Critical Connectedness of Thin Arithmetical Discrete Planes

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Abstract. The critical thickness of an arithmetical discrete plane refers to the infimum thickness that preserves its 2-connectedness. This infimum thickness can be computed thanks to a multidimensional continued fraction algorithm, namely the fully subtractive algorithm. We provide a characterization of the discrete planes with critical thickness that contain the origin and that are 2-connected.

1 Introduction

This paper studies the connectedness of thin arithmetic discrete planes in the three-dimensional space. We focus on the notion of 2-connectedness, and we restrict ourselves to planes with zero intercept that have critical thickness, that is, planes whose thickness is the infimum of the set of all the $\omega \in \mathbb{R}_+$ such that the plane of thickness ω is 2-connected (see Definitions 2.1 and 2.5). Let us recall that standard arithmetic discrete planes are known to be 2-connected, whereas naive ones are too thin to be 2-connected. We thus consider planes with a thickness that lies between the naive and the standard cases.

The problem of the computation of the critical thickness was raised in [6]. It has been answered in [9,10,7], with an algorithm that can be expressed in terms of a multidimensional continued fraction algorithm, namely the so-called fully subtractive algorithm. This algorithm explicitly yields the value of the critical thickness when it halts, and this value can be computed when the algorithm enter a loop (possibly infinite). Furthermore, the set $\mathcal{F}_3$ of vectors for which the algorithm enters an infinite loop has Lebesgue measure zero, as a consequence of results of [12] in the context of a percolation model defined by rotations on the unit circle. Our main result is that a discrete plane with intercept zero and critical thickness is 2-connected when its normal vector belongs to $\mathcal{F}_3$. We also prove that vectors in $\mathcal{F}_3$ are the only ones for which critical arithmetical discrete planes are 2-connected.

Our methods rely on a combinatorial generation method based on the notion of susbtitution for the planes under study (see Section 2.3 for more details). In Section 3 we construct a sequence of finite patterns $(\mathbf{T}_n)_n$ of the planes with critical thickness, and we prove that these patterns are all 2-connected when

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the parameters belong to $\mathcal{F}_3$. We then relate these finite patterns with thinner patterns $\mathbf{P}_n$ that belong to the naive discrete plane with same parameters. These pattern are generated in terms of a geometric interpretation of the fully subtractive algorithm. Next, the thinner patterns $\mathbf{P}_n$ are proved in Section 4 to generate the full naive plane. This yields the 2-connectedness of the critical plane (see Section 5). In other words, we use the fact that the underlying naive plane provides a relatively dense skeleton of the critical plane.

2 Basic Notions and Notation

2.1 Discrete and Stepped Planes

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the canonical basis of $\mathbb{R}^3$, and let $\langle \cdot, \cdot \rangle$ stand for the usual scalar product on $\mathbb{R}^3$. Given $\mathbf{v} \in \mathbb{R}^3$ and $i \in \{1, 2, 3\}$, we let $\mathbf{v}_i = \langle \mathbf{v}, \mathbf{e}_i \rangle$ denote the i^{th}-coordinate of $\mathbf{v}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Definition 2.1 (Arithmetical discrete plane [13,1]). *Given* $\mathbf{v} \in \mathbb{R}^3_+$*,* $\mu \in \mathbb{R}$ *and* $\omega \in \mathbb{R}_+$*, the* arithmetical discrete plane *with* normal vector $\mathbf{v}$*,* intercept μ*, and* thickness ω *is the set* $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ *defined as follows:*

$$\mathfrak{P}(\mathbf{v}, \mu, \omega) = \left\{ \mathbf{x} \in \mathbb{Z}^3 : 0 \leqslant \langle \mathbf{x}, \mathbf{v} \rangle + \mu < \omega \right\}.$$

If $\omega = \|\mathbf{v}\|_\infty = \max\{|\mathbf{v}_1|, |\mathbf{v}_2|, |\mathbf{v}_3|\}$ *(resp.* $\omega = \|\mathbf{v}\|_1 = |\mathbf{v}_1| + |\mathbf{v}_2| + |\mathbf{v}_3|$*), then* $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ *is said to be a* naive arithmetical discrete *plane (resp.* standard arithmetical discrete plane*).*

Even if any finite subset of a digitized plane can be represented as a subset of an arithmetical discrete plane with integer parameters, we do not restrict ourselves here with finite sets, and we consider general arithmetical discrete with possibly non-integer parameters.

We will also deal with another discrete approximation of Euclidean planes, namely *stepped planes*. They can be considered as a more geometrical version, in the sense that they consist of *unit faces* instead of just points of $\mathbb{Z}^3$.

Definition 2.2 (Unit faces, stepped planes). *A* unit face $[\mathbf{x}, i]^\star$ *is defined as:*

$$[\mathbf{x}, 1]^\star = \{\mathbf{x} + \lambda \mathbf{e}_2 + \mu \mathbf{e}_3 : \lambda, \mu \in [0, 1]\} =$$ 
$$[\mathbf{x}, 2]^\star = \{\mathbf{x} + \lambda \mathbf{e}_1 + \mu \mathbf{e}_3 : \lambda, \mu \in [0, 1]\} =$$ 
$$[\mathbf{x}, 3]^\star = \{\mathbf{x} + \lambda \mathbf{e}_1 + \mu \mathbf{e}_2 : \lambda, \mu \in [0, 1]\} =$$ 

where $i \in \{1, 2, 3\}$ *is the* type *of* $[\mathbf{x}, i]^\star$*, and* $\mathbf{x} \in \mathbb{Z}^3$ *is the* distinguished vertex *of* $[\mathbf{x}, i]^\star$*. Let* $\mathbf{v} \in \mathbb{R}^3_+$*. The* stepped plane $\mathbf{\Gamma}_\mathbf{v}$ *is the union of unit faces defined by:*

$$\mathbf{\Gamma}_\mathbf{v} = \{[\mathbf{x}, i]^\star : 0 \leqslant \langle \mathbf{x}, \mathbf{v} \rangle < \langle \mathbf{e}_i, \mathbf{v} \rangle\}.$$

The notation $\mathbf{x} + [\mathbf{y}, i]^\star$ stands for the unit face $[\mathbf{x} + \mathbf{y}, i]^\star$.

Remark 2.3. The set of distinguished vertices of $\mathbf{\Gamma}_\mathbf{v}$ is the naive arithmetical discrete plane $\mathfrak{P}(\mathbf{v}, 0, \|\mathbf{v}\|_\infty)$, whereas the set of all vertices of the faces of $\mathbf{\Gamma}_\mathbf{v}$ is the standard arithmetical discrete plane $\mathfrak{P}(\mathbf{v}, 0, \|\mathbf{v}\|_1)$.

2.2 Connecting Thickness and the Fully Subtractive Algorithm

Definition 2.4 (Adjacency, connectedness). *Two distinct elements* $\mathbf{x}$ *and* $\mathbf{y}$ *of* $\mathbb{Z}^3$ *are* 2-adjacent *if* $\|\mathbf{x}-\mathbf{y}\|_1 = 1$.

A subset $A \subseteq \mathbb{Z}^3$ *is 2-connected if for every* $\mathbf{x}, \mathbf{y} \in A$, *there exist* $\mathbf{x}_1, \ldots, \mathbf{x}_n \in A$ *such that* $\mathbf{x}_i$ *and* $\mathbf{x}_{i+1}$ *are 2-adjacent for all* $i \in \{1, \ldots, n-1\}$, *with* $\mathbf{x}_1 = \mathbf{x}$ *and* $\mathbf{x}_n = \mathbf{y}$.

Definition 2.5 (Connecting thickness [9]). *Given* $\mathbf{v} \in \mathbb{R}^3_+$ *and* $\mu \in \mathbb{R}$, *the* connecting thickness $\Omega(\mathbf{v}, \mu)$ *is defined by:*

$$\Omega(\mathbf{v}, \mu) = \inf \left\{\omega \in \mathbb{R}_+ : \mathfrak{P}(\mathbf{v}, \mu, \omega) \textit{ is 2-connected}\right\}.$$

In order to compute $\Omega(\mathbf{v}, \mu)$ we may assume without loss of generality that $0 \leqslant \mathbf{v}_1 \leqslant \mathbf{v}_2 \leqslant \mathbf{v}_3$. We thus restrict ourselves in the sequel to the set of parameters $\mathcal{O}_3^+ = \left\{\mathbf{v} \in \mathbb{R}^3 : 0 \leqslant \mathbf{v}_1 \leqslant \mathbf{v}_2 \leqslant \mathbf{v}_3\right\}$.

A first approximation of $\Omega(\mathbf{v}, \mu)$ is provided by $\|\mathbf{v}\|_\infty \leqslant \Omega(\mathbf{v}, \mu) \leqslant \|\mathbf{v}\|_1$ (see Corollary 10 of [2]). Given $\mathbf{v} \in \mathcal{O}_3^+$ and $\varepsilon > 0$, it follows from the definition of $\Omega(\mathbf{v}, \mu)$ that $\mathfrak{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}) - \varepsilon)$ is not 2-connected and that $\mathfrak{P}(\mathbf{v}, \mu, \Omega(\mathbf{v}) + \varepsilon)$ is 2-connected.

It is shown in [10] how to compute $\Omega(\mathbf{v}, \mu)$ from the expansion of the vector $\mathbf{v}$ according to the *ordered fully subtractive algorithm* $\mathbf{F} : \mathcal{O}_3^+ \to \mathcal{O}_3^+$ defined by:

$$\mathbf{F}(\mathbf{v}) = \begin{cases} (\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1) & \text{if } \mathbf{v}_1 \leqslant \mathbf{v}_2 - \mathbf{v}_1 \leqslant \mathbf{v}_3 - \mathbf{v}_1 \\ (\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1) & \text{if } \mathbf{v}_2 - \mathbf{v}_1 \leqslant \mathbf{v}_1 \leqslant \mathbf{v}_3 - \mathbf{v}_1 \\ (\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1, \mathbf{v}_1) & \text{if } \mathbf{v}_2 - \mathbf{v}_1 \leqslant \mathbf{v}_3 - \mathbf{v}_1 \leqslant \mathbf{v}_1. \end{cases} \quad (1)$$

Theorem 2.6 ([10]). *Let* $\mathbf{v} \in \mathcal{O}_3^+$ *and* $\mu \in \mathbb{R}$. *The arithmetical discrete plane* $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ *is 2-connected if and only if so is* $\mathfrak{P}(\mathbf{F}(\mathbf{v}), \mu, \omega - \mathbf{v}_1)$. *Consequently,* $\Omega(\mathbf{v}, \mu) = \Omega(\mathbf{F}(\mathbf{v}), \mu) + \mathbf{v}_1$.

Let us fix $\mathbf{v} \in \mathcal{O}_3^+$, and consider the expansion of $\mathbf{v}$ according to the ordered fully subtractive algorithm $\mathbf{F}$ (Eq. 1). We recover a possibly infinite sequence of matrices $(\mathbf{M}_n)_{n \in \mathbb{N}}$ with values in $\{\mathbf{M}_1^{\mathrm{FS}}, \mathbf{M}_2^{\mathrm{FS}}, \mathbf{M}_3^{\mathrm{FS}}\}$ where

$$\mathbf{M}_1^{\mathrm{FS}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{M}_2^{\mathrm{FS}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{M}_3^{\mathrm{FS}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and a sequence of nonzero vectors $(\mathbf{v}^{(n)})_{n \in \mathbb{N}}$ with nonnegative entries such that, for all $n \in \mathbb{N}^\star$, $\mathbf{v} = \mathbf{M}_1 \ldots \mathbf{M}_n \cdot \mathbf{v}^{(n)}$, with $\mathbf{v}^{(n)} = \mathbf{F}^n(\mathbf{v})$. We set $\mathbf{v}^{(0)} = \mathbf{v}$.

The set of parameters $\mathbf{v}$ for which $\mathbf{v}_1^{(n)} + \mathbf{v}_2^{(n)} > \mathbf{v}_3^{(n)}$ for all n has been shown in [11] to play here a particular role: indeed $\lim_{n \to \infty} \mathbf{v}^{(n)} = \mathbf{0}$ if and only if $\mathbf{v}_1^{(n)} + \mathbf{v}_2^{(n)} \geqslant \mathbf{v}_3^{(n)}$ for all n. We thus introduce the following notation:

$$\mathcal{F}_3 = \left\{\mathbf{v} \in \mathcal{O}_3^+ : \mathbf{v}_1^{(n)} + \mathbf{v}_2^{(n)} > \mathbf{v}_3^{(n)}, \text{ for all } n \in \mathbb{N}\right\}.$$

Remark 2.7 ([14]). If $\mathbf{v} \in \mathcal{F}_3$ then $\dim_{\mathbb{Q}}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = 3$.

Let us illustrate the interest of working with this set of parameters.

Proposition 2.8. *If* $\mathbf{v} \in \mathcal{F}_3$*, then* $\Omega(\mathbf{v}, \mu) = \sum_{n=0}^{\infty} \mathbf{v}_1^{(n)} = \frac{\|\mathbf{v}\|_1}{2}$.

Proof. According to Theorem 2.6, we have: $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 - 2\Omega(\mathbf{v}, \mu) = \mathbf{v}_1^{(i)} + \mathbf{v}_2^{(i)} + \mathbf{v}_3^{(i)} - 2\Omega(\mathbf{v}^{(i)}, \mu)$ for all $i \in \{1, \ldots, n\}$. Since $\Omega(\mathbf{v}^{(n)}, \mu) \leqslant \|\mathbf{v}^{(n)}\|_1$ and $\lim_{n\to\infty} \mathbf{v}^{(n)} = \mathbf{0}$, then $\lim_{n\to\infty} \Omega(\mathbf{v}^{(n)}, \mu) = 0$ and the result follows. □

In particular, if $\mathbf{v} \in \mathcal{F}_3$, then $\Omega(\mathbf{v}, \mu)$ does not depend on μ. Hence, from now on, we consider only $\mathbf{v} \in \mathcal{F}_3$ and we refer to $\Omega(\mathbf{v}, \mu)$ as $\Omega(\mathbf{v})$.

2.3 Substitutions and Dual Substitutions

Let $\mathcal{A} = \{1, 2, 3\}$ be a finite alphabet and $\mathcal{A}^\star$ be the set of finite words over $\mathcal{A}$.

Definition 2.9 (Substitution). *A substitution over* $\mathcal{A}$ *is a morphism of the free monoid* $\mathcal{A}^\star$*,* i.e., *a function* $\sigma : \mathcal{A}^\star \to \mathcal{A}^\star$ *with* $\sigma(uv) = \sigma(u)\sigma(v)$ *for all words* $u, v \in \mathcal{A}^\star$*.*

Given a substitution σ over $\mathcal{A}$, the *incidence matrix* $\mathbf{M}_\sigma$ of σ is the square matrix of size 3×3 defined by $\mathbf{M}_\sigma = (m_{ij})$, where $m_{i,j}$ is the number of occurrences of the letter i in $\sigma(j)$. A substitution σ is *unimodular* if $\det \mathbf{M}_\sigma = \pm 1$.

Definition 2.10 (Dual substitution [3]). *Let* $\sigma : \{1, 2, 3\}^\star \longrightarrow \{1, 2, 3\}^\star$ *be a unimodular substitution. The* dual substitution $\mathbf{E}_1^\star(\sigma)$ *is defined as*

$$\mathbf{E}_1^\star(\sigma)([\mathbf{x}, i]^\star) = \mathbf{M}_\sigma^{-1}\mathbf{x} + \bigcup_{(p,j,s)\in\mathcal{A}^\star\times\mathcal{A}\times\mathcal{A}^\star \;:\; \sigma(j)=pis} [\mathbf{M}_\sigma^{-1}\ell(s), j]^\star,$$

where $\ell : w \mapsto (|w|_1, |w|_2, |w|_3) \in \mathbb{Z}^3$ *is the* Parikh map *counting the occurrences of each letter in a word* w*. We extend the above definition to any union of unit faces:* $\mathbf{E}_1^\star(\sigma)(P_1 \cup P_2) = \mathbf{E}_1^\star(\sigma)(P_1) \cup \mathbf{E}_1^\star(\sigma)(P_2)$*.*

Note that $\mathbf{E}_1^\star(\sigma \circ \sigma') = \mathbf{E}_1^\star(\sigma') \circ \mathbf{E}_1^\star(\sigma)$ for unimodular σ and σ' (see [3]).

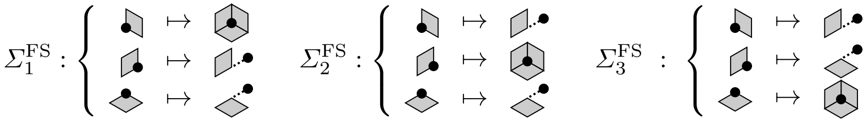
Proposition 2.11 ([3,8]). *We have* $\mathbf{E}_1^\star(\sigma)(\mathbf{\Gamma}_\mathbf{v}) = \mathbf{\Gamma}_{{}^t\mathbf{M}_\sigma \mathbf{v}}$ *for every stepped plane* $\mathbf{\Gamma}_\mathbf{v}$ *and unimodular substitution* σ*. Furthermore, the images of two distinct faces of* $\mathbf{\Gamma}_\mathbf{v}$ *have no common unit face.*

We now introduce the substitutions associated with the ordered fully subtractive algorithm, which will be our main tool. Let

$$\sigma_1^{\mathrm{FS}} = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{cases} \quad \sigma_2^{\mathrm{FS}} = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 12 \\ 3 \mapsto 32 \end{cases} \quad \sigma_3^{\mathrm{FS}} = \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 13 \\ 3 \mapsto 23 \end{cases}$$

The matrices occurring in the expansion of $\mathbf{v}$ according to the ordered fully subtractive algorithm are the transposes of the matrices of incidence of the σ_i^{FS}, that is, $\mathbf{M}_{\sigma_i^{\mathrm{FS}}} = {}^t\mathbf{M}_i^{\mathrm{FS}}$ for $i \in \{1, 2, 3\}$.

We denote by Σ_i^{FS} the three dual substitutions $\mathbf{E}_1^\star(\sigma_i^{\mathrm{FS}})$ for $i \in \{1,2,3\}$. They can be represented as follows, where the black dot respectively stands for the distinguished vector of a face and its image.



3 Properties of the Patterns $\mathbf{T}_n$

Let $(\mathbf{M}_n)_{n\in\mathbb{N}^\star}$ be the sequence of matrices with values in $\{\mathbf{M}_1^{\mathrm{FS}}, \mathbf{M}_2^{\mathrm{FS}}, \mathbf{M}_3^{\mathrm{FS}}\}$ such that $\mathbf{v} = \mathbf{M}_1 \cdots \mathbf{M}_n \cdot \mathbf{F}^n(\mathbf{v}) = \mathbf{M}_1 \cdots \mathbf{M}_n \cdot \mathbf{v}_n$ for all n. Let $(\sigma_n)_{n\in\mathbb{N}^\star}$ be the sequence of corresponding substitutions with values in $\{\sigma_1^{\mathrm{FS}}, \sigma_2^{\mathrm{FS}}, \sigma_3^{\mathrm{FS}}\}$, such that $\mathbf{M}_n = {}^{\mathrm{t}}\mathbf{M}_{\sigma_n}$ for all n.

Definition 3.1 (Generation by translations). *Let $(\mathbf{T}_n)_{n\in\mathbb{N}}$ be the sequence of subsets of $\mathbb{Z}^3$ defined as follows for all nonnegative integer n:*

$$\mathbf{T}_0 = \{\mathbf{0}\}, \quad \mathbf{T}_1 = \{\mathbf{0}, \mathbf{e}_1\}, \quad \mathbf{T}_{n+1} = \mathbf{T}_n \cup \left(\mathbf{T}_n + {}^{\mathrm{t}}(\mathbf{M}_1 \ldots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1\right).$$

Note that the second initial condition $\mathbf{T}_1 = \{\mathbf{0}, \mathbf{e}_1\}$ is consistent with the usual convention that an empty product of matrices is equal to the identity matrix.

Proposition 3.2. *We have $\cup_{n=0}^{\infty} \mathbf{T}_n \subseteq \mathfrak{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$.*

Proof (Sketch). We prove by induction that $\langle \mathbf{x}, \mathbf{v} \rangle < \sum_{i=0}^{n} \mathbf{v}_1^{(i)}$ for all $\mathbf{x} \in \mathbf{T}_n$, by noticing that $\langle {}^{\mathrm{t}}(\mathbf{M}_1 \ldots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1, \mathbf{v} \rangle = \langle {}^{\mathrm{t}}(\mathbf{M}_1 \ldots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1, \mathbf{M}_1 \ldots \mathbf{M}_n \cdot \mathbf{v}^{(n)} \rangle = \langle \mathbf{e}_1, \mathbf{v}^{(n)} \rangle = \mathbf{v}_1^{(n)}$. □

Proposition 3.3. *For all $n \in \mathbb{N}$, the set $\mathbf{T}_n$ is 2-connected.*

Proof (Sketch). With the same arguments as in proof of Proposition 3.2, and by using Remark 2.7, we first get by induction that, for all $n \in \mathbb{N}^\star$:

$$\mathbf{T}_n = \left\{ \mathbf{x} \in \mathbb{Z}^3 : \langle \mathbf{x}, \mathbf{v} \rangle = \sum_{i=0}^{n-1} \varepsilon_i \mathbf{v}_1^{(i)} \text{with } \varepsilon_i \in \{0,1\} \text{ for all } i \right\}.$$

Now, for all $n \in \mathbb{N}$, let $\mathbf{x}_n \in \mathbf{T}_n$ such that $\langle \mathbf{x}_n, \mathbf{v} \rangle = \sum_{i=0}^{n-1} \mathbf{v}_1^{(i)}$ (we set $\mathbf{x}_0 = \mathbf{0}$). Let us prove by induction the following property: for all $n \in \mathbb{N}^\star$ there exists $i_n \in \{1,2,3\}$ such that $\mathbf{x}_n - \mathbf{e}_{i_n} \in \mathbf{T}_{n-1}$. This property implies that $\mathbf{x}_n$ is 2-adjacent to $\mathbf{T}_{n-1}$, which implies the 2-connectedness of $\mathbf{T}_n$.

The induction property is true for $n = 1$ with $\mathbf{x}_1 = \mathbf{e}_1$. Let us now assume that the induction hypothesis hold for $n \geqslant 1$. Let $u_1 \cdots u_n \in \{1,2,3\}^{\mathbb{N}^\star}$ be such that $\mathbf{M}^{\mathrm{FS}}_{u_1} \cdots \mathbf{M}^{\mathrm{FS}}_{u_n} \mathbf{v}^{(n)} = \mathbf{v}$. We have $\langle \mathbf{x}_{n+1}, \mathbf{v} \rangle = \langle \mathbf{x}_n, \mathbf{v} \rangle + \mathbf{v}_1^{(n)}$, and by definition of the fully subtractive algorithm (see Eq. 1):

$$\mathbf{v}_1^{(n)} = \begin{cases} \mathbf{v}_1^{(n-1)}, & \text{if } u_n = 1 \\ \mathbf{v}_2^{(n-1)} - \mathbf{v}_1^{(n-1)}, & \text{if } u_n \in \{2,3\}. \end{cases}$$

We distinghuish several cases according to the values taken by $u_1 \cdots u_n$.

Case 1. If $u_n = 1$, then, $\langle \mathbf{x}_{n+1}, \mathbf{v} \rangle = \langle \mathbf{x}_n, \mathbf{v} \rangle + \mathbf{v}_1^{(n-1)}$, and

$$\langle \mathbf{x}_{n+1} - \mathbf{e}_{i_n}, \mathbf{v} \rangle = \underbrace{\langle \mathbf{x}_n - \mathbf{e}_{i_n}}_{\in \mathbf{T}_{n-1}}, \mathbf{v} \rangle + \mathbf{v}_1^{(n-1)} = \sum_{i=1}^{n-2} \varepsilon_i \mathbf{v}_1^{(i)} + \mathbf{v}_1^{(n-1)},$$

where $\varepsilon_i \in \{0,1\}$ for $1 \leqslant i \leqslant n-2$, which implies that $\mathbf{x}_{n+1} - \mathbf{e}_{i_n} \in \mathbf{T}_n$, so taking $i_{n+1} = i_n$ yields the desired result.

Case 2. If $u_n \in \{2,3\}$ and $u_1 \cdots u_{n-1} = 1^k$, then

$$\begin{aligned} \langle \mathbf{x}_{n+1}, \mathbf{v} \rangle &= \langle \mathbf{x}_n, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1)} - \mathbf{v}_1^{(n-1)} = \langle \mathbf{x}_{n-1}, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1)} \\ &= \langle \mathbf{x}_{n-2}, \mathbf{v} \rangle + \mathbf{v}_2^{(n-2)} = \cdots = \langle \mathbf{x}_{n-1-k}, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1-k)} = \mathbf{v}_2^{(0)}, \end{aligned}$$

which implies that $\mathbf{x}_{n+1} - \mathbf{e}_2 \in \mathbf{T}_n$.

Case 3. If $u_n \in \{2,3\}$ and $u_1 \cdots u_{n-1} = \cdots 21^k$ with $0 \leqslant k \leqslant n-2$, then

$$\begin{aligned} \langle \mathbf{x}_{n+1}, \mathbf{v} \rangle &= \langle \mathbf{x}_n, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1)} - \mathbf{v}_1^{(n-1)} = \langle \mathbf{x}_{n-1}, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1)} \\ &= \langle \mathbf{x}_{n-1-k}, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1-k)} = \langle \mathbf{x}_{n-1-k}, \mathbf{v} \rangle + \mathbf{v}_1^{(n-2-k)}, \end{aligned}$$

so $\mathbf{x}_{n+1} - \mathbf{e}_{i_{n-1-k}} \in \mathbf{T}_{n-1-k} \subseteq \mathbf{T}_n$.

Case 4. If $u_n \in \{2,3\}$ and $u_1 \cdots u_{n-1} = w31^k$ with $w \in \{1,2\}^\ell$ and $k \geqslant 0$, then

$$\begin{aligned} \langle \mathbf{x}_{n+1}, \mathbf{v} \rangle &= \langle \mathbf{x}_{n-1-k}, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1-k)} = \langle \mathbf{x}_{n-2-k}, \mathbf{v} \rangle + \mathbf{v}_3^{(n-2-k)} \\ &= \langle \mathbf{x}_{n-2-k-\ell}, \mathbf{v} \rangle + \mathbf{v}_3^{(n-2-k-\ell)} = \mathbf{v}_3^{(0)}, \end{aligned}$$

so $\mathbf{x}_{n+1} - \mathbf{e}_3 \in \mathbf{T}_n$.

Case 5. If $u_n \in \{2,3\}$ and $u_1 \cdots u_{n-1} = \cdots 3w31^k$ with $w \in \{1,2\}^\ell$, $k \geqslant 0$, then

$$\begin{aligned} \langle \mathbf{x}_{n+1}, \mathbf{v} \rangle &= \langle \mathbf{x}_{n-2-k-\ell}, \mathbf{v} \rangle + \mathbf{v}_3^{(n-2-k-\ell)} \\ &= \langle \mathbf{x}_{n-2-k-\ell}, \mathbf{v} \rangle + \mathbf{v}_1^{(n-3-k-\ell)} \end{aligned}$$

so $\mathbf{x}_{n+1} - \mathbf{e}_{i_{n-2-k-\ell}} \in \mathbf{T}_{n-2-k-\ell} \subseteq \mathbf{T}_n$. □

Definition 3.4 (Generation by dual substitutions). *Let* $\mathcal{U} = [\mathbf{0},1]^\star \cup [\mathbf{0},2]^\star \cup [\mathbf{0},3]^\star =$  *and let* $(\sigma_n)_{n \in \mathbb{N}^\star}$ *be the sequence of fully subtractive substitutions generated by* $\mathbf{v}$. *For* $n \in \mathbb{N}$, *let:*

– $P_n = \mathbf{E}_1^\star(\sigma_n \circ \cdots \circ \sigma_1)(\mathcal{U})$ *with* $P_0 = \mathcal{U}$;

– $\mathbf{P}_n$ *be the set of distinguished vertices of the set* P_n *of unit faces.*

Proposition 3.5. *For every* $n \in \mathbb{N}$*, we have* $\mathbf{P}_n \subseteq \mathbf{T}_n$*.*

Proof. We first remark that $\mathbf{E}_1^\star(\sigma_i^{\mathrm{FS}})(\mathcal{U}) = \mathcal{U} \cup [\mathbf{e}_1, 2]^\star \cup [\mathbf{e}_1, 3]^\star =$  for all $i \in \{1, 2, 3\}$. For $n \in \mathbb{N}$, we have

$$\begin{aligned} P_{n+1} &= \mathbf{E}_1^\star(\sigma_{n+1} \circ \cdots \circ \sigma_1)(\mathcal{U}) \\ &= \mathbf{E}_1^\star(\sigma_n \circ \cdots \circ \sigma_1) \circ \mathbf{E}_1^\star(\sigma_{n+1})(\mathcal{U}) \\ &= P_n \cup \mathbf{E}_1^\star(\sigma_n \circ \cdots \circ \sigma_1)([\mathbf{e}_1, 2]^\star \cup [\mathbf{e}_1, 3]^\star), \end{aligned}$$

which implies $P_n \subseteq P_{n+1}$. Since $[\mathbf{e}_1, 2]^\star \cup [\mathbf{e}_1, 3]^\star \subseteq \mathbf{e}_1 + \mathcal{U}$, we have $P_{n+1} \subseteq P_n \cup \mathbf{E}_1^\star(\sigma_n \circ \cdots \circ \sigma_1)(\mathbf{e}_1 + \mathcal{U})$. By Definition 2.10, we then have

$$\begin{aligned} \mathbf{E}_1^\star(\sigma_n \circ \cdots \circ \sigma_1)(\mathbf{e}_1 + \mathcal{U}) &= \mathbf{M}_{\sigma_n \circ \cdots \circ \sigma_1}^{-1} \cdot \mathbf{e}_1 + \mathbf{E}_1^\star(\sigma_n \circ \cdots \circ \sigma_1)(\mathcal{U}) \\ &= ({}^{\mathrm{t}}\mathbf{M}_n \cdots {}^{\mathrm{t}}\mathbf{M}_1)^{-1} \cdot \mathbf{e}_1 + P_n \\ &= {}^{\mathrm{t}}(\mathbf{M}_1 \cdots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1 + P_n, \end{aligned}$$

which proves $P_n \subseteq P_{n+1} \subseteq P_n \cup \left(P_n + {}^{\mathrm{t}}(\mathbf{M}_1 \cdots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1\right)$. The result now follows by induction. □

4 Generation of Naive Planes with Dual Substitutions

The aim of this section is to prove that the patterns $\mathbf{P}_n$ cover the naive plane, by showing that iterations of dual substitutions yield concentric annuli (see Definition 4.5) with increasing radius. The main result of this section is the following.

Proposition 4.1. *If* $\mathbf{v} \in \mathcal{F}_3$*, then* $\bigcup_{n=0}^{\infty} \mathbf{P}_n = \mathfrak{P}(\mathbf{v}, 0, \|\mathbf{v}\|_\infty)$*.*

The proof will be given at the end of Section 4.3. The remaining of this section is devoted to the development of specific tools used in this proof. Such tools have also been used in [5] to study other multidimensional continued fraction algorithms.

4.1 Covering Properties and Annuli

A *pattern* is a union of unit faces. In the rest of this section we will consider some sets of connected patterns ($\mathcal{L}$, $\mathcal{L}_{\mathrm{edge}}$ and $\mathcal{L}_{\mathrm{FS}}$) that will be needed in order to define (strong) coverings. The patterns contained in these sets are considered up to translation only, as it is all that matters for the definitions below (see Figure 1).

Definition 4.2 ($\mathcal{L}$-cover). *Let* $\mathcal{L}$ *be a set of patterns. A pattern* P *is* $\mathcal{L}$-covered *if for all faces* $e, f \in P$*, there exist* $Q_1, \ldots, Q_n \in \mathcal{L}$ *such that:*

1. $e \in Q_1$ *and* $f \in Q_n$*;*
2. $Q_k \cap Q_{k+1}$ *contains at least one face, for all* $k \in \{1, \ldots, n-1\}$*;*
3. $Q_k \subseteq P$ *for all* $k \in \{1, \ldots, n\}$*.*

Lemma 4.3. *Let P be an $\mathcal{L}$-covered pattern, Σ a dual substitution and $\mathcal{L}$ a set of patterns such that $\Sigma(Q)$ is $\mathcal{L}$-covered for all $Q \in \mathcal{L}$. Then $\Sigma(P)$ is $\mathcal{L}$-covered.*

We will need *strong* coverings to ensure that the image of an annulus is an annulus. We denote by $\mathcal{L}_{\text{edge}}$ the set of all the twelve edge-connected two-face patterns (up to translation).

Definition 4.4 (Strong $\mathcal{L}$-cover). *Let $\mathcal{L}$ be a set of edge-connected patterns. A pattern P is* strongly $\mathcal{L}$-covered *if*

1. *P is $\mathcal{L}$-covered;*
2. *for every pattern $X \in \mathcal{L}_{\text{edge}}$ such that $X \subseteq P$, there exists a pattern $Y \in \mathcal{L}$ such that $X \subseteq Y \subseteq P$.*

The intuitive idea behind the notion of strong $\mathcal{L}$-covering is that every occurrence of a pattern of $\mathcal{L}_{\text{edge}}$ in P is required to be "completed within P" by a pattern of $\mathcal{L}$.

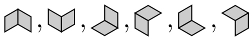
Definition 4.5 (Annulus). *Let $\mathcal{L}$ be a set of edge-connected patterns and $\mathbf{\Gamma}$ be a stepped plane. An* $\mathcal{L}$-annulus *of a pattern $P \subseteq \mathbf{\Gamma}$ is a pattern $A \subseteq \mathbf{\Gamma}$ such that:*

1. *P, $A \cup P$ and $\mathbf{\Gamma} \setminus (A \cup P)$ are $\mathcal{L}$-covered;*
2. *A is strongly $\mathcal{L}$-covered;*
3. *A and P have no face in common;*
4. *$P \cap \overline{\mathbf{\Gamma} \setminus (P \cup A)} = \varnothing$.*

The notation $\overline{\mathbf{\Gamma} \setminus (P \cup A)}$ stands for the topological closure of $\mathbf{\Gamma} \setminus (P \cup A)$.

Conditions 1 and 2 are combinatorial properties that we will use in the proof of Lemma 4.11 in order to prove that the image of an $\mathcal{L}_{\text{FS}}$-annulus by a Σ_i^{FS} is an $\mathcal{L}_{\text{FS}}$-annulus. Conditions 3 and 4 are properties of topological nature that we want annuli to satisfy.

4.2 Covering Properties for $\boldsymbol{\Sigma_1, \Sigma_2, \Sigma_3}$

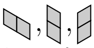
Let $\mathcal{L}_{\text{FS}}$ be the set of patterns containing  $\in \mathcal{L}_{\text{edge}}$ and

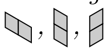
 $= [\mathbf{0}, 2]^\star \cup [(1, 0, 0), 2]^\star \cup [\mathbf{0}, 3]^\star$,

 $= [\mathbf{0}, 3]^\star \cup [(1, 0, 0), 3]^\star \cup [\mathbf{0}, 2]^\star$,

 $= [\mathbf{0}, 3]^\star \cup [(0, 1, 0), 3]^\star \cup [\mathbf{0}, 1]^\star$.

Lemma 4.6 ($\mathcal{L}_{\text{FS}}$-covering). *Let P be an $\mathcal{L}_{\text{FS}}$-covered pattern. Then the pattern $\Sigma_i(P)$ is $\mathcal{L}_{\text{FS}}$-covered for every $i \in \{1, 2, 3\}$.*

Lemma 4.7. *Let $\mathbf{\Gamma}$ be a stepped plane that does not contain any translate of one of the patterns*  $\in \mathcal{L}_{\text{edge}}$ *and*  $= [\mathbf{0}, 3]^\star \cup [(1, 1, 0), 3]^\star$. *Then no translate of any of these four patterns appears in $\Sigma_i(\mathbf{\Gamma})$.*

Lemma 4.8 (Strong $\mathcal{L}_{\text{FS}}$-covering). *Let P be a strongly $\mathcal{L}_{\text{FS}}$-covered pattern which is contained in a stepped plane that avoids*  *and* . *Then $\Sigma_i(P)$ is strongly $\mathcal{L}_{\text{FS}}$-covered for every $i \in \{1, 2, 3\}$.*

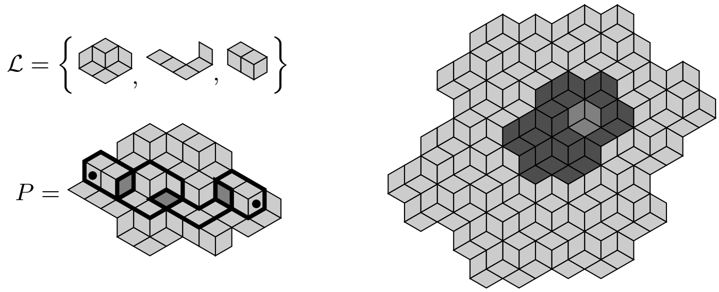


Fig. 1. On the left, the pattern P is $\mathcal{L}$-covered. Two faces of P are connected via a sequence of patterns from $\mathcal{L}$. On the right, examples of $\mathcal{L}_{\mathrm{FS}}$-annulus. Patterns $P_0 \subsetneq P_4 \subsetneq P_7$ are defined by $P_0 = \mathcal{U}$ and $P_{i+1} = \Sigma_3^{\mathrm{FS}}(P_i)$. The lighter pattern $P_7 \setminus P_4$ is a $\mathcal{L}_{\mathrm{FS}}$-annulus of P_4 and the darker pattern $P_4 \setminus P_0$ is an $\mathcal{L}_{\mathrm{FS}}$-annulus of P_0.

4.3 Annuli and Dual Substitutions

The proof of the following proposition (by induction) relies on Lemma 4.10 (base case) and on Lemma 4.11 (induction step). We recall that $\mathcal{U} = [\mathbf{0}, 1]^\star \cup [\mathbf{0}, 2]^\star \cup [\mathbf{0}, 3]^\star =$ 

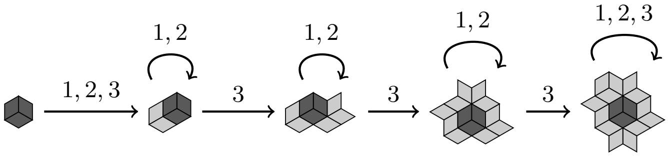
Proposition 4.9. *Let $(\Sigma_i)_{n\in\mathbb{N}}$ be a sequence with values in $\{\Sigma_1^{\mathrm{FS}}, \Sigma_2^{\mathrm{FS}}, \Sigma_3^{\mathrm{FS}}\}$ such that Σ_3^{FS} occurs infinitely often, and let $k \in \mathbb{N}$ such that $(\Sigma_1, \ldots, \Sigma_k)$ contains Σ_3^{FS} at least four times.*

Then for every $\ell \geqslant 1$, $\Sigma_1 \cdots \Sigma_{k+\ell}(\mathcal{U}) \setminus \Sigma_1 \cdots \Sigma_\ell(\mathcal{U})$ is an $\mathcal{L}_{\mathrm{FS}}$-annulus of $\Sigma_1 \cdots \Sigma_\ell(\mathcal{U})$ in the stepped plane $\Sigma_1 \cdots \Sigma_{k+\ell}(\mathbf{\Gamma}_{(1,1,1)})$.

Proof. We prove the result by induction on ℓ. The case $\ell = 0$ (*i.e.*, $\Sigma_1 \cdots \Sigma_k(\mathcal{U}) \setminus \mathcal{U}$ is an annulus of $\mathcal{U}$) is settled by Lemma 4.10. Now, assume that the induction property holds for some $\ell \in \mathbb{N}$. The pattern $\Sigma_1 \cdots \Sigma_{k+\ell}(\mathcal{U})$ is contained in the stepped plane $\Sigma_{k+\ell}(\mathbf{\Gamma}_{(1,1,1)})$, so it does not contain any of the patterns forbidden by Lemma 4.7. We can then apply Lemma 4.11 to deduce that $\Sigma_1 \cdots \Sigma_{k+\ell+1}(\mathcal{U}) \setminus \Sigma_1 \cdots \Sigma_{\ell+1}(\mathcal{U})$ is an $\mathcal{L}_{\mathrm{FS}}$-annulus of $\Sigma_1 \cdots \Sigma_{\ell+1}(\mathcal{U})$. □

Lemma 4.10. *Let Σ be a product of Σ_1, Σ_2 and Σ_3 such that Σ_3 appears at least four times. Then $\Sigma(\mathcal{U}) \setminus \mathcal{U}$ is an $\mathcal{L}_{\mathrm{FS}}$-annulus of $\mathcal{U}$ in $\Sigma(\mathbf{\Gamma}_{(1,1,1)})$.*

Proof. Below, "$P \xrightarrow{i} Q$" means that $Q \subseteq \Sigma_i(P)$ so the result follows.



□

Lemma 4.11. *Let $\mathbf{\Gamma}$ be a stepped plane that avoids* , ,  *and* . *Let $A \subseteq \mathbf{\Gamma}$ be an $\mathcal{L}_{\mathrm{FS}}$-annulus of a pattern $P \subseteq \mathbf{\Gamma}$, and let $\Sigma = \Sigma_i$ for some $i \in \{1, 2, 3\}$. Then $\Sigma(A)$ is an $\mathcal{L}_{\mathrm{FS}}$-annulus of $\Sigma(P)$ in the stepped plane $\Sigma(\mathbf{\Gamma})$.*

Proof. We must prove the following:

1. $\Sigma(P)$, $\Sigma(A) \cup \Sigma(P)$ and $\mathbf{\Gamma} \setminus (\Sigma(A) \cup \Sigma(P))$ are $\mathcal{L}_{\mathrm{FS}}$-covered;
2. $\Sigma(A)$ is strongly $\mathcal{L}_{\mathrm{FS}}$-covered;
3. $\Sigma(A)$ and $\Sigma(P)$ have no face in common;
4. $\Sigma(P) \cap \overline{\Sigma(\mathbf{\Gamma}) \setminus (\Sigma(P) \cup \Sigma(A))} = \varnothing$.

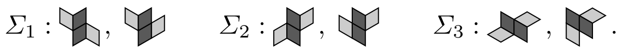
Conditions 1 and 3 hold thanks to Lemma 4.3 and Proposition 2.11 respectively, and 2 holds thanks to Lemma 4.8. It remains to prove that 4 holds.

Suppose that 4 does not hold. This implies that there exist faces $f \in P, g \in \mathbf{\Gamma} \setminus (A \cup P), f' \in \Sigma(f)$ and $g' \in \Sigma(g)$ such that f' and g' have a nonempty intersection. Also, $f \cup g$ must be disconnected because P and $\overline{\mathbf{\Gamma} \setminus (P \cup A)}$ have empty intersection by hypothesis.

The strategy of the proof is as follows: we check all the possible patterns $f \cup g$ and $f' \cup g'$ as above, and for each case we derive a contradiction. This can be done by inspection of a finite number of cases. Indeed, there are 36 possibilities for $f' \cup g'$ up to translation (the number of connected two-face patterns that share a vertex or an edge), and each of these patterns has a finite number of two-face preimages.

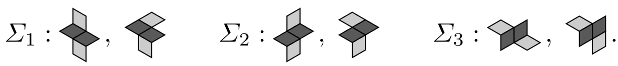
The first patterns $f' \cup g'$ which have disconnected preimages are $f' \cup g' = [\mathbf{0}, 3]^\star \cup [(1,1,0), 3]^\star$ or $[\mathbf{0}, 2]^\star \cup [(1,-1,1), 1]^\star$ or $[\mathbf{0}, 2]^\star \cup [(1,0,1), 2]^\star$. These cases can be ignored thanks to Lemma 4.7: the first case () is forbidden by assumption. In the second case, Definition 2.2 implies that if a stepped plane contains $f' \cup g'$, then it contains the face $[(0,0,1), 2]^\star$ shown in dark gray . This contains a pattern ruled out by Lemma 4.7, which settles this case. The third case can be treated in the same way.

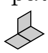
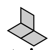
Another possibility is $f' \cup g' = [\mathbf{0}, 2]^\star \cup [(1,-1,1), 3]^\star$, which admits six disconnected preimages (two for each Σ_i). They are shown below (in light gray), together with their only possible completion within a stepped plane (in dark gray), which can be deduced from Definition 2.2:



The patterns that appear in dark gray are forbidden by Lemma 4.7, so this case is settled.

The last two possibilities are $f' \cup g' = [\mathbf{0}, 3]^\star \cup [(1,1,-1), 1]^\star$ or $f' \cup g' = [\mathbf{0}, 3]^\star \cup [(1,1,-1), 2]^\star$. Below (in light gray) are all the possible preimages $f \cup g$ (which are the same for the two possibilities), and in dark gray is shown their only possible completion X within a stepped plane:



Now, we have $X \subseteq A$ because Condition 4 for A and P would fail otherwise (f and g cannot touch). However, this contradicts the fact that strongly $\mathcal{L}_{\mathrm{FS}}$-connected. Indeed, $X \in \mathcal{L}_{\mathrm{edge}}$ but there cannot exist a pattern $Y \in \mathcal{L}_{\mathrm{FS}}$ such that $X \subseteq Y \subseteq A$ because then we must have $Y =$ ,  or , so Y must overlap with f or g, which is impossible because f and g are not in A. □

Proof (Proof of Proposition 4.1). Let $\mathbf{v} \in \mathcal{F}_3$. To prove the proposition, it is enough to prove that $\cup_{n=0}^{\infty} P_n = \mathbf{\Gamma_v}$, thanks to Remark 2.3. Let $P \in \mathbf{\Gamma_v}$ be a finite pattern. The *combinatorial radius* of P is defined to be the length of the smallest path of edge-connected unit faces from the origin to $\mathbf{\Gamma_v} \setminus P$.

Now, since $\mathbf{v} \in \mathcal{F}_3$, Σ_3^{FS} occurs infinitely many often in the sequence $(\Sigma_i)_{i\in\mathbb{N}}$ of the dual substitutions associated with the expansion of $\mathbf{v}$. Hence, we can apply Proposition 4.9 to prove that there exists $k \in \mathbb{N}$ such that for all $\ell \geqslant 0$, the pattern $A_\ell := \Sigma_1 \cdots \Sigma_{k+\ell}(\mathcal{U}) \setminus \Sigma_1 \cdots \Sigma_\ell(\mathcal{U})$ is an $\mathcal{L}_{\mathrm{FS}}$-annulus of $\Sigma_1 \cdots \Sigma_\ell(\mathcal{U})$.

By Condition 1 of Definition 4.5, the pattern $A_\ell \cup \Sigma_1 \cdots \Sigma_\ell(\mathcal{U})$ is simply connected for all $\ell \geqslant 0$, so its combinatorial radius increases at least by 1 when ℓ is incremented, thanks to Conditions 3 4. This proves the required property. □

5 Main Results

In general, we do not know if the set $\{\omega \in \mathbb{R} \mid \mathfrak{P}(\mathbf{v}, \omega, \Omega(\mathbf{v}))$ is 2-connected$\}$ is closed. In fact, $\{\omega \in \mathbb{R} \mid \mathfrak{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))\}$ may be 2-connected or not.

Theorem 5.1. *Let* $\mathbf{v} \in \mathcal{O}_3^+$. *The arithmetical discrete plane* $\mathfrak{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ *is* 2*-connected if and only if* $\mathbf{v} \in \mathcal{F}_3$.

Proof. Let $\mathbf{v} \in \mathcal{F}_3$ and $\mathbf{x} \in \mathfrak{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$, by Proposition 2.8 we have $\Omega(\mathbf{v}) = \|\mathbf{v}\|_1/2$. If $\|\mathbf{v}\|_\infty \leqslant \langle \mathbf{x}, \mathbf{v}\rangle < \|\mathbf{v}\|_1/2$, then $\|\mathbf{v}\|_\infty - \mathbf{v}_1 \leqslant \langle \mathbf{x} - \mathbf{e}_1, \mathbf{v}\rangle < \|\mathbf{v}\|_1/2 - \mathbf{v}_1 < \|\mathbf{v}\|_\infty$, so $\mathbf{x} - \mathbf{e}_1 \in \mathfrak{P}(\mathbf{v}, 0, \|\mathbf{v}\|_\infty)$. In other words, an element $\mathbf{x}$ of $\mathfrak{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ either belongs to $\mathfrak{P}(\mathbf{v}, 0, \|\mathbf{v}\|_\infty)$ or is 2-adjacent to an element of $\mathfrak{P}(\mathbf{v}, 0, \|\mathbf{v}\|_\infty)$.

Now, given $\mathbf{y} \in \mathfrak{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$, both $\mathbf{x}$ and $\mathbf{y}$ belong or are adjacent to $\mathfrak{P}(\mathbf{v}, 0, \|\mathbf{v}\|_\infty)$, so they are 2-connected in $\mathfrak{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ because:

– $\mathfrak{P}(\mathbf{v}, 0, \|\mathbf{v}\|_\infty) \subseteq \cup_{n=0}^{\infty} \mathbf{T}_n$, thanks Propositions 3.5 and 4.1,
– $\cup_{n=0}^{\infty} \mathbf{T}_n$ is 2-connected: it is a increasing union of sets $\mathbf{T}_n$ which are 2-connected thanks to Proposition 3.3,
– $\cup_{n=0}^{\infty} \mathbf{T}_n \subseteq \mathfrak{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$, thanks to Proposition 3.2.

We now prove the converse implication, and we assume that $\mathfrak{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is 2-connected. Assume $\dim_{\mathbb{Q}}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = 1$, $\mathbf{v} \in \mathbb{Z}^3$ with $\gcd\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = 1$. Let $n \in \mathbb{N}$ such that $\mathbf{v}_1^{(n)} = 0$. The plane $\mathfrak{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is 2-connected if and only if so is $\mathfrak{P}(\mathbf{v}^{(n)}, 0, \Omega(\mathbf{v}^{(n)}))$. But $\Omega(\mathbf{v}^{(n)}) = \mathbf{v}_2^{(n)} + \mathbf{v}_3^{(n)} - 1$ so $\mathfrak{P}(\mathbf{v}^{(n)}, 0, \Omega(\mathbf{v}^{(n)}))$ is the translation along $\mathbf{e}_1$ of an arithmetical discrete line strictly thinner than a standard one and cannot be 2-connected. Hence $\dim_{\mathbb{Q}}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} > 1$. If $\mathbf{v} \notin \mathcal{F}_3$, there exists $n \in \mathbb{N}$ such that $\mathbf{v}_1^{(n)} + \mathbf{v}_2^{(n)} \leqslant \mathbf{v}_3^{(n)}$. The plane $\mathfrak{P}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ is 2-connected if and only if so is $\mathfrak{P}(\mathbf{v}^{(n)}, 0, \Omega(\mathbf{v}^{(n)}))$. But $\Omega(\mathbf{v}^{(n)}) = \|\mathbf{v}^{(n)}\|_\infty$, so $\mathfrak{P}(\mathbf{v}^{(n)}, 0, \|\mathbf{v}^{(n)}\|_\infty)$ cannot be 2-connected since $\mathbf{x}$ and $\mathbf{x} + \mathbf{e}_3$ cannot be both in $\mathfrak{P}(\mathbf{v}^{(n)}, 0, \|\mathbf{v}^{(n)}\|_\infty)$. □

The theorem below is a direct consequence of Propositions 3.5 and 4.1, but it is worth mentioning.

Theorem 5.2. *If* $\mathbf{v} \in \mathcal{F}_3$*, then* $\mathfrak{P}(\mathbf{v}, 0, \|\mathbf{v}\|_\infty) \subseteq \bigcup_{n=0}^{\infty} \mathbf{T}_n$*,* i.e.*, the naive plane of normal vector* $\mathbf{v}$ *is included in* $\bigcup_{n=0}^{\infty} \mathbf{T}_n$.

6 Conclusion

We have provided a full understanding of the 2-connectedness of discrete planes with critical thickness, using a combination of tools issued from numeration systems and combinatorics on words. Theorem 5.1 highlights the limit behavior of discrete plane with critical thickness which is reminiscent of similar phenomena occurring in percolation theory. We plan to investigate further the properties of critical planes (as in [4]), their tree structure, and we plan to explore more deeply the connections with Rauzy fractals and numeration systems.

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