Reconstruction of Quantitative Properties from X-Rays*

Fatma Abdmouleh and Mohamed Tajine

LSIIT CNRS UMR 7005, Strasbourg University
Pôle API Boulevard Sébastien Brant 67412 Illkirch-Graffenstaden, France

Abstract. In some applications, the tomographic reconstruction is not an end in itself. When the goal is rather to gather information about the object being studied, the question is if it is more interesting to directly extract these information from the projections without the reconstructing step. We would then know if less projections are needed to directly get the information than to reconstruct the object. In this paper, we address the problem of extracting quantitative information about an object namely an estimation of its area, an upper and a lower bound to the perimeter given its projections from point sources.

1 Introduction

Tomographic reconstruction aims to reconstruct the image of an object given its projections. In some applications, this is done in the purpose of gathering information about this object. This information can be of a qualitative type: the topology (connexity, Euler number, tree of connected components), the geometry (convexity, shape). The information sought can also be of a quantitative type: perimeter, surface area, curvature,etc.

Many researches were lead to know how many projections are needed for the tomographic reconstruction and how this reconstruction is possible in an optimal way. For example, in [6] it is proven that we need three point sources to reconstruct a convex set. The aim of our study is to answer the following question: do we need less projections to directly get the information without reconstructing the image of the object? In this case, it would be more interesting to skip the reconstruction step

In literature, some papers worked on this idea. For example in [5] the smallest possible boundary length of the projected set is estimated from horizontal and vertical projections, in [3] the perimeter of convex sets is estimated from horizontal and vertical projections. In [2], decision trees are used to classify hv-convex sets only from their projections. The cited works consider projections from parallel X-rays, while we address here the problem for projections from point sources. This context is more realistic and general than the parallel X-rays

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since the latter is obtained via an approximation supposing that the point source is at an infinite distance from the object being studied.

This paper is organized as follows. In Section 2 we define the basic tools that will be used in this work. Section 3 is dedicated to the estimation of the surface area from point source projections while in Section 4 an upper and a lower bound to the perimeter are presented.

2 Definition and Notation

In this section we introduce the notation and define the tools that will be useful in the paper. When a statement is true as well in $\mathbb{R}^2$ and in $\mathbb{Z}^2$, we use the notation $\mathbb{S}^2$.

All topological notion used in this paper is considered relatively to the usual topology (Euclidean topology).

With no loss of generality, we consider that the point sources are collinear on the x-axis.

We start by defining the notion of line segment in the continuous and the discrete space:

Definition 1. *Let $a, b \in \mathbb{S}^2$. We define the continuous line segment as $[a, b] = \{\lambda a + (1 - \lambda)b \mid 0 \leq \lambda \leq 1\}$.*

When $a, b \in \mathbb{Z}^2$, we define the discrete line segment as $[\![a, b]\!] = [a, b] \cap \mathbb{Z}^2$.

Let E be a subset of $\mathbb{R}^2$ and a and b be two distinct points of $\mathbb{R}^2$. We use the following notation:

- If E is a finite set, then $|E|$ is the cardinality of E indicating the number of elements of E.
- δE is the boundary of E.
- $\mathring{E}$ is the topological interior of E.
- (ab) is the straight line joining a and b.
- $\mathfrak{P}(E)$ is the powerset of E ($\mathfrak{P}(E) = \{F | F \subseteq E\}$).
- A ray or a half-line R_θ from a point $S = (x_0, y_0)$ in the direction $\overrightarrow{u}_\theta = (u_1, u_2)$ where $||\overrightarrow{u}_\theta|| = 1$, and $\cos\theta = u_1$ and $\sin\theta = u_2$ can be defined in different ways:

$$\begin{aligned} R_\theta &= \left\{(x, y) \in \mathbb{R}^2 \mid u_2(x - x_0) - u_1(y - y_0) = 0 \text{ and } x \geq x_0\right\}; \\ &= \{(x_0, y_0) + \lambda \overrightarrow{u}_\theta \mid \lambda \geq 0\}; \\ &= \left\{M \in \mathbb{R}^2 \mid \widehat{PSM} = \theta\right\}; \end{aligned}$$

  where $\widehat{PSM}$ denotes the angle between (SP) and (SM) with $P = S + (1, 0)$ (see Figure 1.). In what follows, the angle $\widehat{PSM}$ is denoted θ_M.
- For a point $S \in \mathbb{R}^2$, we define the set $\mathcal{K}(S, \mathbb{S}^2)$ formed by all the angles of all the rays issuing from S and passing through points of $\mathbb{S}^2$ with respect to the horizontal line passing through $P = S + (1, 0)$:

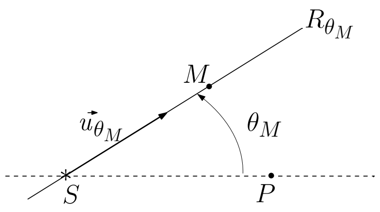

Fig. 1. The ray R_{θ_M} is defined by the source point S and the angle θ_M

$$\mathcal{K}(S, \mathbb{S}^2) = \left\{ \theta_M \in [0, 2\pi[\mid M \in \mathbb{S}^2 \right\}.$$

– $\mathcal{L}(E)$ and $\mathcal{A}(E)$ are respectively the perimeter and the surface area of E.

Definition 2. *Consider a set $E \subset \mathbb{R}^2$, $r > 0$.*

We denote by $r\mathbb{Z}^2$ the discrete grid having a resolution that is equal to $p = 1/r$ called r-grid.

We define the discretization operator Λ_r: $\mathfrak{P}(\mathbb{R}^2) \mapsto \mathfrak{P}(r\mathbb{Z}^2)$ such that $\Lambda_r(E) = E_r = E \cap r\mathbb{Z}^2$.

We now introduce the notions of projections from a point source.

The continuous projection *(or $\mathbb{R}$-projection)* of $E \subset \mathbb{R}^2$ from the point source $S \in \mathbb{R}^2 \setminus E$ denoted $X_{\mathbb{R}}(E, S, .)$: $[0, 2\pi[\mapsto \mathbb{R}$ is:

$$X_{\mathbb{R}}(E, S, \theta) = \int_0^{+\infty} \chi_E(S + t\overrightarrow{u}_\theta)dt.$$

Where $\overrightarrow{u}_\theta = (\cos\theta, \sin\theta)$ and

$$\chi_E(x) = \begin{cases} 1 \; if \; x \in E \\ \\ 0 \; otherwise. \end{cases}$$

Then, $X_{\mathbb{R}}(E, S, \theta) = \mu(E \cap R_\theta)$ where μ is the usual Lebesgue's measure on $\mathbb{R}$.

Let $r > 0$, we define the notion of discrete projection from a point source on the r-grid. Let $S \in \mathbb{R}^2$ and a finite subset $D \subset r\mathbb{Z}^2$ such that $S \notin D$.
D being a finite set, we have a finite number of rays issuing from S and passing through points of D and each of these rays passes through a finite number of points of D. The $r\mathbb{Z}-$projection of D from the point source S is the function $X_{r\mathbb{Z}}(D, S, .)$: $[0, 2\pi[\mapsto \mathbb{N}$ such that:

$$X_{r\mathbb{Z}}(D, S, \theta) = |R_\theta \cap D|.$$

Finally, we define the support of the projection for a point source and the projected set [1].

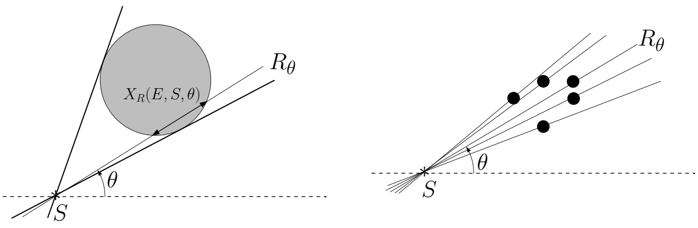

Fig. 2. Continuous(left) and discrete(right) point X-rays

Definition 3. *Let $S \in \mathbb{S}^2$ and $E \subset \mathbb{S}^2$ such that $S \notin E$. The support of $\mathbb{S}-$projections of E for the point source S is the set:*

$$Supp_{\mathbb{S}}(E,S) = \left\{\theta \in \mathcal{K}(S,\mathbb{S}^2) \mid X_{\mathbb{S}}(E,S,\theta) \neq 0\right\}.$$

3 Surface Area Estimation from Point X-Rays

In this section, we aim to find an estimation of the surface area of a set given its projections.

Consider a set $E \subset \mathbb{R}^2$, $r > 0$ and $E_r = \Lambda_r(E)$.

We suppose in this subsection that we have the exact $r\mathbb{Z}$-projections of E_r for any r. The sum of the projections of E_r is the cardinality of E_r and then is the same for any point source. Let n_r be the number of the rays from S passing through all the points of E_r. Given the projections from each of these rays, the number of the points of E_r is given by:

$$|E_r| = \sum_{j=1}^{n_r} s_j^r.$$

where s_j^r is the number of points of $r\mathbb{Z}^2$ lying on the j^{th} ray corresponding to an angle of $Supp_{r\mathbb{Z}}(E_r, S)$.

In all the following, we suppose that we have the boundary $\delta E = \Gamma_1 \cup \Gamma_2$ such that Γ_1 and Γ_2 respectively the graphs of continuous functions $f_1, f_2 : [a,b] \mapsto \mathbb{R}$ with $a, b \in \mathbb{R}$ (see Figure 3 for illustration).

For each point $P = (p_1, p_2)$ of $r\mathbb{Z}^2$, we consider the pixel centered at P: $W(P) = \{(x,y) \in \mathbb{R}^2 \mid |x - p_1| \leq r/2; |y - p_2| \leq r/2\}$. The area of $W(P)$ is then equal to r^2. This will be used to estimate the area of E_r as follows:

$$\mathcal{A}(E_r) = r^2 * \sum_{j=1}^{n_r} s_j^r;$$

We present in the following proposition a new estimator of the area of E:

Proposition 1. *Given a set $E \in \mathbb{R}^2$ with $\delta E = \Gamma_1 \cup \Gamma_2$ such that Γ_1 and Γ_2 respectively the graphs of continuous functions and $E_r = \Lambda_r(E)$ with $r > 0$. We have:*

$$\lim_{r \to 0} \mathcal{A}(E_r) \to \mathcal{A}(E)$$

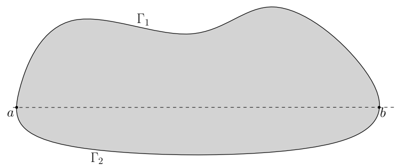

Fig. 3. $\delta E = \Gamma_1 \cup \Gamma_2$ such that Γ_1 and Γ_2 are the graphs of continuous functions

Proof. This proof is based on the Riemann Integral theory. Indeed, consider a line (ab) that divides the boundary of E into Γ_1 and Γ_2 such that Γ_1 and Γ_2 are respectively the graphs of continuous functions $f_1,\ f_2 :\ [a,b] \mapsto \mathbb{R}$. We can suppose with no loss of generality that (ab) is the x-axis. We are then interested in measuring $\mathcal{A}(E) = \int_a^b f_1(x)\,\mathrm{d}x + \int_a^b f_2(x)\,\mathrm{d}x$. Let us show how to estimate $\int_a^b f_1(x)$. The same can be done for f_2. We will cover the considered area with rectangles of width equal to r starting from $a_r = \lceil \frac{a}{r} \rceil \times r$ and ending at $b_r = \lfloor \frac{b}{r} \rfloor \times r$ as illustrated on Figure 4.

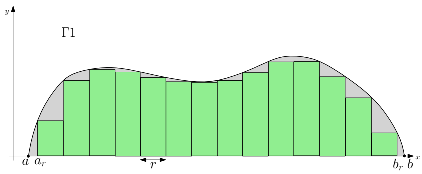

Fig. 4. The area of the considered set is covered by rectangles of width equal to r

By the property of Riemann's integral, we have:

$$\lim_{r \to 0} r \times \sum_i \min_{x \in [i \times r, (i+1) \times r]} f_1(a_r + x) \to \int_a^b f_1(x)\,\mathrm{d}x$$

In our case, we only have the points with coordinates in $r\mathbb{Z}^2$. For a rectangle i, we consider $n_i = \lfloor \frac{f_1(a_r + i \times r)}{r} \rfloor$ (see Figure 5). The additional error induced by considering n_i instead of $f_1(a_r + i \times r)$ on each rectangle of the partition is then $r \times (f_1(a_r + i \times r) - n_i \times r)$ where $(f_1(a_r + i \times r) - n_i \times r) \leq r$. Summing on all the rectangles gives then an error that is at most equal to $(b - a) \times r$. Yet, $\lim_{r \to 0}(b - a) \times r \to 0$.

There remains the parts we neglected when we started the rectangles at a_r and finished at b_r. Since f_1 is continuous on the compact subset $[a, b]$, there exists $M(f_1) = \max_{x \in [a,b]}(f_1(x))$. The area of the neglected part is then at most equal to $2 \times r \times M(f_1)$ and so it tends to 0 when r tends to 0. $\square$

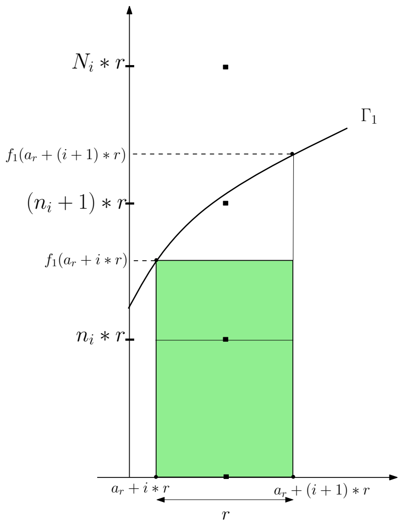

Fig. 5. The error on each rectangle of the partition is $r \times (f_1(a_r + i \times r) - n_i)$

3.1 Example

Let $E = [1,2]^2$ be a square having the sides equal to 1. The projections from any point source $S \in \mathbb{R}^2 \setminus E$ of $E_r = \Lambda_r(E)$, with $r = 1/p > 0$ and $p \in \mathbb{N}*$, verify the following:

$$\sum_{j=1}^{n_r} s_j^r = (\frac{1}{r} + 1)^2$$

The area of the pixel of $r\mathbb{Z}^2$ is equal to r^2, and so :

$$\mathcal{A}(E_r) = (\frac{1}{r} + 1)^2 \times r^1 = 1 + 2r + r^2$$

Then $\lim_{r \to 0} \mathcal{A}(E_r) \to 1 = \mathcal{A}(E)$.

4 Perimeter Estimation from Point Sources

In this part we give two lower bounds and a higher bound to the perimeter of a given set from two projections.

4.1 Lower Bounds of the Perimeter with One Point Source

A lower bound of the perimeter of a set E is given thanks to the following property called the isoperimetric inequality [4]

$$\mathcal{L}^2(E) \geq 4\pi\mathcal{A}(E)$$

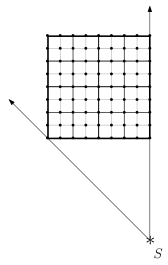

Fig. 6. The discretization of the square at a resolution $1/r$ contains $(\frac{1}{r}+1)^2$ points of $r-\mathbb{Z}^2$

where $\mathcal{A}$ is the measure of the area enclosed by a curve of length L. When E is a circle, we obtain the isoperimetric equality: $\mathcal{L}^2(E) = 4\pi\mathcal{A}(E)$.

From the isoperimetric inequality we can then deduce the following:

Proposition 2. *Given $E \subset \mathbb{R}^2$ and a point source $S \in \mathbb{R}^2$. The perimeter $\mathcal{L}(E)$ of E necessarily verifies the following inequality:*

$$\mathcal{L}^2(E) \geq 4\pi \lim_{r\to 0} \mathcal{A}(E_r) \tag{1}$$

When E is convex, another lower bound of the perimeter can be given thanks to the Crofton Formula:

Proposition 3. ***[Crofton Formula]*** *Let $\gamma : [0,1] \mapsto \mathbb{R}^2$ be a planar curve. Then the length of γ is given by*

$$l(\gamma) = 1/2 \iint_{\mathbb{P}} \eta_\gamma(\rho,\theta)\, \mathrm{d}\rho d\theta$$

where $\mathbb{P} = \mathbb{R}^+ \times [0, 2\pi[$ and for all $(\rho,\theta) \in \mathbb{P}$, $\eta_\gamma(\rho,\theta) = |\gamma([0,1]) \cap D(\rho,\theta)| \in \mathbb{N} \cap \{\infty\}$ which is the number of intersection points of the curve γ with the straight line $D(\rho,\theta)$ as represented in Figure 7.

Then, if $\eta_E(\rho,\theta)$ is the number of points of E on the straight line $D(\rho,\theta)$, we have

$$\mathcal{L}(E) = 1/2 \iint_{\mathbb{P}} \eta_E(\rho,\theta)\, \mathrm{d}\rho d\theta$$

Yet any straight line intersects a convex set in 0, 1 (if it is a tangent line) or 2 points.

With one point source we have :

Proposition 4. *Given a convex set $E \subset \mathbb{R}^2$ and a point source $S \in \mathbb{R}^2$. The perimeter $\mathcal{L}(E)$ of E necessarily verifies the following inequality:*

$$\mathcal{L}^2(E) \geq \iint_{I_r, Supp_{\mathbb{R}}(E,S)} \mathrm{d}\rho d\theta = |I_\rho| * |Supp_{\mathbb{R}}(E,S)| \geq 1/2|\cos(\theta_u) - \cos(\theta_d)| * |\theta_u + \theta_d|$$

Where $I_\rho = [\rho_{min}, \rho_{max}]$, θ_u and θ_d as represented in Figure 8.

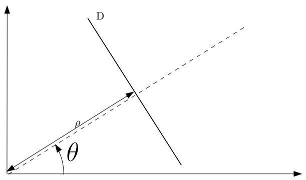

Fig. 7. Representation of a straight line with $(\rho, \theta) \in \mathbb{R}^2 \times [0, 2\pi[$

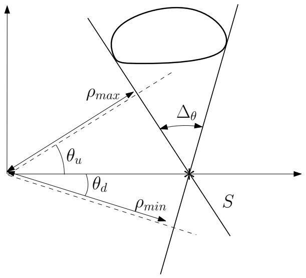

Fig. 8. Representation of the rays of a point source S. $\Delta_\theta = \theta_u + \theta_p$

4.2 Higher Bound of the Perimeter

To find a higher bound of the perimeter of a given convex set $E \subset \mathbb{R}^2$, we need two point sources S and S' of $\mathbb{R}^2$. Similarly to the notation for S, we denote θ' the angles of the rays issuing from S'.

Let θ_1, θ_n, θ'_1 and θ'_m be such that

$$\theta_1 = \min\{\theta \in \text{Supp}_{\mathbb{R}}(E, S)\},\ \theta_n = \max\{\theta \in \text{Supp}_{\mathbb{R}}(E, S)\},$$
$$\theta'_1 = \min\{\theta' \in \text{Supp}_{\mathbb{R}}(E, S')\},\ \theta'_m = \max\{\theta' \in \text{Supp}_{\mathbb{R}}(E, S')\}.$$

The following result is true only when both extreme rays of S (R_{θ_1} and R_{θ_n}) intersect with both extreme rays of S' ($R_{\theta'_1}$ and $R_{\theta'_m}$). In this situation, let us consider A, B, C and D the intersection points of the extreme rays of S and S' see Figure 9. It is evident that we have $E \subseteq ABCD$. Thus we can prove the following result.

Proposition 5. *Let $E \subset \mathbb{R}^2$ and two point sources $S, S' \in \mathbb{R}^2$. The perimeter of $\mathcal{L}(E)$ of E necessarily verifies the following inequality:*

$$\mathcal{L}(E) \leq \mathcal{L}(ABCD)$$

where A, B, C and D are the intersection points of the extreme rays of S and S'.

Proof. As proven in [3], since $ABCD$ is a convex, E is a convex as well, and $E \subseteq ABCD$, then $\mathcal{L}(E) \leq \mathcal{L}(ABCD)$.

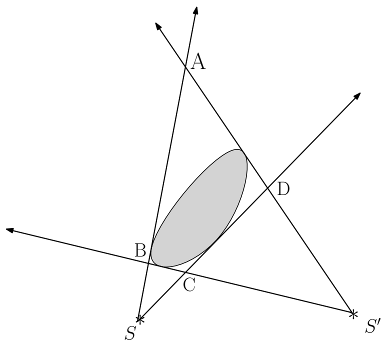

Fig. 9. A, B, C and D the intersection points of the extreme rays of S and S'

4.3 Example

We consider the same square $E = [1, 2]^2$ as for the last section. We suppose that the vertices are $(1, 1)$, $(2, 1)$, $(1, 2)$ and $(2, 2)$. We have $\mathcal{L}(E) = 4$.

- First lower bound:
  Using the information about the surface area $\mathcal{A}(E)$ we have: $\mathcal{L}^2(E) \geq 4\pi \times 1 = 12.566$ and so $\mathcal{L} \geq \sqrt{12.566} = 3.54$.
- Second lower bound:
  To apply the Crofton formula we consider a point source $S = (2, 0)$ (see Figure 10).
  We have then $\theta_u = 45$ and $\theta_d = 0$. Thus:

$$\mathcal{L}^2(E) \geq \frac{1}{2}|\frac{\sqrt{2}}{2} - 1| * 45 = 6.59$$

- Upper bound:
  To compute the upper bound to the perimeter, we consider a second point source $S' = (0, 1)$ as illustrated on Figure 11.
  We have then $\mathcal{L}(E) \leq 3 + 2\sqrt{2} = 5.82$.

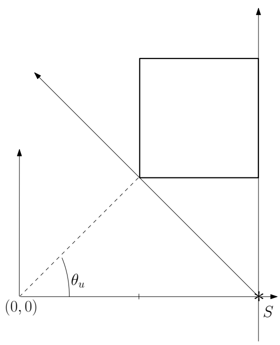

Fig. 10. The square E and a point source $S = (2, 0)$. $\mathcal{L}(E) = 4$.

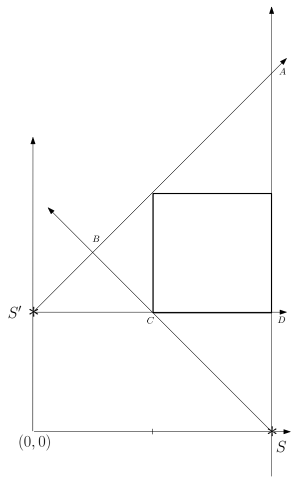

Fig. 11. Computation of an upper bound to the square. $ABCD = 3 + 2\sqrt{2}$

The perimeter verifies then :

$$3.54 \leq \mathcal{L}(E) = 4 \leq 5.82.$$

5 Conclusion

We presented in this paper a new method of extracting some information about sets given the projections from point sources. With one point source, we can estimate the surface area of the projected set and find two lower bounds for its perimeter. An additional point source is needed in order to have a higher bound of the perimeter. This quantitative information is deduced directly from the projection with no reconstruction step and with less point sources than for the reconstruction.

A question remains about the possibility of estimating the perimeter from projections with two point sources. Another interesting perspective to this work is the extraction of qualitative information from projections such as topological information.

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