One-to-One Disjoint Path Covers in DCell

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Abstract. DCell has been proposed for one of the most important data center networks as a server centric data center network structure. DCell can support millions of servers with outstanding network capacity and provide good fault tolerance by only using commodity switches. In this paper, we prove that there exist r vertex disjoint paths $\{P_i|1 \leq i \leq r\}$ between any two distinct vertices u and v of $DCell_k$ $(k \geq 0)$ where $r = n+k-1$ and n is the vertex number of $DCell_0$. The result is optimal because of any vertex in $DCell_k$ has r neighbors with $r = n + k - 1$.

Keywords: DCell, Data Center Network, Disjoint Path Covers, Hamiltonian.

1 Introduction

Data centers become more and more important with the development of cloud computing. Specifically, in recent years, data centers are critical to the business of companies such as Amazon, Google, FaceBook, and Microsoft, which have already owned tremendous data centers with more than hundreds of thousands of servers. Their operations are important to offer both many on-line applications such as web search, on-line gaming, email, cloud disk and infrastructure services such as GFS [1], Map-reduce [2], and Dryad [3].

Researches showed that the traditional tree-based data center networks [4] have issues of bandwidth bottleneck, failure of single switch, etc.. In order to solve the defects of tree-based data center networks, there are many data center networks which have been proposed such as DCell [4], BCube [5], and FiConn [6, 7]. DCell has many good properties including exponential scalability, high network capacity, small diameter, and high fault tolerantly. In comparison with good capabilities of DCell, BCube is meant for container-based data center networks which only supports thousands of servers, and FiConn is not a regularly network which may raises the construction complexity.

DCells use servers as routing and forwarding infrastructure, and the multicast routing frequency between servers are quite high in data center networks. Multi-cast routing algorithms in DCells can be based on the Hamiltonian model as methods on [8, 9]. One-to-one disjoint path covers (also named spanning connectivity [10, 11]) are the extension of the Hamiltonian-connectivity which could

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as well as used on multi-cast routing algorithms in DCells to largely decrease deadlock and congestion, compared with tree-based multi-cast routing. However, the problem of finding disjoint path covers is NP-complete [13]. Therefore, a large amount researches on problems of disjoint path covers focused on different special networks, such as hypercubes [13–16], their variants [17–19], and others [20–22].

So far there is no work reported about the one-to-one disjoint path cover properties of DCell. In this paper, we prove that there exist r vertex disjoint paths $\{P_i | 1 \leq i \leq r\}$ between any two distinct vertices u and v of $DCell_k$ ($k \geq 0$) where n is the vertex number of $DCell_0$ and $r = n + k - 1$. The result is optimal because of any vertex in $DCell_k$ has r neighbors with $r = n + k - 1$.

This work is organized as follows. Section 2 provides the preliminary knowledge. Some basic one-to-one disjoint path covers properties are given in Section 3. We make a conclusion in Section 4.

2 Preliminaries

A data center network can be represented by a simple graph $G = (V(G), E(G))$, where $V(G)$ represents the vertex set and $E(G)$ represents the edge set, and each vertex represents a server and each edge represents a link between servers (switches can be regarded as transparent network devices [4]). The edge from vertex u to vertex v is denoted by (u, v). In this paper all graphs are simple and undirected.

We use $G_1 \cup G_2$ to denote the subgraph induced by $V(G_1) \cup V(G_2)$ of G. For $U \subseteq V(G)$, we use $G[U]$ to denote the subgraph induced by U in G, i.e., $G[U] = (U, E')$, where $E' = \{(u, v) \in E(G) | u, v \in U\}$. A path in a graph is a sequence of vertices, $P :< u_0, u_1, \ldots, u_j, \ldots u_{n-1}, u_n >$, in which no vertices are repeated and u_j, u_{j+1} are adjacent for $0 \leq j < n$. Let $V(P)$ denote the set of all vertices appearing in P. We call u_0 and u_n the terminal vertices of P. P can be denoted by $P(u_0, u_n)$, which is a path beginning with u_0 and ending at u_n. Let P_1 denote $< u_1, u_2, \ldots, u_{k-1}, u_k >$ and P_2 denote $< u_k, u_{k+1}, \ldots, u_{k+n} >$, then $P_1 + P_2$ denotes the path $< u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_{k+n} >$. If $e = (u_k, u_{k+1})$, then $P_1 + e$ denote the path $< u_1, u_2, \ldots, u_k, u_{k+1} >$. Furthermore, if $e = (u_{k-1}, u_k)$, $P_1 - e$ denote the path $< u_1, u_2, \ldots, u_{k-1} >$.

A path in a graph G containing every vertex of G is called a Hamiltonian path (HP). $HP(u, v, G)$ can be denoted by a Hamiltonian path beginning with a vertex u and ending with another vertex v in graph G. Obviously, if $(v, u) \in E(G)$, then $HP(u, v, G) + (v, u)$ is a Hamiltonian cycle in G. A Hamiltonian graph is a graph containing a Hamiltonian cycle. G is called a Hamiltonian-connected graph if there exists a Hamiltonian path between any two different vertices of G. Obviously, if G is a Hamiltonian-connected graph, then G must be the Hamiltonian graph. Suppose that u and v are two vertices of a graph G. We say a set of r paths between u and v is an r-disjoint path cover in G if the r paths do not contain the same vertex besides u and v and their union covers all vertices of G. An r-disjoint path cover is abbreviated as an r-DPC for simplicity.

A graph G is one-to-one r-disjoint path coverable (r-DPC-able for short) if there is an r-DPC between any two vertices of G. In this paper G is r-DPC-able is not same as G is $(r+1)$-DPC-able.

For any other fundamental graph theoretical terminology, please refer to [12].

DCell uses recursively-defined structure to interconnect servers. Each server connects to different levels of DCell through multiple links. We build high-level DCell recursively form many low-level ones. Due to this structure, DCell uses only mini-switches to scale out instead of using high-end switches to scale up, and it scales doubly exponentially with server vertex degree.

We use $DCell_k$ to denote a k-dimension DCell ($k \geq 0$), $DCell_0$ is a complete graph on n vertices ($n \geq 2$). Let t_0 denote the number of vertices in a $DCell_0$, where $t_0 = n$. Let t_k denote the number of vertices in a $DCell_k$ ($k \geq 1$), where $t_k = t_{k-1} \times (t_{k-1}+1)$. The vertex of $DCell_k$ can be labeled by $[\alpha_k, \alpha_{k-1}, \cdots, \alpha_i, \cdots, \alpha_0]$, where $\alpha_i \in \{0, 1, \cdots, t_{i-1}\}$, $i \in \{1, 2, \cdots, k\}$, and $\alpha_0 \in \{0, 1, \cdots, t_0 - 1\}$. According to the definition of $DCell_k$ [4, 23], we provide the recursive definition as Definition 1.

Definition 1. The k-dimensional DCell, $DCell_k$, is defined recursively as follows.

(1) $DCell_0$ is a complete graph consisting of n vertices labeled with $[0],[1],\cdots,[n-1]$.

(2) For any $k \geq 1$, $DCell_k$ is built from $t_{k-1}+1$ disjoint copies $DCell_{k-1}$, according to the following steps.

(2.1) Let $DCell_{k-1}^0$ denote the graph obtained by prefixing the label of each vertex of one copy of $DCell_{k-1}$ with 0. Let $DCell_{k-1}^1$ denote the graph obtained by prefixing the label of each vertex of one copy of $DCell_{k-1}$ with 1. $\cdots$. Let $DCell_{k-1}^{t_{k-1}}$ denote the graph obtained by prefixing the label of each vertex of one copy of $DCell_{k-1}$ with t_{k-1}. Clearly, $DCell_{k-1}^0 \cong DCell_{k-1}^1 \cong \cdots \cong DCell_{k-1}^{t_{k-1}}$.

(2.2) For any $\alpha_k, \beta_k \in \{0, 1, \cdots, t_{k-1}\}$ and $\alpha_k \geq \beta_k$ (resp. $\alpha_k < \beta_k$), connecting the vertex $[\alpha_k, \alpha_{k-1}, \cdots, \alpha_i, \cdots, \alpha_1, \alpha_0]$ of $DCell_{k-1}^{\alpha_k}$ with the vertex $[\beta_k, \beta_{k-1}, \cdots, \beta_i, \cdots, \beta_1, \beta_0]$ of $DCell_{k-1}^{\beta_k}$ as follow:

$$\begin{cases} \alpha_k = \beta_0 + \sum_{j=1}^{k-1} (\beta_j \times t_{j-1}) + 1 \\ \beta_k = \alpha_0 + \sum_{j=1}^{k-1} (\alpha_j \times t_{j-1}) \end{cases} \tag{1}$$

(resp.),

$$\begin{cases} \alpha_k = \beta_0 + \sum_{j=1}^{k-1} (\beta_j \times t_{j-1}) \\ \beta_k = \alpha_0 + \sum_{j=1}^{k-1} (\alpha_j \times t_{j-1}) + 1 \end{cases} \tag{2}$$

where $\alpha_i, \beta_i \in \{0, 1, \cdots, t_{i-1}\}$, $i \in \{1, 2, \cdots, k\}$, and $\alpha_0, \beta_0 \in \{0, 1, \cdots, t_0 - 1\}$.

By Definition 1, $DCell_{k-1}^{\alpha_k}$ is a subgraph of $DCell_k$, where $\alpha_k \in \{0, 1, \cdots, t_{k-1}\}$.

Figure 1(1), 1(2), and 1(3) demonstrate $DCell_0$, $DCell_1$, and $DCell_2$ with $t_0 = 2$ respectively. 1(4) and 1(5) demonstrate $DCell_0$ and $DCell_1$ with $t_0 = 3$ respectively.

3 Main Results

In this section, we will study one-to-one disjoint path cover properties of DCell.

Theorem 1. $DCell_k$ $(k \geq 0)$ is Hamiltonian-connected with $t_0 \geq 2$ except for $DCell_1$ with $t_0 = 2$. In other word, $DCell_k$ $(k \geq 0)$ is 1-DPC-able with $t_0 \geq 2$ except for $DCell_1$ with $t_0 = 2$.

Proof. We omit the proof due to the page limitation. □

Theorem 2. $DCell_k$ is a Hamiltonian graph for any $k \geq 0$. In other word, $DCell_k$ is 2-DPC-able for any $k \geq 0$.

Proof. We omit the proof due to the page limitation. □

Lemma 1. $DCell_0$ is $(t_0 - 1)$-DPC-able with $t_0 \geq 2$.

Proof. The lemma holds for $DCell_0$ which is a complete graph [12]. □

Lemma 2. $DCell_1$ is 2-DPC-able with $t_0 = 2$.

Proof. $DCell_1$ is a cycle with 6 vertices. Therefore, $DCell_1$ is 2-DPC with $t_0 = 2$ [12]. □

Lemma 3. $DCell_2$ is 3-DPC-able with $t_0 = 2$.

Proof. For $t_0 = 2$, we use construction method to proof this lemma. We can construct an 3-DPC between u and v in $DCell_2$ for any pair of vertices $\{u, v\} \in V(DCell_2)$.

For example, the 3-DPC $\{P_1, P_2, P_3\}$ (resp. $\{R_1, R_2, R_3\}$, $\{T_1, T_2, T_3\}$, $\{S_1, S_2, S_3\}$, $\{U_1, U_2, U_3\}$) from $[0, 0, 0]$ to $[0, 0, 1]$ (resp. $[0, 1, 0]$, $[0, 1, 1]$, $[0, 2, 0]$, $[0, 2, 1]$) whose union covers $V(DCell_2)$ with $t_0 = 2$ are listed below (Similarly for the other cases).

$P_1 =< [0, 0, 0], [0, 0, 1] >$,

$P_2 =< [0, 0, 0], [0, 1, 0], [0, 1, 1], [0, 2, 1], [0, 2, 0], [0, 0, 1] >$,

$P_3 =< [0, 0, 0], [1, 0, 0], [1, 0, 1], [2, 0, 1], [2, 2, 0], [2, 2, 1], [2, 1, 1], [4, 1, 0], [4, 0, 0], [4, 0, 1], [4, 2, 0], [5, 2, 0], [5, 2, 1], [6, 2, 1], [6, 1, 1], [6, 1, 0], [6, 0, 0], [6, 0, 1], [6, 2, 0], [4, 2, 1], [4, 1, 1], [3, 1, 1], [3, 2, 1], [3, 2, 0], [5, 1, 1], [5, 1, 0], [5, 0, 0], [5, 0, 1], [1, 2, 0], [1, 2, 1], [1, 1, 1], [1, 1, 0], [3, 0, 1], [3, 0, 0], [3, 1, 0], [2, 1, 0], [2, 0, 0], [0, 0, 1] >$.

$R_1 =< [0, 0, 0], [0, 1, 0] >$,

$R_2 =< [0, 0, 0], [0, 0, 1], [0, 2, 0], [0, 2, 1], [0, 1, 1], [0, 1, 0] >$,

$R_3 =< [0, 0, 0], [1, 0, 0], [1, 0, 1], [1, 2, 0], [1, 2, 1], [1, 1, 1], [1, 1, 0], [3, 0, 1],$

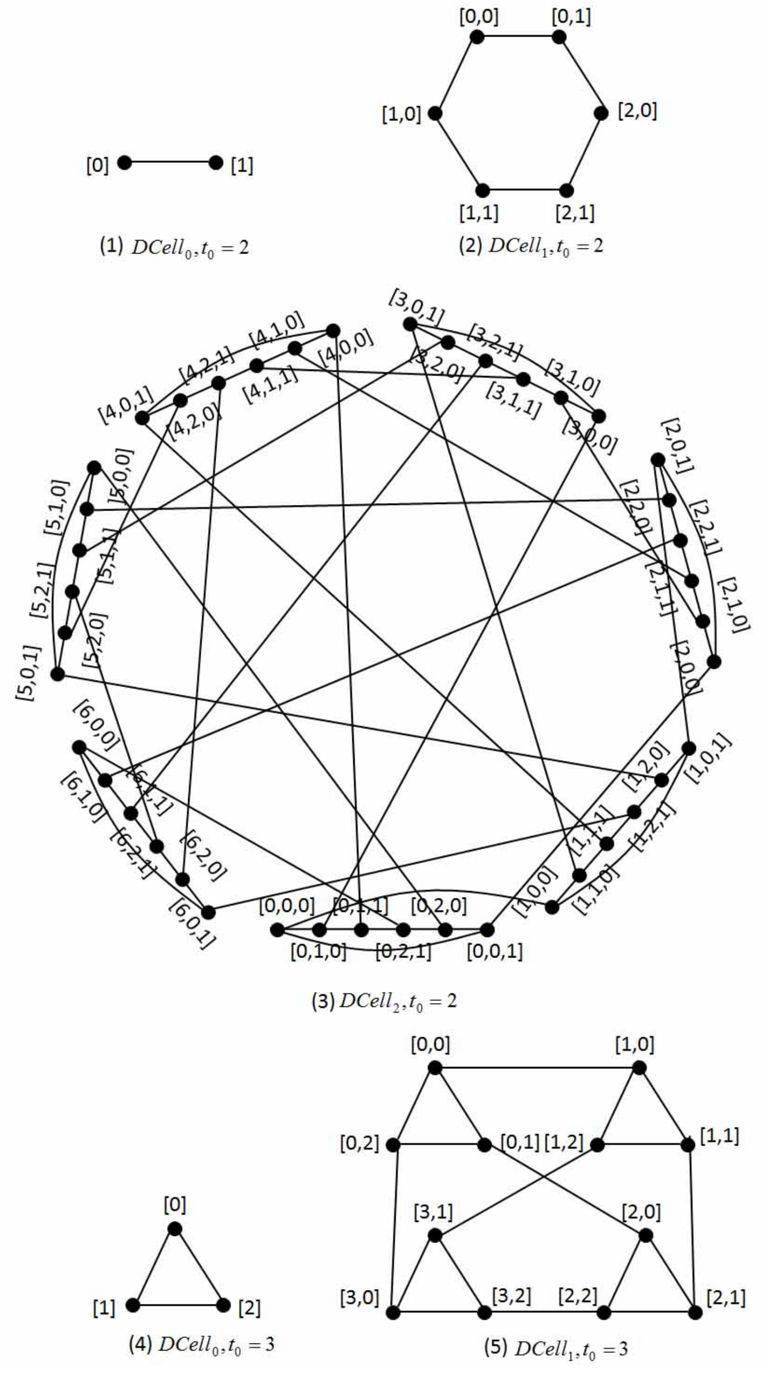

Fig. 1. (1), (2), and (3) demonstrate $DCell_0$, $DCell_1$, and $DCell_2$ with $t_0 = 2$ respectively. (4) and (5) demonstrate $DCell_0$ and $DCell_1$ with $t_0 = 3$ respectively.

$[3,2,0],[3,2,1],[3,1,1],[4,1,1],[4,2,1],[6,2,0],[6,0,1],[6,0,0],[6,1,0],[6,1,1],$
$[6,2,1],[5,2,1],[5,1,1],[5,1,0],[5,0,0],[5,0,1],[5,2,0],[4,2,0],[4,0,1],[4,0,0],$
$[4,1,0],[2,1,1],[2,2,1],[2,2,0],[2,0,1],[2,0,0],[2,1,0],[3,1,0],[3,0,0],[0,1,0]>.$

$T_1 =< [0,0,0],[0,1,0],[0,1,1]>,$

$T_2 =< [0,0,0],[0,0,1],[0,2,0],[0,2,1],[0,1,1]>,$

$T_3 =< [0,0,0],[1,0,0],[1,0,1],[1,2,0],[1,2,1],[1,1,1],[1,1,0],[3,0,1],$
$[3,0,0],[3,1,0],[2,1,0],[2,0,0],[2,0,1],[2,2,0],[2,2,1],[2,1,1],[4,1,0],[4,1,1],$
$[3,1,1],[3,2,1],[3,2,0],[5,1,1],[5,1,0],[5,0,0],[5,0,1],[5,2,0],[5,2,1],[6,2,1],$
$[6,1,1],[6,1,0],[6,0,0],[6,0,1],[6,2,0],[4,2,1],[4,2,0],[4,0,1],[4,0,0],[0,1,1]>.$

$S_1 =< [0,0,0],[0,0,1],[0,2,0]>,$

$S_2 =< [0,0,0],[0,1,0],[0,1,1],[0,2,1],[0,2,0]>,$

$S_3 =< [0,0,0],[1,0,0],[1,0,1],[1,2,0],[1,2,1],[1,1,1],[1,1,0],[3,0,1],$
$[3,0,0],[3,1,0],[3,1,1],[4,1,1],[4,1,0],[4,0,0],[4,0,1],[4,2,0],[4,2,1],[6,2,0],$
$[6,0,1],[6,0,0],[6,1,0],[2,2,1],[2,1,1],[2,1,0],[2,0,0],[2,0,1],[2,2,0],[5,1,0],$
$[5,1,1],[3,2,0],[3,2,1],[6,1,1],[6,2,1],[5,2,1],[5,2,0],[5,0,1],[5,0,0],[0,2,0]>.$

$U_1 =< [0,0,0],[0,0,1],[0,2,0],[0,2,1]>,$

$U_2 =< [0,0,0],[0,1,0],[0,1,1],[0,2,1]>,$

$U_3 =< [0,0,0],[1,0,0],[1,0,1],[1,2,0],[1,2,1],[1,1,1],[1,1,0],[3,0,1],$
$[3,0,0],[3,1,0],[2,1,0],[2,0,0],[2,0,1],[2,2,0],[5,1,0],[5,0,0],[5,0,1],[5,2,0],$
$[4,2,0],[4,0,1],[4,0,0],[4,1,0],[2,1,1],[2,2,1],[6,1,0],[6,1,1],[6,2,1],[5,2,1],$
$[5,1,1],[3,2,0],[3,2,1],[3,1,1],[4,1,1],[4,2,1],[6,2,0],[6,0,1],[6,0,0],[0,2,1]>.$
□

Lemma 4. For any $\alpha, \beta \in \{0,1,\cdots,t_k\}$, $m \in \{1,2,\cdots,t_k-3\}$, and $\alpha \neq \beta$, let $x \in V(DCell_k^\alpha)$ be an arbitrary white vertex, $y \in V(DCell_k^\beta)$ be an arbitrary black vertex, and $G_0 = DCell_k^\alpha \cup DCell_k^\beta \cup (\bigcup_{\theta=0}^m DCell_k^{\omega_\theta})$, where $DCell_k^\alpha$, $DCell_k^\beta$, $DCell_k^{\omega_0}, \cdots, DCell_k^{\omega_i}, \cdots, DCell_k^{\omega_m}$ are internally vertex-independent with $i \in \{0,1,\cdots,m\}$ and $\omega_i \in \{0,1,\cdots,t_k\}$. Then there exists a path between x and y that containing every vertex in $DCell_k[V(G_0)]$ where $k \geq 1$ and $t_0 = 2$.

Proof. Let $G_1 = DCell_k^\alpha \cup DCell_k^\beta$. Select $z \in V(DCell_k^\alpha)$ and $u \in V(DCell_k^\gamma)$, such that $z \neq x$, $(u,z) \in E(DCell_k)$, and $DCell_k^\gamma \subseteq G_0$, where two graphs G_1 and $DCell_k^\gamma$ are internally vertex-independent. Select $\omega \in V(DCell_k^\beta)$ and $v \in V(DCell_k^\delta)$, such that $\omega \neq y$, $(\omega, v) \in E(DCell_k)$, and $DCell_k^\delta \subseteq G_0$ where three graphs G_1, $DCell_k^\gamma$, and $DCell_k^\delta$ are internally vertex-independent. According to Theorem 1, there exists a path P from x to z that containing every vertex in $DCell_k^\alpha$ and a path Q from ω to y that containing every vertex in $DCell_k^\beta$. Let $G_2 = G_0[V(\bigcup_{\theta=0}^m DCell_k^{\omega_\theta})]$. We can construct a path S from u to v that containing every vertex in G_2 which is similar to Theorem 1. Then there exists a path $P + (z,u) + S + (v,\omega) + Q$ between x and y that containing every vertex in $DCell_k[V(G_0)]$ where $k \geq 1$ and $t_0 = 2$. □

Lemma 5. $DCell_k$ is $(k+1)$-DPC-able with $k \geq 2$ and $t_0 = 2$.

Proof. We will prove this lemma by induction on the dimension k of DCell. By lemma 3, the lemma holds for $t_0 = 2$ and $k = 2$. For $t_0 = 2$, supposing that the lemma holds for $k = \tau$ ($\tau \geq 2$), we will prove that the lemma holds for $k = \tau + 1$.

For any vertex $x, y \in V(DCell_{\tau+1})$ with $x \neq y$. Let $x \in V(DCell_\tau^\alpha)$ and $y \in V(DCell_\tau^\beta)$ with $\alpha, \beta \in \{0, 1, \cdots, t_\tau\}$. We can identity α and β as follows.

Case 1. $\alpha = \beta$. There exist $(\tau + 1)$ vertex disjoint paths $\{P_i | 1 \leq i \leq \tau + 1\}$ between any two distinct vertices x and y of $DCell_\tau^\alpha$. Select $u \in V(DCell_\tau^\gamma)$ and $v \in V(DCell_\tau^\delta)$, such that $(x, u), (y, v) \in E(DCell_{\tau+1})$, where three graphs $DCell_\tau^\alpha$, $DCell_\tau^\gamma$, and $DCell_\tau^\delta$ are internally vertex-independent. According to Lemma 4, there exists a path $P_{\tau+2}$ from u to v that visits every vertex in $DCell_{\tau+1}[V(DCell_{\tau+1} - DCell_\tau^\alpha)]$. Then there exist $(\tau+2)$ vertex disjoint paths $\{P_i | 1 \leq i \leq \tau + 2\}$ between any two distinct vertices x and y of $DCell_{\tau+1}$.

Case 2. $\alpha \neq \beta$ and $(x, y) \in E(DCell_{\tau+1})$. Let $P_1 =< x, y >$. Select $x_0 \in V(DCell_\tau^\alpha)$ (resp. $y_0 \in V(DCell_\tau^\beta)$), such that $(x, x_0) \in E(DCell_\tau^\alpha)$ (resp. $(y, y_0) \in E(DCell_\tau^\beta)$). According to the induction hypothesis, there exist $(\tau + 1)$ vertex disjoint paths $\{P'_i | 2 \leq i \leq \tau + 2\}$ (resp. $\{Q'_j | 2 \leq j \leq \tau + 2\}$) between any two distinct vertices x and x_0 (resp. y_0 and y) in $DCell_\tau^\alpha$ (resp. $DCell_\tau^\beta$). Let $P''_2 =< x, x_0 >$ (resp. $Q''_2 =< y_0, y >$), $P'_i =< x, \cdots, x_i, x_0 >$ (resp. $Q'_j =< y_0, y_j, \cdots, y >$), and $P''_i = P'_i - (x_i, x_0)$ (resp. $Q''_j = Q'_j - (y_0, y_j)$) with $3 \leq i \leq \tau+2$ (resp. $3 \leq j \leq \tau+2$). Furthermore, let $z_i \in V(DCell_\tau^{\gamma_i})$ (resp. $w_j \in V(DCell_\tau^{\delta_j})$) with $2 \leq i \leq \tau+2$ (resp. $2 \leq j \leq \tau+2$) and $(x_i, z_i) \in E(DCell_{\tau+1})$ (resp. $(y_i, w_j) \in E(DCell_{\tau+1})$). Let $W_0 = \bigcup_{\theta=2}^{\tau+2} DCell_\tau^{\gamma_\theta}$, $W_1 = \bigcup_{\theta=2}^{\tau+2} DCell_\tau^{\delta_\theta}$ and $W = W_0 \cup W_1 \cup DCell_\tau^\alpha \cup DCell_\tau^\beta$. For $2 \leq i \leq \tau + 2$, we can claim the following two subcases with respect to $DCell_\tau^{\gamma_i}$.

Case 2.1. $DCell_\tau^{\gamma_i} \subseteq W_1$. Select $w_j \in V(DCell_\tau^{\gamma_i})$ such that $2 \leq j \leq \tau + 2$. According to Theorem 1, there exists path a S from z_i to w_j in $DCell_\tau^{\gamma_i}$. Furthermore, let $W = W \cup DCell_\tau^{\gamma_i}$ and $P_i = P''_i + (x_i, z_i) + S + (w_j, y_j) + Q''_j$.

Case 2.2. $DCell_\tau^{\gamma_i} \not\subseteq W_1$. Select $DCell_{\tau+1}^{\delta_j} \not\subseteq W$ such that $2 \leq j \leq \tau + 2$. Then, choose $DCell_\tau^p$ and $DCell_\tau^q$, such that three graphs $DCell_\tau^p$, $DCell_\tau^q$, and W are are internally vertex-independent with $p, q \in \{0, 1, \cdots, t_k\}$. Let $W'_i = DCell_\tau^{\gamma_i} \cup DCell_\tau^{\delta_j} \cup DCell_\tau^p \cup DCell_\tau^q$, according to Lemma 4, there exists a path S from z_i to w_j in $DCell_\tau[W'_i]$. Furthermore, let $W = W \cup W'_i$ and $P_i = P''_i + (x_i, z_i) + S + (w_j, y_j) + Q''_j$.

Furthermore, select P_i, such that $z_i \notin V(W_1)$ and $w_j \in V(W'_i)$ where $2 \leq i \leq \tau + 2$ and $2 \leq j \leq \tau + 2$. According to Lemma 4, there exists path S from z_i to w_j in $DCell_{\tau+1}[V(W'_i) \cup (V(DCell_{\tau+1}) - V(W))]$. Furthermore, let $P_i = P''_i + (x_i, z_i) + S + (w_j, y_j) + Q''_j$.

According to above discussions, there exist $(\tau+2)$ vertex disjoint paths $\{P_i | 1 \leq i \leq \tau + 2\}$ between any two distinct vertices x and y of $DCell_{\tau+1}$.

Case 3. $\alpha \neq \beta$ and $(x, y) \notin E(DCell_{\tau+1})$. Select $u \in V(DCell_{\tau+1})$ (resp. $v \in V(DCell_{\tau+1})$), such that $(x, u) \in E(DCell_{\tau+1})$ (resp. $(y, v) \in E(DCell_{\tau+1})$), $u \in DCell_\tau^\phi$ (resp. $v \in DCell_\tau^\psi$), and $\phi, \psi \in \{0, 1, \cdots, t_k\}$, where $DCell_\tau^\alpha$ and $DCell_\tau^\phi$ (resp. $DCell_\tau^\beta$ and $DCell_\tau^\psi$) are internally vertex-independent. We can claim the following three subcases with respect to u and v.

Case 3.1. $u \in V(DCell_\tau^\beta)$. Select $x_0 \in V(DCell_\tau^\alpha)$, such that $(x, x_0) \in E(DCell_\tau^\alpha)$. Let $y_0 = u$. According to the induction hypothesis, there exist $(\tau+1)$ vertex disjoint paths $\{P'_i | 2 \le i \le \tau+2\}$ (resp. $\{Q'_j | 1 \le j \le \tau+1\}$) between any two distinct vertices x and x_0 (resp. y_0 and y) in $DCell_\tau^\alpha$ (resp. $DCell_\tau^\beta$). Let $P_1 = (x, y_0) + Q'_1$ and $Q''_{\tau+2} = \emptyset$. Then, let $P''_2 =< x, x_0 >$, $P'_i =< x, \cdots, x_i, x_0 >$ (resp. $Q'_j =< y_0, y_j, \cdots, y >$), and $P''_i = P'_i - (x_i, x_0)$ (resp. $Q''_j = Q'_j - (y_0, y_j)$) with $3 \le i \le \tau+2$ (resp. $2 \le j \le \tau+1$). Furthermore, let $z_i \in V(DCell_\tau^{\gamma_i})$ (resp. $w_j \in V(DCell_\tau^{\delta_j})$), where $2 \le i \le \tau+2$ (resp. $1 \le j \le \tau+1$), $w_{\tau+2} = v \in V(DCell_\tau^{\delta_{\tau+2}})$, and $(x_i, z_i) \in E(DCell_{\tau+1})$ (resp. $(y_i, w_j) \in E(DCell_{\tau+1})$). The required $\{P_i | 2 \le i \le \tau+2\}$ paths can be derived by the similar approach as the Case 2, so we skip it.

According to discussions in Case 3 and Case 3.1, there exist $(\tau+2)$ vertex disjoint paths $\{P_i | 1 \le i \le \tau+2\}$ between any two distinct vertices x and y of $DCell_{\tau+1}$.

Case 3.2. $v \in V(DCell_\tau^\alpha)$. The required paths can be derived by the similar approach as the Case 3.1, so we skip it.

Case 3.3. $u \notin V(DCell_\tau^\beta)$ and $v \notin V(DCell_\tau^\alpha)$. Let $P''_1 = Q''_1 = \emptyset$, $x_1 = x$, $z_1 = u$, $w_1 = v$ and $y_1 = y$. Select $x_0 \in V(DCell_\tau^\alpha)$ (resp. $y_0 \in V(DCell_\tau^\beta)$), such that $(x, x_0) \in E(DCell_\tau^\alpha)$ (resp. $(y, y_0) \in E(DCell_\tau^\beta)$). According to the induction hypothesis, there exist $(\tau+1)$ vertex disjoint paths $\{P'_i | 2 \le i \le \tau+2\}$ (resp. $\{Q'_j | 2 \le j \le \tau+2\}$) between any two distinct vertices x and x_0 (resp. y_0 and y) in $DCell_\tau^\alpha$ (resp. $DCell_\tau^\beta$). Let $P''_2 =< x, x_0 >$ (resp. $Q''_2 =< y_0, y >$), $P'_i =< x, \cdots, x_i, x_0 >$ (resp. $Q'_j =< y_0, y_j, \cdots, y >$), and $P''_i = P'_i - (x_i, x_0)$ (resp. $Q''_j = Q'_j - (y_0, y_j)$) with $3 \le i \le \tau+2$ (resp. $3 \le j \le \tau+2$). Furthermore, let $z_i \in V(DCell_{\tau+1}^{\gamma_i})$ (resp. $w_j \in V(DCell_\tau^{\delta_j})$), where $2 \le i \le \tau+2$ (resp. $2 \le j \le \tau+2$) and $(x_i, z_i) \in E(DCell_{\tau+1})$ (resp. $(y_i, w_j) \in E(DCell_{\tau+1})$). The required $\{P_i | 1 \le i \le \tau+2\}$ paths can be derived by the similar approach as the Case 2, so we skip it.

According to discussions in Case 3 and Case 3.3, there exist $(\tau+2)$ vertex disjoint paths $\{P_i | 1 \le i \le \tau+2\}$ between any two distinct vertices x and y of $DCell_{\tau+1}$.

In summary, for any two distinct vertices $x, y \in V(DCell_{\tau+1})$, there exist $(\tau+2)$ vertex disjoint paths $\{P_i | 1 \le i \le \tau+2\}$ between any two distinct vertices x and y of $DCell_{\tau+1}$. □

Lemma 6. For any $t_0 \ge 3$ and $k \ge 0$, $DCell_k$ is $(k + t_0 - 1)$-DPC-able.

Proof. We will prove this lemma by induction on the dimension k of DCell. For any $t_0 \ge 3$, by Lemma 1, the lemma holds for $k = 0$. For any $t_0 \ge 3$, supposing that the lemma holds for $k = \tau$, where $\tau \ge 0$, the proof that the lemma holds for $k = \tau + 1$ is similar to that of lemma 5 and thus omitted. □

Theorem 3. $DCell_k$ is $(k + t_0 - 1)$-DPC-able with $k \ge 0$.

Proof. By Lemma 1, the theorem holds for $k = 0$ and $t_0 \geq 2$. By Lemma 2, the theorem holds for $k = 1$ and $t_0 = 2$. By Lemma 5, the theorem holds for $k \geq 2$ and $t_0 = 2$. By Lemma 6, the theorem holds for $t_0 \geq 3$ and $k \geq 0$. □

4 Conclusions

DCell has been proposed for one of the most important data center networks and can support millions of servers with outstanding network capacity and provide good fault tolerance by only using commodity switches. In this paper, we prove that there exist r vertex disjoint paths $\{P_i|1 \leq i \leq r\}$ between any two distinct vertices u and v of $DCell_k$ ($k \geq 0$) where n is the vertex number of $DCell_0$ and $r = n + k - 1$. The result is optimal because of any vertex in $DCell_k$ has r neighbors with $r = n + k - 1$. According to our result, the method in [8, 9] can be used to decrease deadlock and congestion in multi-cast routing in DCell.

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