The Persistence Space in Multidimensional Persistent Homology

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Abstract. Multidimensional persistent modules do not admit a concise representation analogous to that provided by persistence diagrams for real-valued functions. However, there is no obstruction for multidimensional persistent Betti numbers to admit one. Therefore, it is reasonable to look for a generalization of persistence diagrams concerning those properties that are related only to persistent Betti numbers. In this paper, the *persistence space* of a vector-valued continuous function is introduced to generalize the concept of persistence diagram in this sense. Furthermore, it is presented a method to visualize topological features of a shape via persistence spaces. Finally, it is shown that this method is resistant to perturbations of the input data.

1 Introduction

Analyzing and interpreting digital images and shapes are challenging issues in computer vision, computer graphics and pattern recognition [24,25]. Topological persistence – including the theory of persistent homology [13] and size theory [17] – has been promisingly linked to the aforementioned research fields [2].

The classical persistence setting is continuous. Data can be modeled as a pair (X, f), with X a topological space and $f : X \to \mathbb{R}$ a continuous function called a *filtering function*. The role of X is to represent the data under study, while f is a descriptor of some properties which are considered relevant to analyze data. The main idea of persistence is to topologically explore the evolution of the sublevel sets of f in X. These sets, being nested by inclusion, produce a filtration of X. Focusing on the occurrence of important topological events along this filtration – such as the birth and death of connected components, tunnels and voids – it is possible to obtain a global description of data, which can be formalized via an algebraic structure called a *persistence module*. Such information can be encoded in a parameterized version of the Betti numbers, known in the literature as *persistent Betti numbers* [14], a *rank invariant* [6] and – for the $0th$ homology – a *size function* [26]. The key point is that these descriptors can be represented in a very simple and concise way, by means of multi-sets of points called *persistence*

R. Gonzalez-Diaz, M.-J. Jimenez, B. Medrano (Eds.): DGCI 2013, LNCS 7749, pp. 180–191, 2013.

diagrams. Moreover, they are stable with respect to the *bottleneck distance*, thus implying resistance to noise [12].

In concrete applications, where images and shapes are digital, the input data is necessarily discrete. According to the problem at hand, spaces can be modeled by discrete structures such as triangle meshes or cubical complexes. Filtering functions are usually taken to be piecewise-linear. Persistence fits nicely in this discrete framework with none or very little changes. In particular, the case of gray-scale images is treated in [23] and [15].

Thanks to this property, persistence is a viable option for analyzing data from the topological perspective, as shown in a number of concrete problems concerning shape comparison and retrieval [1,10], segmentation [19,20], denoising [11], 3D image simplification [23] and reconstruction [27], visualization [21,22].

A common scenario in applications is to deal with multi-parameter information. An example is given by photometric properties, which are usually taken into account for digital image segmentation. Another instance is the analysis of 4D time-varying CT scans in medical imaging. Further examples can be found in contexts such as computational biology and scientific simulations of natural phenomena. In all these cases, the use of vector-valued filtering functions would enable the study of multi-parameter filtrations, whereas a scalar-valued function only gives a one-parameter filtration. Therefore, Frosini and Mulazzani [18] and Carlsson and Zomorodian [6] proposed *multidimensional persistence* to analyze richer and more complex data. Also in this case the passage from continuous to discrete input data works finely, as shown in [7].

A major issue in multidimensional persistence is that, when filtrations depend on multiple parameters, it is not possible to provide a complete and discrete representation for multidimensional persistence modules analogous to that provided by persistence diagrams for one-dimensional persistence modules [6]. This theoretical obstruction discouraged so far the introduction of a multidimensional analogue of the persistence diagram.

Given the importance of persistence diagrams for the use of persistence in concrete tasks, one can immediately see that the lack of such an analogue is a severe drawback for the actual application of multidimensional persistence. Therefore a natural question we may ask is the following one: In which other sense may we hope to construct a generalization of a persistence diagram for the multidimensional setting?

The persistence diagram is known to satisfy the following important properties [12] (see also [8,16]):

- it can be defined via *multiplicities* obtained from persistent Betti numbers;
- it allows to completely reconstruct persistent Betti numbers;
- the coordinates of its off-diagonal points are homological critical values.

Therefore, it is reasonable to require that a generalization of a persistence diagram for the multidimensional setting satisfies all these properties. We underline that, because of the aforementioned impossibility result in [6], no generalization of a persistence diagram exists that can achieve the goal of representing completely a persistence module, but only its persistent Betti numbers. For this reason, in this paper we will only study persistent Betti numbers and not persistence modules.

The main contribution of the present work is the introduction of a *persistence space*. We show that it generalizes the notion of a persistence diagram in the aforementioned sense. More precisely, we define a persistence space as a multiset of points defined via multiplicities (Definition 3). In the one-dimensional case it coincides with persistence diagrams. Moreover, it allows for a complete reconstruction of multidimensional persistent Betti numbers (Multidimensional Representation Theorem 2), and the coordinates of its off-diagonal points are multidimensional homological critical values (Theorem 5).

Having established these properties (Section 3), the next step is to use persistence spaces to analyze shapes. The tasks we consider are visualization of a summary of topological information of a shape, and comparison of shapes. Indeed these are the main tasks where persistence diagrams are employed. Therefore, as further contributions of this paper, we show that persistence spaces can be visualized and stably compared.

To the best of our knowledge, so far it was impossible to visualize in a single structure the information contained in multidimensional persistent Betti numbers (although a line-by-line visualization method is given, e.g., in [8]). Our visualization method relies on a projection of points of the persistence space onto a lower dimensional space. Moreover, each point is enriched with the persistence value of the topological feature it represents (color-coded). Our visualization procedure is presented in Section 4.

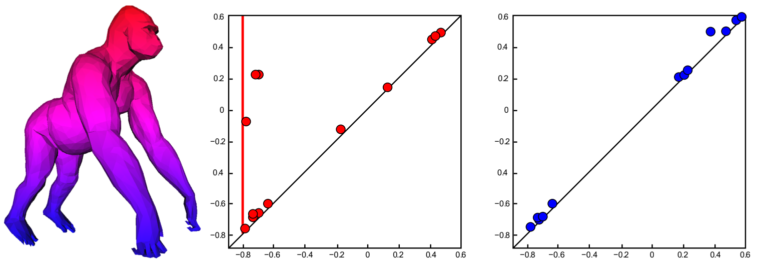
We devote Section 5 to show that the persistence space is resistant to noise. Indeed, persistence spaces can be compared using the multidimensional matching distance introduced in [4]. This comparison turns out to be stable as a simple consequence of the stability of multidimensional persistent Betti numbers [8].

Section 6 concludes the paper with a discussion on the results obtained in this paper and some questions for future research.

2 Background

Let us first consider the case when the filtering function f is real-valued. We can consider the sublevel sets of f to define a family of subspaces $X_u = f^{-1}((-\infty, u])$, $u \in \mathbb{R}$, nested by inclusion, i.e. a *filtration of* X. Homology may be applied to derive some topological information about the filtration of X induced by f. The first step is to define persistent homology groups as follows. Given $u < v \in \mathbb{R}$, we consider the inclusion of X_u into X_v. This inclusion induces a homomorphism of homology groups $H_k(X_u) \to H_k(X_v)$ for every $k \in \mathbb{Z}$. Its image consists of the k-homology classes that live at least from $H_k(X_u)$ to $H_k(X_v)$ and is called the *kth persistent homology group of* (X, f) *at* (u, v). If X satisfies some mild conditions [5] – which will be assumed to hold throughout this paper – this group is finitely generated: Its rank is called a *kth persistent Betti number of* (X, f), and is denoted by $\beta_f(u, v)$ (references to X and k are omitted for simplicity).

A simple and compact description for the kth persistent Betti numbers of (X, f) is provided by the corresponding persistence diagrams. Figure 1(b) and



(*a*) (*b*) (*c*)

Fig. 1. (*a*) A model from the TOSCA dataset [3] with the “height” filtering function color-coded (left), and the corresponding 0th (*b*) and 1st (*c*) persistence diagram

Figure 1(c) show respectively the 0th and the 1st persistence diagrams which are obtained from the 0th and the 1st persistent Betti numbers of a surface model – the space X – filtered by the height function (Fig. 1(a)).

As shown in Fig. 1, persistence diagrams can be seen as multi-sets of points lying in the half-plane $\Delta^+ = \{(u,v) \in \mathbb{R} \times \mathbb{R} : u < v\}$. For each point, the u-coordinate represents the *birth* – in terms of the values of the filtering function – of a topological feature (e.g., connected components in the case of Fig. 1(b), tunnels in the case of Fig. 1(c)), whereas the v-coordinate represents its *death*. In particular, the red line in Fig. 1(b) can be interpreted as a point at infinity, representing a connected component that *will never die*, i.e. its v-component is equal to $+\infty$. The distance of a point from the diagonal $\Delta : u = v$ represents the *lifespan* of the associated topological feature, which in turn reflects its importance: points far from the diagonal describe important or global features, i.e. the long-lived ones, whereas points close to the diagonal describe local information such as smaller details and noise. For example, consider the three red points in Fig. 1(b) which are farthest from Δ. Together with the red line, they reveal the existence of four meaningful features: The limbs of the gorilla (Fig. 1(b)) born at the four minima of the height function.

A persistence diagram can be formally defined via the notion of *multiplicity* [13,16]. In what follows, the symbol Δ^* denotes the set $\Delta^+ \cup \{(u,\infty) : u \in \mathbb{R}\}$. Since we assume that the homology degree and the topological space are fixed, we keep omitting any reference to k and X.

Definition 1 (Multiplicity). *The* multiplicity $\mu_f(u,v)$ *of* $(u,v) \in \Delta^+$ *is the finite, non-negative number given by*

$$\min_{\substack{\varepsilon>0 \\ u+\varepsilon<v-\varepsilon}} \beta_f(u+\varepsilon, v-\varepsilon) - \beta_f(u-\varepsilon, v-\varepsilon) - \beta_f(u+\varepsilon, v+\varepsilon) + \beta_f(u-\varepsilon, v+\varepsilon).$$

The multiplicity $\mu_f(u,\infty)$ *of* (u,∞) *is the finite, non-negative number given by*

$$\min_{\varepsilon>0,\, u+\varepsilon<v} \beta_f(u+\varepsilon, v) - \beta_f(u-\varepsilon, v).$$

Definition 2 (Persistence Diagram). *The persistence diagram* $Dgm(f)$ *is the multiset of all points* $p \in \Delta^*$ *such that* $\mu_f(p) > 0$*, counted with their multiplicity, union the points of* Δ*, counted with infinite multiplicity.*

Definition 2 implies that, given the persistent Betti numbers of f, β_f, it is possible to completely and uniquely determine $\mathrm{Dgm}(f)$. The converse is also true, as shown by the following result [16], also known as *k-triangle Lemma* [14].

Theorem 1 (Representation Theorem). *For every* $(\bar{u}, \bar{v}) \in \Delta^+$ *it holds that*

$$\beta_f(\bar{u}, \bar{v}) = \sum_{u \leq \bar{u},\, v > \bar{v}} \mu_f(u, v) + \sum_{u \leq \bar{u}} \mu_f(u, \infty).$$

2.1 The Multidimensional Setting

If the considered filtering function is vector-valued, i.e. $f : X \to \mathbb{R}^n$, providing the multidimensional analogue of persistent homology groups and Betti numbers is straightforward. For $u, v \in \mathbb{R}^n$, with $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, we say that $u \prec v$ (resp. $u \preceq v$, $u \succ v$) iff $u_i < v_i$ (resp. $u_i \leq v_i$, $u_i > v_i$) for every $i = 1, \ldots, n$. Given $u \prec v$, the *multidimensional kth persistent homology group* of (X, f) at (u, v) is defined as the image of the homomorphism $H_k(X_u) \to H_k(X_v)$ induced in homology by the inclusion of $H_k(X_u)$ into $H_k(X_v)$. Its rank, still denoted by $\beta_f(u, v)$, is called a *multidimensional persistent Betti number*.

What is not straightforward is to generalize Definition 1 and Definition 2. As a consequence, even the multidimensional counterpart of the Representation Theorem 1 cannot be directly deduced from its one-dimensional version. These are actually the main goals of the next section.

3 The Persistence Space of a Multi-parameter Filtration

In this section we present the main theoretical results of the paper. Proving most of them is rather technical, and requires a number of intermediate results which, for the sake of clarity, we prefer not to recall here. For more details the reader is referred to the extended version of this work [9].

We start by observing that, in general, $\beta_f(u, v)$ can be seen as the number of homology classes of cycles "born" no later than u and "still alive" at v. Having this in mind, it is easy to figure out that, for u', u'' with $u' \preceq u''$, the homology classes of cycles "born" no later than u' are necessarily not larger in number than the ones "born" no later than u''. Indeed, it holds that

$$\beta_f(u'', v) - \beta_f(u', v) \geq 0 \tag{1}$$

for every $v \in \mathbb{R}^n$ with $u'' \prec v$. Analogously we can argue that, given $v', v'' \in \mathbb{R}^n$ with $u' \preceq u'' \preceq v' \preceq v''$, the number of homology classes born between u' and u'' and still alive at v' is certainly not smaller than the number of those still alive at v''. More formally,

$$\beta_f(u'', v') - \beta_f(u', v') \geq \beta_f(u'', v'') - \beta_f(u', v''). \tag{2}$$

Let us now set $\Delta_n^+ = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^n : u \prec v\}$. The next step is to recall some properties of the discontinuity points for β_f, which is considered here as a function taking each $(u,v) \in \Delta^+$ to $\beta_f(u,v)$. The next two propositions give some constraints on the presence of discontinuity points for β_f. For every $\bar{u} \in \mathbb{R}^n$, we denote by $\mathbb{R}^n_\pm(\bar{u})$ the subset of $\mathbb{R}^n$ given by $\{u \in \mathbb{R}^n : u \prec \bar{u} \ \vee \ u \succ \bar{u}\}$.

Proposition 1. *Let $p = (\bar{u}, \bar{v}) \in \Delta_n^+$. A real number $\varepsilon > 0$ exists, such that*

$$W_\varepsilon(p) = \{(u,v) \in \mathbb{R}^n_\pm(\bar{u}) \times \mathbb{R}^n_\pm(\bar{v}) : \|u - \bar{u}\|_\infty < \varepsilon, \, \|v - \bar{v}\|_\infty < \varepsilon\}$$

is an open subset of Δ_n^+, and does not contain any discontinuity point for β_f.

Proposition 2. *Let $\bar{u} \in \mathbb{R}^n$. A real number $\varepsilon > 0$ exists, such that*

$$V_\varepsilon(\bar{u}) = \{(u,v) \in \mathbb{R}^n_\pm(\bar{u}) \times \mathbb{R}^n : \|u - \bar{u}\|_\infty < \varepsilon, \, v_i > \frac{1}{\varepsilon}, \, i = 1, \ldots, n\}$$

is an open subset of Δ_n^+, and does not contain any discontinuity point for β_f.

We can now introduce the multidimensional analogue of Definition 1. For every $(u,v) \in \Delta_n^+$ and $e \in \mathbb{R}^n$ with $e \succ 0$ and $u + e \prec v - e$, we consider the number

$$\mu_f^e(u,v) = \beta_f(u+e, v-e) \ - \ \beta_f(u-e, v-e) + \\ - \ \beta_f(u+e, v+e) + \ \beta_f(u-e, v+e). \tag{3}$$

Since we are assuming that the persistent homology groups of (X,f) are finitely generated, we have that $\mu_f^e(u,v)$ is an integer number, and by (2) it is non-negative. Once again by (2), if $\eta \in \mathbb{R}^n$ with $0 \prec e \preceq \eta$, then $\mu_f^e(u,v) \leq \mu_f^\eta(u,v)$, i.e. $\mu_f^e(u,v)$ is non-decreasing in e. Moreover, by Proposition 1 each term in the sum defining $\mu_f^e(u,v)$ is constant for every $e \succ 0$ in $\mathbb{R}^n$ with $\|e\|_\infty$ sufficiently close to 0. Similarly, the number

$$\beta_f(u+e, v) - \beta_f(u-e, v) \tag{4}$$

is certainly an integer number, non-negative by (1). It is also non-decreasing in e and non-increasing in v, as easily implied by (1) and (2), respectively. Moreover, by Proposition 2 each term in (4) is constant for every for $e \succ 0$ in $\mathbb{R}^n$ with $\|e\|_\infty$ sufficiently close to 0, and every $v \in \mathbb{R}^n$ with $v_i > \frac{1}{\|e\|_\infty}$ for all $i = 1, \ldots, n$. The previous remarks justify the following definition.

Definition 3 (Multiplicity). *The* multiplicity $\mu_f(u,v)$ *of $(u,v) \in \Delta_n^+$ is the finite, non-negative number defined by setting*

$$\mu_f(u,v) = \min_{\substack{e \succ 0 \\ u+e \prec v-e}} \mu_f^e(u,v). \tag{5}$$

The multiplicity $\mu_f(u,\infty)$ *of (u,∞) is the finite, non-negative number given by*

$$\mu_f(u,\infty) = \min_{e \succ 0, \, u+e \prec v} \beta_f(u+e, v) - \beta_f(u-e, v). \tag{6}$$

Having extended the notion of multiplicity to a multidimensional setting, the definition of persistence space is now completely analogous to the one of persistence diagram for a real-valued filtering function. Set $\Delta_n^* = \Delta_n^+ \cup \{(u, \infty) : u \in \mathbb{R}^n\}$ and $\Delta_n = \partial \Delta_n^+$.

Definition 4 (Persistence Space). *The persistence space $Spc(f)$ is the multiset of all points $p \in \Delta_n^*$ such that $\mu_f(p) > 0$, counted with their multiplicity, union the points of Δ_n, counted with infinite multiplicity.*

Persistence spaces can be reasonably thought as the analogue – in the case of a multi-parameter filtration – of persistence diagrams. This is due to a number of properties they have in common.

First, it is quite easy to see that, if $f : X \to \mathbb{R}^n$ and $n = 1$, then Definition 2 and Definition 4 coincide as a simple consequence of the equivalence between Definition 1 and Definition 3.

Second, similarly to the one-dimensional case, a persistence space is completely and uniquely determined by the corresponding persistent Betti numbers. Moreover, even in the multi-parameter situation the converse is true as well, since it is possible to prove the following Multidimensional Representation Theorem. In what follows, $\langle e \rangle$ denotes the line in $\mathbb{R}^n$ spanned by e.

Theorem 2 (Multidimensional Representation Theorem). *Let $(\bar{u}, \bar{v}) \in \Delta_n^+$. For every $e \in \mathbb{R}^n$ with $e \succ 0$, it holds that*

$$\beta_f(\bar{u}, \bar{v}) = \sum_{\substack{u \preceq \bar{u},\, v \succ \bar{v} \\ \bar{u} - u,\, v - \bar{v} \in \langle e \rangle}} \mu_f(u, v) + \sum_{\substack{u \preceq \bar{u} \\ \bar{u} - u \in \langle e \rangle}} \mu_f(u, \infty). \tag{7}$$

A further analogy between persistence diagrams and persistence spaces concerns points with positive multiplicity. In both cases, such points can be characterized via the notion of *homological critical value*, introduced in [12] for real-valued functions and in [7] for vector-valued functions.

Definition 5 (Homological critical value). *We say that $u \in \mathbb{R}^n$ is a* homological critical value *of $f : X \to \mathbb{R}^n$ if, for every sufficiently small real value $\varepsilon > 0$, there exist $u', u'' \in \mathbb{R}^n$ such that $u' \prec u \prec u''$, $\|u' - u\|_\infty \le \varepsilon$, $\|u'' - u\|_\infty \le \varepsilon$, and the homomorphism $H_k(X_u) \to H_k(X_v)$ induced by inclusion is not an isomorphism for some integer k.*

It is well known that the coordinates of points in a persistence diagram are homological critical (real) values [12]. The same holds for the points of a persistence space. In fact, the following result can be proved.

Theorem 3. *Let p be a point of $Spc(f)$, with $\mu_f(p) > 0$. If $p = (u, v)$ then both u and v are homological critical values for f. If $p = (u, \infty)$ then u is a homological critical value for f.*

4 Visualization of Multidimensional Persistence

One of the main applications of persistence is in visualization and data analysis, mainly due to the fact that the persistence diagram allows us to visualize the appearance and disappearance of topological features in a filtration in a way that is resistant to noise. This is especially important in application areas, where data often come from noisy measurements.

In this section we address the problem of visualizing appearance and disappearance of topological features in a multi-filtration. Our main tool is the persistence space introduced in the previous section.

The idea underlying our visualization method is motivated by the following observations. Recall that, by definition, the persistence space consists of points of $\Delta_n^* = \Delta_n^+ \cup \Delta_n$, where n is the number of parameters in the filtration. In particular, these points are exactly those with a positive multiplicity. For simplicity we can think that this multiplicity is exactly equal to 1, since higher values for multiplicity correspond to non-generic situations. Moreover, intuition suggests that points are arranged on patches of $2(n-1)$-dimensional manifolds that intersect each other. For example, for $n = 1$, we have exactly a set of isolated points in $\mathbb{R}^2$, for $n = 2$ we have patches of 2-dimensional surfaces in $\mathbb{R}^4$. In general, since $\Delta_n^* \subseteq \mathbb{R}^n \times \mathbb{R}^n$, in order to visualize these points, we project them into a lower dimensional space.

Since each point of the persistence space captures the birth and death of a topological feature along the filtration, how can we visualize its life-time or persistence? In the one-dimensional setting, i.e. for persistence diagrams, this is achieved simply by considering the distance from the diagonal: the larger the distance, the more persistent the feature. In the multidimensional setting, persistence can analogously be defined as the distance from Δ_n. However, in the visualization process, when we project onto a lower dimensional space, distances are deformed. Therefore, our idea is to color-code points in the projection according to the persistence of the topological feature they represent.

The result of our method is illustrated in Figure 4 for the case of 1-homology of triangular meshes of a cube and a sphere, respectively. In both cases the considered filtering function on the vertices of the mesh is 2-dimensional and has the first component equal to $|x|$ (left, color-coded) and the second component equal to $|y|$ (center, color-coded), and it is interpolated on the other faces. The corresponding persistence spaces are visualized on the right. The darker a point, the more persistent the feature it represents. The projection from $\mathbb{R}^4$ to $\mathbb{R}^3$ is obtained using an implementation of Sammon's algorithm.

Having given the main idea of the method, we now explain it step-by-step for a filtering function $f : X \to \mathbb{R}^n$. Basically, the method consists of the following steps:

Step 1: Compute a sample of the points $p = (u, v)$ of the persistence space such that $u \preceq v$.

Step 2: Compute the persistence of each such point (u, v) as $\|v - u\|_\infty$, and the reciprocal distance – in the max-norm – between these points in $\mathbb{R}^{2n}$.

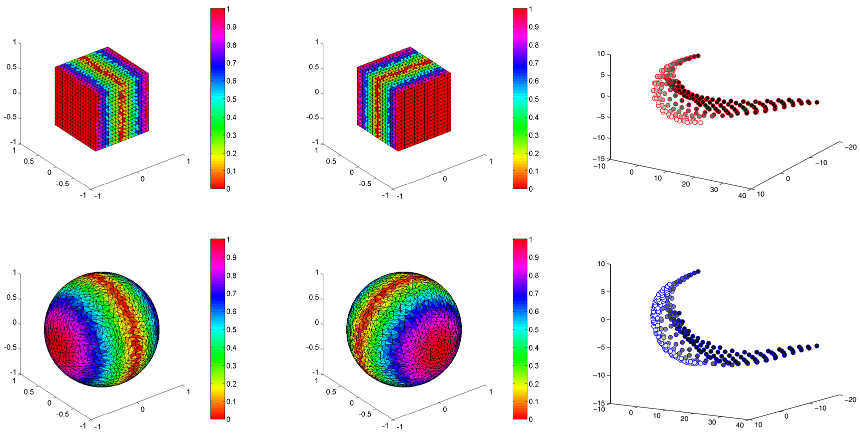


Fig. 2. Triangular meshes of a cube and a sphere, endowed with the filtering function $|x|$ (left), $|y|$ (center) and a visualization of the 1-homology persistence space for the 2-dimensional function $(|x|, |y|)$ (right)

Step 3: Project these points onto a lower dimensional space by an algorithm that tries to preserve the structure of inter-point distances computed in Step 2 in the lower-dimension projection (e.g., Sammon's algorithm).

Step 4: Plot points coloring them according to the persistence computed in Step 2.

Computations in Step 1 can be accomplished as follows, using the one-dimensional reduction described in [4]. Intuitively, the idea is to consider the set of the so-called *admissible lines.* A line parameterized by s with equation $u = sm + b$ is admissible if $b = (b_1, \ldots, b_n)$ is such that $\sum_{i=1}^{n} b_i = 0$, and $m = (m_1, \ldots, m_n)$ is such that $m_i > 0$ for each i and $\sum_{i=1}^{n} m_i^2 = 1$. The filtration obtained by sweeping each such line correspond to a persistence diagram: Gluing together all diagrams gives us the persistence space. It follows that, by taking a finite set of admissible lines, we get an approximation of a persistence space. More in details:

- Chose k admissible lines $L_1, \ldots, L_k$ in $\mathbb{R}^n$.
- For $h = 1, \ldots k$, consider the line L_h. Assume its parametric equation is $u = sm^h + b^h$. Compute the points $(s,t) \in \mathbb{R}^2$ of the persistence diagram of the one-dimensional filtration induced by the real-valued function $F^{L_h}(x) = m_* \cdot \max_{i=1,\ldots,n} \left\{ \frac{f_i(x) - b_i^h}{m_i^h} \right\}$, with $m_* = \min_i m_i$.
- For each h and for each point (s,t) in the persistence diagram of F^{L_h}, compute the corresponding point $(u,v) \in \mathbb{R}^n \times \mathbb{R}^n$ of the persistence space of f by the formulas $u = sm^h + b$, $v = tm^h + b$.

In particular, the above procedure is justified by the correspondence existing between the points in $\mathrm{Spc}(f)$ and the ones in the persistence diagrams $\mathrm{Dgm}(F^{L_h})$ [9, Lemma 3.16].

In the next section we shall see that persistence spaces enjoy stability with respect to perturbations of f. We observe that stability is inherited by the projections obtained according to the proposed method, provided that the point-projection procedure in step 3 preserves the structure of inter-point distances.

5 Stable Comparison of Persistence Spaces

A crucial property for applications is the stability of persistence spaces, which means that close functions in the sup-norm should have close persistence spaces in a natural metric.

The reason why our method is stable intuitively is that each admissible line $L : u = sm + b$ can be associated with a 1-parameter filtration: By sweeping L, the birth and the death of a topological event occur according to the values of the function F^L previously defined, whose sublevel sets represent the subspaces of the filtration. It follows that each admissible line L can be associated to a persistence diagram $\mathrm{Dgm}(F^L)$. Moreover, it happens that the collection of persistence diagrams associated with the set $\mathcal{L}_n$ of all possible admissible lines turns out to be a complete descriptor of the persistent Betti numbers β_f. Therefore, given another filtering function $g : X \to \mathbb{R}^n$, it is possible to define the *multidimensional matching distance* D_{match} between β_f and β_g by comparing line by line the associated persistence diagrams via the bottleneck distance d_B [4]:

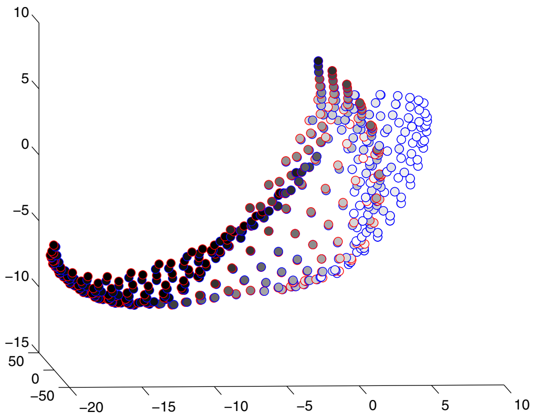


Fig. 3. Superimposition of the 1-homology persistence spaces of the cube (red points) and the sphere (blue points) of Figure 4

$$D_{match}(\beta_f, \beta_g) = \sup_{L \in \mathcal{L}_n} d_B(\mathrm{Dgm}(F^L), \mathrm{Dgm}(G^L)).$$

Such a distance is stable with respect to perturbations of the considered vector-valued filtering function: In [8] it has been proved that

$$D_{match}(\beta_f, \beta_g) \le \max_{x \in X} \|f(x) - g(x)\|_\infty.$$

Now, we can conclude that, by the Multidimensional Representation Theorem 2, persistence spaces inherit stability from multidimensional persistent Betti numbers.

In Figure 5 stability of persistence spaces is illustrated by displaying the 1-homology persistence spaces of the cube (red points) and the sphere (blue points) of Figure 4 (superimposed). It is clearly visible that points with higher persistence (dark colored) are almost overlapping, while differences are appreciable for points with lower persistence (light colored).

6 Conclusions

We have shown that persistence spaces provide a representation of the topological properties of vector-valued functions, and have described how persistence spaces can be visualized. Finally, we have explained how stability of multidimensional persistent Betti numbers implies stability of persistence spaces.

Some questions remain unanswered. Is it possible to further improve this representation by using only a finite set of points, at least in simple cases such as for functions interpolated on vertices of simplicial complexes? Is it possible to explicitly define a bottleneck distance between persistence spaces?

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