigned a coordinate in a high dimensional space i.e. each 2D point in image I_1 is mapped from 2D image-plane into an m dimensional hyperspace, and each 2D point in image I_2 is mapped from 2D image-plane into an n dimensional hyperspace. This mapping is performed independently for each image, and when the shapes of the images are similar, the corresponding feature points coincide in the hyperspace.

The eigenvectors or the modes of a single image having m features x_i are computed from a square proximity matrix H. The matrix H is created, recording the affinity between each pair of feature points within the image.

$$H_{ij} = e^{-d_{ij}^2/2\sigma^2} \tag{1}$$

where $d_{ij}^2 = ||x_i - x_j||^2$ is the squared Euclidean distance between each pair of feature points. H is a symmetric matrix and its diagonal entries are unity. The parameter σ controls the interaction between feature points. For small σ the interaction is local, while for large σ each feature point is more globally aware of its surroundings. The next step is to compute the eigenvalues λ_i and the eigenvectors e_i of the matrix H, i.e. by solving

$$He_i = \lambda_i e_i, i = 1 \ldots m,$$

The eigenvectors form an orthonormal basis as the eigenvectors are of unit length and are mutually orthogonal. In matrix form

$$H = \Phi \Lambda \Phi^T \tag{2}$$

where the diagonal matrix Λ contains the eigenvalues along its diagonal in decreasing order. The modal matrix Φ is orthogonal and has the eigenvectors as its column vectors i.e. $\Phi = [e_1|\ldots|e_m]$. Each row of Φ can be thought of a feature vector F_i, containing the m model coordinates of feature point i.

$$\Phi = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{bmatrix}$$

This computation is done for both images I_1 (with m feature points) and I_2 (with n feature points). Two sets of *feature vectors* are obtained, $F_{i,1}$ and F_{j2}, one for each

image respectively. The final step is to compute the association matrix Z. The elements of Z_{ij} show the confidence in the match between the feature points x_i and y_j. The least significant $|m–n|$ eigenvectors and feature vectors are discarded from the larger modal matrix, in the case where the two modal matrices are of different sizes. The best matches are given by the elements of the association matrix Z which are smallest in their row and column. The value Z_{ij} is the squared Euclidean distance between feature vectors,

$$Z_{ij} = ||F_{i,1} - F_{j,2}||^2 \tag{3}$$

However, before computing the association matrix Z, the direction of the both sets of the eigenvectors must be made consistent with each other. The sign of each eigenvector is not unique as the signs of the eigenvectors returned by numerical solvers are assigned arbitrary and switching its direction does not violate the orthogonality of the basis. When calculating the distance between two feature vectors in equation (3), signs play a critical role. In case of inconsistent eigenvector signs, we need to change the sign of feature vector components in one of the two eigenvector matrices to be consistent with the other.

If H_1 represents the proximity matrix of a set of points and H_2 represents the proximity matrix of the same set of points after reordering the labels of the points, the two proximity matrices will contain the same measures but at different locations. Consequently, the eigenvalues obtained from the two matrices will be the same except that their components will be in different order. When $m \neq n$, the eigenvalues obtained from the two proximity matrices H_1 and H_2 are both order from the largest to the smallest. Similarly, the eigenvectors of H_1 and H_2 are reordered so that their order match the order of their eigenvalues. Then m eigenvectors are used to create the feature vectors, from which the values of matrix Z are calculated.

3 Sign Correction Methods

Several researchers have proposed different methods to correct the direction of the eigenvectors. For instance,

1. Park et al. [6] have suggested a method to correct the direction of the eigenvectors. Let V and V' be the modal matrices with e and e' as its eigenvectors respectively. In [6] each eigenvector e_i is compared with it counterpart e'_i and the sign of e_i is corrected so that

$$e_i := \begin{cases} e_i, & \text{if } ||e_i + e'_i|| > ||e_i - e'_i|| \\ -e_i, & \text{otherwise} \end{cases}$$

   where e_i and e'_i are the corresponding eigenvectors of the two adjacency matrices computed from the two images respectively.
2. Umeyama [11] has handled the problem of eigenvector sign correction by taking the absolute values of the components of the eigenvector of both the modal matrices. Umeyama's method works fine under three conditions. First, when noise is sufficiently low. Second, when the eigenvalues of the proximity matrices are not very close to each other. Third, when the rows of the absolute modal matrices are sufficiently different from each other.

3. Caelli and Kosinov [1] find the number of positive and negative components for each eigenvector. The eigenvector is multiplied by -1 if the number of negative components is greater than the number of positive components. This is essentially a dominant sign correction, always ensuring that there are more positive entries in each eigenvector. However, this is highly unreliable since specific to spectral correspondence, the eigenvectors tend to have about the same number of positive and negative entries due to orthogonality to a constant eigenvector.
4. Shapiro and Brady [9] suggested a greedy approach to correct the direction of the eigenvectors. They treat one modal matrix as reference basis and proceed to orient the axes of the other modal matrix one at a time by optimizing for a correspondence cost, choosing for each that direction which maximally aligns the two sets of feature vectors [8].

4 Symmetric Polynomials

A symmetric polynomial is a polynomial $S(x_1, x_2, \ldots, x_n)$ in n variables, such that if any of the variables are interchanged, the same polynomial is obtained. A symmetric polynomial is invariant under permutation of the variable indices. There is a special set of symmetric polynomials referred to as the *elementary symmetric polynomial (S)* that form a basis set for symmetric polynomial. The elementary symmetric polynomials are the most fundamental symmetric polynomials. Any symmetric polynomial can be expressed as a polynomial function of the elementary symmetric polynomials. For a set of variables $x_1, x_2, \ldots, x_n$ the elementary symmetric polynomials can be defined as:

$$
\begin{aligned}
S_1(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n} x_i \\
S_2(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} x_i x_j \\
&\vdots \\
S_r(x_1, x_2, \ldots, x_n) &= \sum_{i_1 < i_2 < \ldots < i_r} x_{i1} x_{i2} \ldots x_{ir} \\
&\vdots \\
S_n(x_1, x_2, \ldots, x_n) &= \prod_{i=1}^{n} x_i
\end{aligned}
$$

The power symmetric polynomial functions (P) are defined as

$$
\begin{aligned}
P_1(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n} x_i \\
P_2(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n} x_i^2 \\
&\vdots \\
P_r(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n} x_i^r
\end{aligned}
$$

$$\vdots$$
$$P_n(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^n$$

The elementary symmetric polynomials can be efficiently computed from the coefficients of the power symmetric polynomials using the Newton-Girard formula

$$S_r = \frac{(-1)^{r+1}}{r} \sum_{k=1}^{r} (-1)^{k+r} P_r S_{r-k} \tag{4}$$

here the shortcut S_r is used for $S_r(x_1, x_2, \ldots, x_n)$ and P_r is used for $P_r(x_1, x_2, \ldots, x_n)$.

4.1 Proposed Method

Our proposed method for eigenvector direction correction is based on the use of the coefficients of the elementary symmetric polynomials. For any two eigenvectors the corresponding odd coefficients have opposite sign if their directions are not consistent with each other. Our approach is similar to that of Shapiro and Brady, i.e. we treat one modal matrix as reference basis and proceed to orient the axes of the other model matrix one eigenvector at a time, by comparing their corresponding coefficients of symmetric polynomials. If the corresponding odd coefficients for the two eigenvectors have opposite sign then we multiply one of the eigenvectors by -1 to flip its direction. Any odd coefficients will work, for instance, using only the first coefficients should work. Which is essentially the sum of the eigenvector components. i.e. $S_1(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i$. However, if the sum of the eigenvector components is very close to zero then we move to the next odd coefficients to compare.

The Algorithm. The following steps show how to correct the sign of eigenvectors for correspondence matching.

Input: Proximity matrices A and B

1. Find eigen-decomposition, $A = V_A \Lambda V_A^T$ and $B = V_B \Lambda V_B^T$
2. Discard the least significant eigenvectors and the feature vectors are discarded from the larger modal matrix in the case where A and B are of different sizes. Let N be the size of the smaller matrix.
3. Compute the coefficients symmetric polynomial of each column of matrix V_A and V_B, Let S_A and S_B be the matrices containing the coefficients in its columns, computed from the corresponding columns of the model matrices V_A and V_B respectively.
4. for v := 1 to N
   i := 1
   flipflag := False
   while (i < N and not flipflag) do
   if (S_A[i, v] $*$ S_B[i,v]<0) then
   Flip the sign of the vth eigenvector in matrix V_B
   flipflag := True

```
                end if
                i := i + 2
            end while
        end for
```

4.2 Eigenvector Sign Correction for EM Algorithm

In [2], Carcassoni and Hancock have presented a point pattern alignment method. They have shown how the modal structure of point-sets can be embedded into the statistical framework of the EM algorithm. The global structural properties of the point-sets are represented by the eigenvalues and eigenvectors of the proximity matrices. The algorithm performs very well and offers a powerful means of estimating the transformation parameters. However, there is one problem that restrict the automatic use of the method, which is the need to initialise the parameters. The quality of the corresponding matching results very much depends on a good choice of initial values of the parameters. The authors have used classical multidimensional scaling on the pairwise dissimilarity matrices S_D (for data-points) and S_M (for model-points) computed from the given point-sets. MDS embeds the dissimilarity matrices onto a low dimensional space using the eigen-decomposition. The embedded coordinates for both the point-sets are used to compute the initial correspondence probabilities for the EM algorithm to proceed.

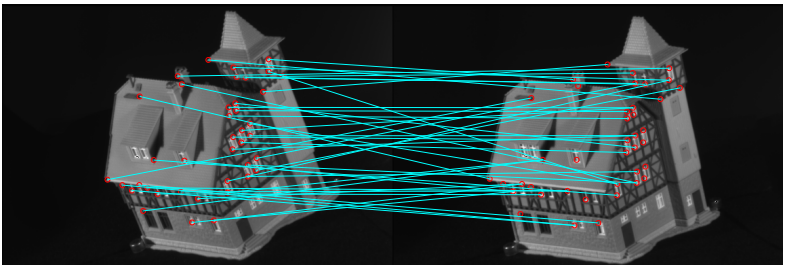
Since, MDS uses the eigen-decomposition to embed the point-set, therefore, the embedded coordinates for both the point-sets may not align properly because of the eigenvector sign flipping problem. The EM alignment algorithm developed by Carcassoni produces very bad results if the eigenvectors are not made consistent with each other by correcting their signs. This is illustrated in figure 2.

Table 1. Performance of sign correction methods on the CMU/VASC house sequence. The first image frame has been matched against the $10^{th}, 20^{th}, 30^{th}, 40^{th}, 50^{th}$ and 60^{th} frame.

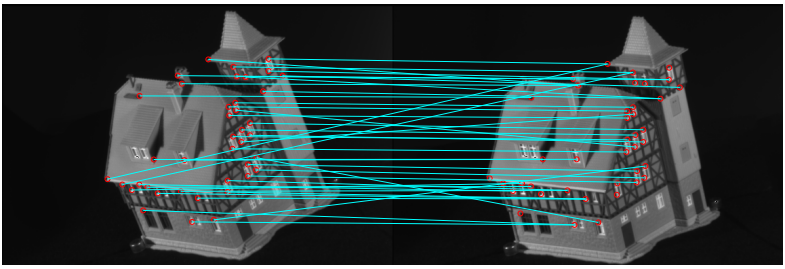
	Number of incorrect matches					
Frames	1 – 10	1 – 20	1 – 30	1 – 40	1 – 50	1 – 60
Park et al.	21	29	34	37	38	37
Caelli & Kosinov	5	7	16	19	19	21
Umeyamma	0	1	5	8	13	16
Symmetric Polynomials	0	0	0	3	7	9

5 Experimental Results

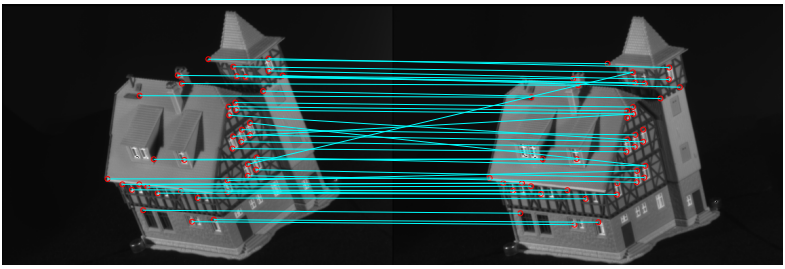
In this section, we provide some experimental results of the correspondence matching affected by the problem of eigenvector sign flipping. We use different techniques for the eigenvector sign correction detailed in Section 3. We extract 40 feature points from



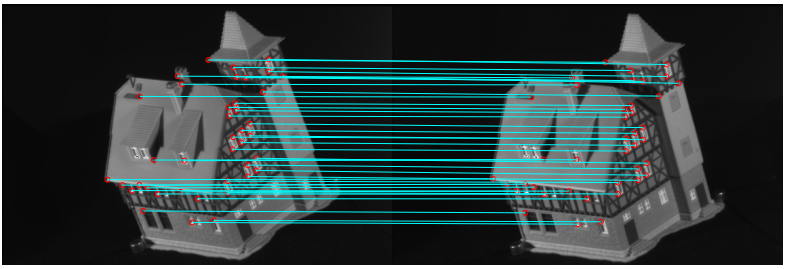
(a)



(b)



(c)



(d)

Fig. 1. Comparing different eigenvector direction correction methods, a) Park et al. b) Caelli & Kosinov. c) Umeyamma. d) Symmetric Polynomials.

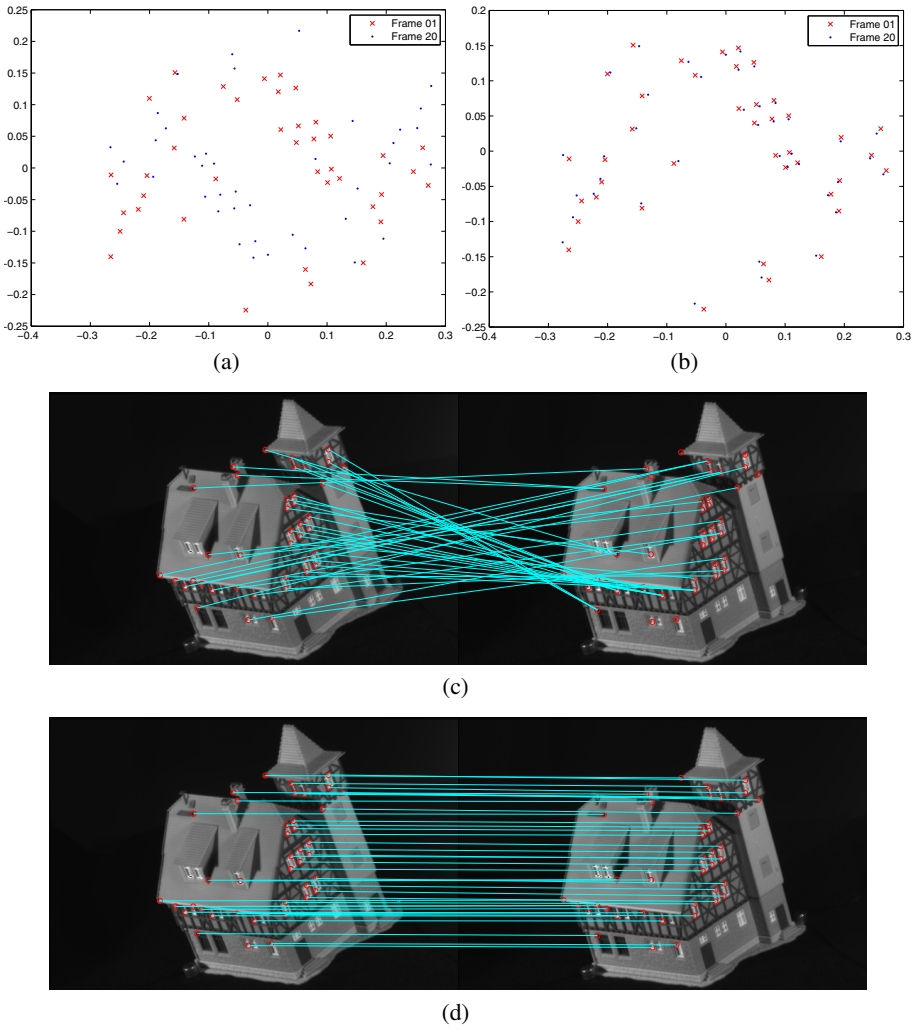


Fig. 2. Effect of the eigenvector sign correction on Carcassoni's alignment EM algorithm, a) Embedded point without sign corrections. b) Embedded point after sign corrections. c) Correspondence matching without sign corrections. d) Correspondence matching after sign corrections.

the 1^{st}, 10^{th}, 20^{th}, 30^{th}, 40^{th}, 50^{th} and 60^{th} frame of the CMU/VASC model house sequence. Table 1 shows the number of incorrect correspondences obtained matching the first frame with the other frames mentioned above. Figure 1 shows correspondence matching between of the feature points extracted from frame 1 and frame 30 of the CMU/VASC model house sequence, after using four different method to correct the sign of the eigenvectors. The first image (figure 1(a)) is the result of the method proposed by Park et al. [6]. The second image (figure 1(b)) is the result of the method proposed by Caelli and Kosinov [1], the third image (figure 1(c)) shows the result of

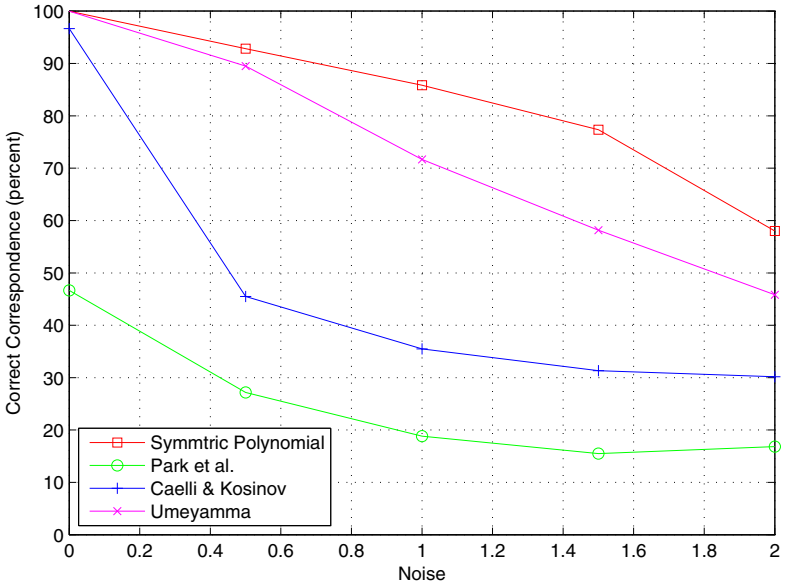


Fig. 3. Effect of increasing noise on correct correspondences using different eigenvector sign correction strategies

the correspondence after the eigenvector sign corrections using the method proposed by Umeyamma [11] and the final image (figure 1(d)) shows the result of the correspondence after the eigenvector sign corrections using the coefficients of symmetric polynomials. It is clear from the figure that out of 40 correspondences, the method proposed by Park et al. produces 34 wrong correspondences, the method developed by Caelli and Kosinov produces 16 wrong correspondences, the method proposed by Umeyamma produces 5 wrong correspondence while our proposed method (Symmetric Polynomials) produces 100% correct correspondences.

In the next experiment, we compare the performance of the different eigenvectors sign correction strategies against the Gaussian noise added in the point positions. Figure 3 shows the fraction of correct correspondences of the four eigenvectors sign correction strategies as a function of the increasing random point-position jitter. Random position jitter is simulated by adding randomly generated position error sampled from a 2D Gaussian distribution to the data point-set. The performance of all the methods decreases with the increase in the noise level. However, the best performance is obtained by using the coefficients of symmetric polynomials. The next best performance is obtained by Umeyamma's strategy. The poorest performance is produced by the method of Park et al.

In the final experiment, we show the results of the correspondence matching by the EM algorithm proposed by Carcassoni [2] with and without the sign correction. 40 points are extracted from the 1st and 20th frame of the CMU/VASC model house sequence for matching. Figure 2(a) and 2(b) show the embedding points of the two frames. Red crosses are the embedded points from Frame 1 while the blue dots represent the

embedded points from Frame 20. Figure 2(a) shows the embedded points when the signs of the eigenvectors have not been corrected. The resulting correspondences are shown in Figure 2(c). Figure 2(b) and Figure 2(d) show the embedded points and the resulting correspondences respectively when the signs of the eigenvectors are corrected. Note that without correcting the signs of the eigenvectors the EM alignment algorithm can not compute the correct correspondences.

6 Conclusions

In this paper we have investigated the problem of the eigenvector sign correction for correspondences matching. The eigenvector sign correction is an important step in all graph spectral correspondence matching techniques. If the sign of the eigenvector are not corrected properly, the robust alignment algorithms like the one developed by Carcassoni [2] can fail to produce good results.

We used the coefficients of the symmetric polynomials to solve the problem. We also compared our method to some other methods already proposed in the literature, and found that using the coefficients of the symmetric polynomials solved the problem better than the others.

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