Proportional Contact Representations of 4-Connected Planar Graphs

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Abstract. In a contact representation of a planar graph, vertices are represented by interior-disjoint polygons and two polygons share a non-empty common boundary when the corresponding vertices are adjacent. In the weighted version, a weight is assigned to each vertex and a contact representation is called proportional if each polygon realizes an area proportional to the vertex weight. In this paper we study proportional contact representations of 4-connected internally triangulated planar graphs. The best known lower and upper bounds on the polygonal complexity for such graphs are 4 and 8, respectively. We narrow the gap between them by proving the existence of a representation with complexity 6. We then disprove a 10-year old conjecture on the existence of a Hamiltonian canonical cycle in a 4-connected maximal planar graph, which also implies that a previously suggested method for constructing proportional contact representations of complexity 6 for these graphs will not work. Finally we prove that it is **NP**-hard to decide whether a 4-connected planar graph admits a proportional contact representation using only rectangles.

1 Introduction

Contact graph representations for planar graphs are a well-studied alternative to the traditional node-link diagram. In most contact graph representations, vertices are represented by geometric objects such as circles, triangles and rectangles, while edges correspond to two objects touching in some specified fashion.

Here we consider contact representations of planar graphs, with vertices represented by simple interior-disjoint polygons and edges represented by non-empty shared boundaries between corresponding polygons. In the weighted version, the input is a planar graph $G = (V, E)$ and a weight function $w : V \to \mathbb{R}^+$. A *proportional contact representation* of G is one where each vertex v is represented by a polygon with $w(v)$ area. When the polygons are made of only axis-aligned sides, the unweighted and weighted representations are called *rectilinear duals* and *rectilinear cartograms*, respectively.

Contact representations have practical applications in cartography [15], geography [17], sociology [11] and floor-planning for VLSI layout [19]. Other applications are in visualization of relational data, where using the adjacency of regions to represent edges in a graph can lead to a more compelling visualization than just drawing a line segment between two points [3]. In this context it is often desirable, for aesthetic,

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practical and cognitive reasons, to limit the *polygonal complexity* of the representation, measured by the maximum number of sides in a polygon. Similarly, it is also desirable to minimize the unused area, also known as *holes* in floor-planning and VLSI layouts. With these considerations in mind, we study the problem of constructing hole-free proportional contact representations with minimal polygonal complexity.

1.1 Related Work

Koebe's theorem [12] is an early example of a contact representation, showing that any planar graph can be represented by touching circles. Any planar graph also has a representation with triangles [7] and with cubes in 3D [6]. However all these results yield representations that contain point-contacts between adjacent polygons. When adjacent polygons must share non-trivial common boundaries (also known as side-contact representation) it has been shown that convex hexagons are sometimes necessary and always sufficient [4]. The rectilinear variant of the side-contact representation problem was first studied in graph theoretic context, and then with renewed interest in VLSI layouts and floor planning. It is known that 8 sides are sometimes necessary and always sufficient for rectilinear duals of maximal planar graphs [10, 14, 19]. Characterizations for planar graphs with rectilinear duals of complexity 4 and 6 are also known [13, 16, 18].

While all the above results deal with the unweighted version of the problem, the weighted version was first studied back in 1934 when Raisz described rectangular cartograms, i.e., rectilinear cartograms that use only rectangles [15]. In the general rectilinear setting it has been recently shown that 8 sides are always sufficient for a rectilinear cartogram of maximal planar graphs [2] while it has always been known that 8 sides are sometimes necessary. It has been shown that 7 sides are sometimes necessary and always sufficient if we drop the rectilinear restriction while still requiring proportional side-contact representations [1]. Note that the representation with 7 sides comes at the expense of many holes, when compared to the representation with 8 sides.

In this paper we study proportional contact representations for 4-connected internally-triangulated planar graphs. There are several earlier results on contact representation for this large graph class and for some subclasses thereof. It is known that the class of graphs that have rectilinear duals with rectangles and without any degree-4 points in the representations are exactly the class of 4-connected planar graphs with triangular internal faces and a non-triangular outerface [13, 18]. However, the same cannot be said about proportional contact representations; there are instances of 4-connected planar graphs with triangular internal faces and a non-triangular outerface that have no proportional contact representations with rectangles for some weight functions. Recently, Eppstein *et al.* [5] characterized the class of *area-universal* rectangular duals, i.e., rectangular duals that can realize any specified area for the rectangles. In summary, the best known lower and upper bounds for hole-free proportional contact representations of 4-connected planar graphs are 4 and 8, respectively [2, 4].

1.2 Our Contributions

We are interested in narrowing the gap between the known lower and upper bounds on the polygonal complexity in proportional contact representations for 4-connected planar graphs in various settings (rectilinear or not, hole-free or with holes). We also present some computational complexity results about a natural recognition problem, and disprove a 10-year old conjecture. To summarize:

(1) We describe an algorithm for constructing a hole-free proportional contact representation of a 4-connected internally triangulated planar graph using 6-sided polygons with arbitrarily small cartographic error[1]. We then prove the existence of a representation without cartographic error. Note that this result also improves the known upper bound on the polygonal complexity of representations where holes are also allowed.

(2) We disprove a conjecture, posed independently by two sets of authors [2, 8], about the existence of a Hamiltonian canonical order (a canonical order that induces a Hamiltonian cycle) in a 4-connected maximal planar graph. In particular, this shows that a method suggested in [2] for constructing 6-sided rectilinear proportional contact representations for 4-connected graphs will not work.

(3) We show that it is **NP**-hard to decide whether a 4-connected planar graph has a proportional contact representation with rectangles.

2 Proportional Contact Representation of Complexity 6

In this section we give an algorithm for constructing a 6-sided proportional contact representation of an internally triangulated 4-connected planar graph with arbitrarily small cartographic error. With the help of this algorithm, we then prove the existence of a representation with complexity 6 and no cartographic error.

2.1 Representations with Cartographic Error

We prove the following main theorem in the rest of this section.

Theorem 1. *Let $G = (V, E)$ be an internally triangulated 4-connected plane graph and let $w : V \mapsto \mathbb{R}^+$ be a weight function on the vertices of G. Then for any $\epsilon > 0$, G has a contact representation where each vertex v of G is represented by a 6-sided polygon with area in the range $[w(v), w(v) + \epsilon]$.*

Assume first that the outer face of G is not a triangle. Then G admits a *rectangular dual* Γ (a rectilinear dual that uses only rectangles) due to [13, 18]. We need the following definitions for a rectangular dual Γ of G. The graph G is the *dual* of Γ. Since each polygon in Γ is a rectangle, the adjacency between two rectangles in Γ representing an edge of G can occur in one of two ways: (i) through a shared horizontal segment (*horizontal adjacency*), or (ii) through a shared vertical segment (*vertical adjacency*). We say Γ is *topologically equivalent* to another rectangular layout Γ' when both Γ and Γ' have the same dual graph G and each edge of G is represented by the same type of adjacency (horizontal or vertical) in both Γ and Γ'. A *line-segment* in Γ is the union of inner edges of Γ forming a consecutive part of a straight-line. A line-segment not contained in any other line-segment is *maximal*. A maximal line-segment s is called *one-sided* if it forms a full side of at least one rectangular face, or in other words, if the

[1] *Cartographic error* in a representation of a graph G is the maximum over the vertices v in G of the value $|A(v) - w(v)|$, where $A(v)$ is the area for v and $w(v)$ is its weight.

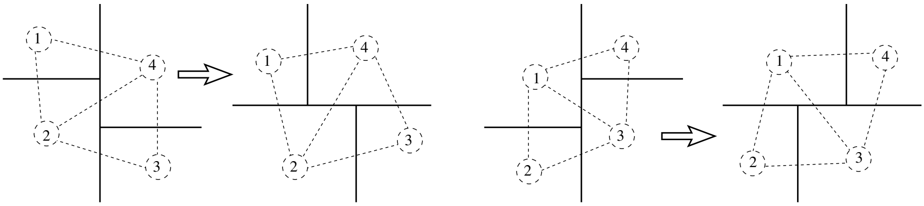

Fig. 1. Illustration for the proof of Lemma 1

perpendicular line segments that attach to its interior are all on one side of s. Otherwise, s is *two-sided*. Eppstein *et al.* proved that if all the maximal segments in a rectangular dual Γ are one-sided then Γ is *area-universal*, which means that any distribution of areas to the rectangles in Γ can be realized with a topologically equivalent layout [5]. Unfortunately, not every internally triangulated 4-connected plane graph has a rectangular dual that is also area-universal. With the next lemma we can prove a slightly weaker statement which can help us reduce the polygonal complexity. Specifically, we can show that for any such graph with non-triangle outerface, there exists a rectangular dual where all two-sided maximal segments are horizontal.

Lemma 1. *Let G be an internally triangulated 4-connected plane graph with a non-triangle outerface. Then G has a rectangular dual with no vertical two-sided segment.*

Proof Sketch. We start by computing a (possibly two-sided) rectangular dual Γ of G [13]. If G is one-sided or has only horizontal 2-sided maximal segments we are done. Let s be a vertical maximal segment in Γ. Call every degree-3 point on s a *junction* on s. If s is not one-sided, then going from bottom to top along s there will be at least one of the two configurations in Fig. 1. In both cases, we modify the layout locally[2], as illustrated in Fig. 1. If we repeatedly apply this operation for each vertical two-sided segment in Γ, there will be no more vertical two-sided segment. □

Once we have a rectangular dual of a planar graph G with no two-sided vertical maximal segments, we modify the representation into a contact representation with 6-sided polygons to realize any set of weights on the vertices of G, at the expense of ϵ-cartographic error, $\epsilon > 0$.

Lemma 2. *Let $G = (V, E)$ be an internally triangulated plane graph and let Γ be a rectangular dual of G such that Γ contains no vertical two-sided maximal segment. Then G admits a contact representation Λ such that each vertex v of G is represented by a polygon of complexity at most 6 with area in the range $[w(v), w(v) + \epsilon]$, where $w : V \mapsto \mathbb{R}^+$ is an arbitrary weight function and $\epsilon > 0$.*

Proof Sketch. If Γ contains no two-sided maximal segment, then it can realize any weight function [5] and we are done. Otherwise, for each horizontal two-sided maximal segment s, we replace s by a rectangle with a small height (< the minimum feature size of Γ) and with horizontal span same as s; see Fig. 2(a)–(b). It is easy to see that

[2] These operations are well-known as "flips" in much of the related work; see [9] for example.

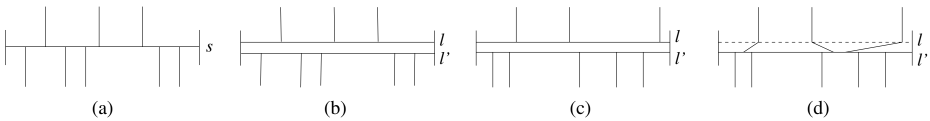

Fig. 2. Illustration for the proof of Lemma 2

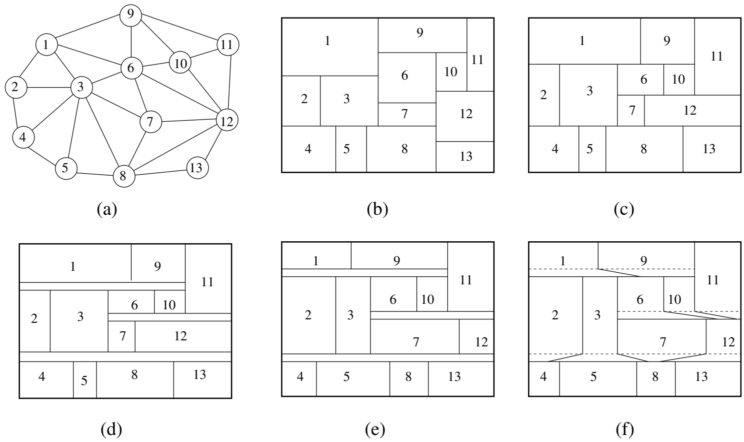

Fig. 3. Illustration of the construction with 6-sided polygons and ϵ-cartographic error

this modification makes all maximal segments one-sided, which makes the resulting representation Γ' area-universal. Let Γ'' be a rectangular layout, where the area of each newly formed rectangle is ϵ and the areas for all other rectangles realize the weight function w; see Fig. 2(c). Suppose s was a horizontal two-sided segment in Γ and let l and l' be the top and bottom side of the corresponding rectangle in Γ' (also in Γ''). We select some points on l' corresponding to all the junctions on l so that the order of all these junctions defined by s is respected. We then add an edge from each junction on l to its corresponding point on l'. In this way the area of the rectangle defined by l and l' which had area ϵ is distributed among the rectangles above l. For each rectangle at most two additional corners are thus added making it a polygon with at most 6 sides. $\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1. If the outerface of G is a not triangle, then by Lemma 1 and Lemma 2, G admits a desired representation. We thus assume that the outerface of G is a triangle abc. Then G admits no rectangular dual. However, G admits a rectilinear dual where one of the outer vertices, say a, is represented by a 6-sided "L-shaped" rectilinear polygon and all other vertices are represented by rectangles [16]. Furthermore, the representation is contained inside a rectangle and so is the union of the rectangles representing all vertices of $G' = G - \{a\}$. We then obtain a contact representation of G' with 6-sided polygons by Lemma 1 and Lemma 2. Since the boundary of the rectangular dual of G' is not changed, we can still add the **L**-shaped polygon for a around it with the desired area and correct adjacencies. $\square$

Figure 3 illustrates the construction of a contact representation of a 4-connected plane graph G with 6-sided polygons using the above procedure.

2.2 Representations without Cartographic Error

Here we prove that an internally triangulated 4-connected plane graph has a proportional contact representation of complexity 6 with no cartographic error for any weight function. We begin with a representation that has ϵ-cartographic error and argue that it can be modified so as to remove the errors, while preserving the topology of the layout.

Let $G = (V, E)$ be a graph with weight function $w : V \mapsto \mathbb{R}^+$. Consider a vertex order $v_1, v_2, \ldots, v_n$ of the vertices of G. Let $\Lambda = \langle \lambda_1, \lambda_2, \ldots, \lambda_n \rangle$ be a list of n non-negative real numbers.Then the *Λ-vicinity* of w is the set of all weight functions $w' : V \mapsto \mathbb{R}^+$ such that $|w(v_i) - w'(v_i)| \leq \lambda_i$ for each vertex v_i of G. If $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \lambda$, then the Λ-vicinity of w is also called the *λ-vicinity* of w. We have the following lemma, whose proof is ommitted in this extended abstract.

Lemma 3. *Let $G = (V, E)$ be an internally triangulated plane graph and let Γ be a rectangular cartogram of G for a weight function $w : V \mapsto \mathbb{R}^+$ where Γ contains no vertical two-sided segment. Then there exists a sufficiently small $\lambda > 0$ such that for any weight function $w' : V \mapsto \mathbb{R}^+$ in the λ-vicinity of w, there is a rectangular cartogram $\Gamma(w')$ of G with respect to w', where $\Gamma(w')$ is topologically equivalent to Γ.*

Using the above lemma, we can prove the existence of a cartogram for a internally triangulated 4-connected graph without any cartographic error.

Theorem 2. *Let $G = (V, E)$ be an internally triangulated 4-connected plane graph and let $w : V \mapsto \mathbb{R}^+$ be a weight function on the vertices of G. Then G has a proportional contact representation of complexity 6 with respect to w.*

Proof. Here we use the algorithm from Theorem 1. We assume that G has a non-triangle outerface since the case with triangle outerface can be handled in the same way as in Theorem 1. Let Γ be a representation of G obtained by this algorithm. Each polyline between two horizontal segments in Γ consists of two segments: a vertical segment followed by a segment with an arbitrary slope. Call each such polyline a *vertical 2-line* and call the common point between the two segments of a vertical 2-line a *pivot point*. For example, in Fig. 2(d), there are three such vertical 2-lines. Each pivot point can be moved up or down, resulting in the increase or decrease of the areas of its adjacent polygons. We use this flexibility to find a representation without cartographic error.

Let Γ_0 denote the rectangular layout when we set $\epsilon = 0$, where each pivot point has the same y-coordinate as the bottommost point of its vertical 2-line. Note that Γ_0 may no longer represent G because the adjacencies between rectangles on opposite sides of a horizontal segment may change. We now modify the weights for the rectangles in Γ_0 so that the total weight for all rectangles remains the same and we can move the pivot points to correct the error created by this change. We choose the new weight function w' to be in the λ-vicinity of w, for some $\lambda > 0$, by applying Lemma 3, so that we can find a rectangular cartogram Γ' topologically equivalent to Γ_0. We set the width and height of

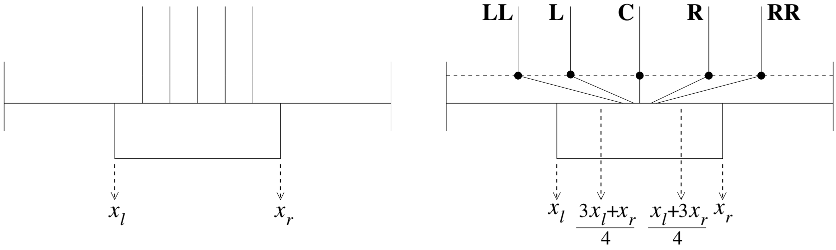

Fig. 4. Illustration for Theorem 2

this layout to be B and H, where $B \times H = \sum_{v \in V} w(v)$. Then $B_{min} = (w_{min} + \lambda)/H$ and $H_{min} = (w_{min} + \lambda)/B$ are the minimum width and height of a rectangle in Γ', respectively, where $w_{min} = min_{v \in V} w(v)$.

We now define the weights for all the rectangles above each two-sided horizontal maximal segment s as follows. Let us consider the set S of all the vertical 2-lines incident to the top-side of a particular rectangle R. Fig. 4 illustrates a set of such vertical 2-lines where for convenience of visualization the pivot points and the bottommost points are placed at different y-coordinates. Let x_l and x_r be the x-coordinates of the left and right side of the rectangle to the bottom of the horizontal segment. In the final representation with 6-sided polygons, we want the bottommost point of each of these vertical 2-lines to be placed between one-fourth and three-fourth of its horizontal span; i.e., between $(3x_l + x_r)/4$ and $(x_l + 3x_r)/4$. In this respect we partition the vertical 2-lines in S into five classes as follows, depending on the x-coordinates of the pivot points; see Fig. 4(b) (pivot points are highlighted by black dots).

(i) **LL**-*lines*: pivot points are to the left of x_l
(ii) **L**-*lines*: pivot points are between x_l and $(3x_l + x_r)/4$
(iii) **C**-*lines*: pivot points are between $(3x_l + x_r)/4$ and $(x_l + 3x_r)/4$
(iv) **R**-*lines*: pivot points are between $(x_l + 3x_r)/4$ and x_r
(v) **RR**-*lines*: pivot points are to the right of x_r

By a left-to-right scan, we set the weight for all the rectangles to the left of a **LL**-line and a **L**-line in S. For each **LL**-line, we set the weight of the rectangle to its left such that the total weight for all the rectangles to its left is decreased by an amount $\epsilon > 0$ where $\epsilon < \lambda$ and $\epsilon < (B_{min} \times H_{min})/8$. For each **L**-line, we set the weight of the rectangle to its left so that the weight of all rectangles to its left is decreased by $\epsilon > 0$ where $\epsilon < \lambda$ and $\epsilon < (B_{min} \times H_{min})/(8|S|)$. Similarly by a right-to-left scan, we set the weight for each the rectangle to the left of a **RR**-line (resp. **R**-line) so that the weight of all rectangles to the left of the polyline is increased by $\epsilon > 0$ where $\epsilon < \lambda$ and $\epsilon < (B_{min} \times H_{min})/8$ (resp. $\epsilon < (B_{min} \times H_{min})/(8|S|)$). For each **C**-line, we set the weight of the rectangle to its left so that the total weight of all the rectangles to its left remains the same. Once we compute ϵ for all the vertical 2-lines, we can calculate the exact weights to be assigned to each rectangle in S. By Lemma 3, we then find a rectangular cartogram Γ' with the new weight function such that Γ' is topologically equivalent to Γ. By shifting the pivot points up as needed, we realize exact weights for each rectangle. By the weight distribution, it is never required to shift any pivot point more than H_{min} distance, the minimum vertical distance between two horizontal segments. By choosing ϵ small enough, we make sure that the segments between the pivot points and bottommost points of vertical 2-lines do not cross. □

3 Hamiltonian Canonical Cycles

Let $G = (V, E)$ be a maximal plane graph with outer vertices u, v, w in clockwise order. A *canonical order* of the vertices $v_1 = u$, $v_2 = v$, v_3, ..., $v_n = w$ of G, is one that meets the following criteria for every $4 \leq i \leq n$, where G_i is the subgraph of G induced by the vertices $v_1, v_2, \ldots, v_{i-1}$:

- $G_{i-1} \subseteq G$ is biconnected, and its outer-cycle C_{i-1} contains the edge (u, v).
- vertex v_i is in the outerface of G_{i-1}, and its neighbors in G_{i-1} form an (at least 2-element) subinterval of the path $C_{i-1} - (u, v)$.

A *Hamiltonian Canonical Cycle* in a maximal planar graph G is a canonical order $v_1, v_2, \ldots, v_n$ of the vertices of G such that $v_1 v_2 \ldots v_n$ is also a Hamiltonian cycle of G. Whether every 4-connected maximal planar graph has a Hamiltonian canonical cycle is a question asked at least two times by two sets of authors in different contexts [2, 8]. In fact, an algorithm in [2] produces 6-sided rectilinear proportional contact representation of any maximal planar graph that has a Hamiltonian canonical cycle. If the above conjecture were true, that would suffice to lower the current best known upper bound on the polygonal complexity for rectilinear cartograms of 4-connected maximal planar graphs from 8 sides to 6 sides. Unfortunately, we show that the conjecture is not true by constructing a 4-connected maximal planar graph with no Hamiltonian canonical cycle.

Theorem 3. *There exist 4-connected maximal planar graphs that do not have any Hamiltonian canonical cycle in any embedding.*

To prove this claim we construct a 4-connected graph G where there are two internal faces of length 4 and the remaining faces (including the outerface) are triangles. We put one isomorphic copy of the graph K in Fig. 5 inside each of the faces of length 4 in G (so that the four vertices on the faces are superimposed with a, b, c and d). In any embedding of this graph, at least one copy of K will retain its embedding. Thus in order to prove Theorem 3, it suffices to prove that there is no Hamiltonian canonical cycle of any plane graph that contains an isomorphic copy of K with the fixed embedding.

Lemma 4. *Let G be a maximal plane graph with an isomorphic copy of the graph K in Fig. 5 as an embedded subgraph. Then G has no Hamiltonian canonical cycle.*

The graph K is very symmetric and it contains four isomorphic copies of the graph L in Fig. 6. Lemma 5 shows that any Hamiltonian canonical cycle of K must follow a restricted path inside L. This can be used to prove Lemma 4.

Let $\mathcal{C} = v_1, v_2, \ldots, v_n$ be a Hamiltonian canonical cycle of a maximal planar graph G. Since $\mathcal{C}$ induces a canonical order, we can consider it as a directed cycle where the edge (v_i, v_{i+1}) is directed towards v_{i+1} for $1 \leq i \leq n-1$ and the edge (v_n, v_1) is directed towards v_1. This also induces a direction for any subpath of $\mathcal{C}$.

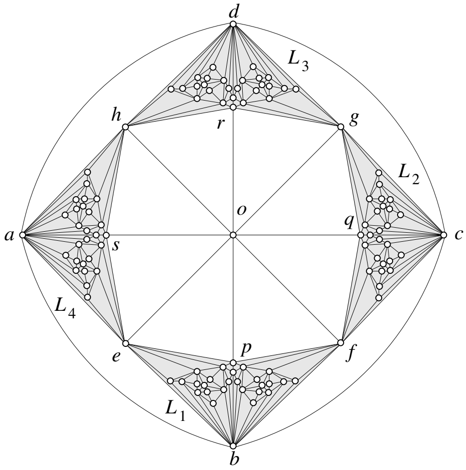

Fig. 5. The graph K used in Theorem 3 and Lemma 4

Lemma 5. *Let G be a maximal plane graph that contains an isomorphic copy of graph L of Fig. 6 as an embedded subgraph. Then there exists no Hamiltonian canonical cycle $\mathcal{C}$ of G such that it contains a subpath P where: (i) the first two vertices on P are from the set $\{A, B\}$, (ii) the third vertex on P is also from the subgraph L, and (iii) either the last vertex of P is Z and F is after Z in $\mathcal{C}$ or the last vertex of P is F.*

Proof Sketch. Assume for a contradiction that there exists a Hamiltonian canonical cycle $\mathcal{C}$ of G with a subpath P such that the conditions (i)–(iii) hold. Denote the vertex set of L by S. First note that the vertex set $T = \{A, B, Z, F\}$ defines a separating cycle of G. Let L' be the graph induced by $S - T$. Then P can enter and exit L' only once. The first two vertices on P are A and B. Without loss of generality, a_1 is the third vertex. Since C is the only vertex for P to go between vertices on different sides of the polyline BCF, C must appear on P after all the vertices from the left of BCF have appeared. Then since $\mathcal{C}$ is a Hamiltonian cycle as well as a canonical order, a careful observation shows that the initial subpath of P is the one drawn by the thick black polyline, followed by the one drawn by the dotted grey polyline in Fig. 6. On the other hand, the subpath of P ending at C must be the one drawn by the thick gray solid polyline. However there is no edge to go between these two subpaths, a contradiction. □

We are now ready to prove Lemma 4. We are only giving the proof sketch here.

Proof Sketch of Lemma 4. Assume for a contradiction that G has a Hamiltonian canonical cycle $\mathcal{C}$. For any embedded subgraph H of G bounded by a separating cycle C_H, call the subgraph induced by the vertices of $H - C_H$ the *inside* of H. Since $\mathcal{C}$ is a canonical order, withour loss of generality the first vertex on $\mathcal{C}$ inside K is e. Then both a and b appear before e and $\mathcal{C}$ can enter and exit the inside of K only once and either c or d is the only exit vertex. Call the subpath of $\mathcal{C}$ between the entry and the exit vertex P. Assume due to symmetry that P enters in the inside of L_1 after e. Since either c or d is the exit vertex, P must enter and exit the inside of L_4 through s and h, respectively

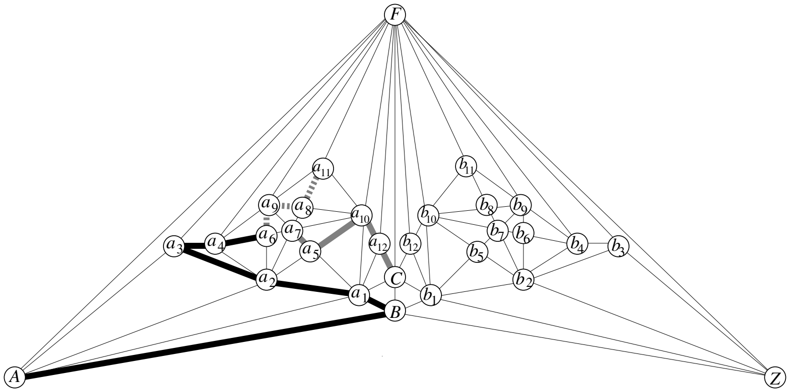

Fig. 6. The graph L used in Lemmas 4 and 5

and o must immediately preceed s on P. Looking back at L_1, in case P goes from either p or f to o, it must go from h to the inside of L_3 and eventually exit the inside of K through c. However, this is not possible by Lemma 5. Thus P must visit all the vertices of L_1 and exit through f and go via the inside of L_2, to c. Since a, b, c have all been already visited by P, it must then exit though d from the inside of K; which is also not possible due to Lemma 5. □

The proof of Theorem 3 follows from Lemma 4 since we could construct a graph that contains an isomorphoc copy of the graph K as an embedded graph.

4 NP-Hardness for Rectangular Representations

Here we consider the following problem. Given a 4-connected plane graph $G = (V, E)$ with triangle and quadrangle internal faces and a non-triangle outerface and a weight function $w : V \mapsto \mathbb{R}^+$, we want to determine whether G has a rectangular cartogram with respect to w. Let us call the problem **RectangleCartogram** (**RC**). We now show that this problem is **NP**-hard by a reduction from the well-known **NP**-hard problem **Partition** defined as follows. Given a set of integers $S = \{x_1, \ldots, x_n\}$ with $\sum_{i=1}^{n} x_n = 2A$ for some integer A, we want to find a subset I of S such that $\sum_{x_i \in I} x_i = A$.

Given an instance of **Partition**, we construct an instance of **RC** as follows. For each integer x_i of S, we have a subgraph with eight vertices: $X_i, p_i, p_{i+1}, q_i, q_{i+1}, a_i, b_i, c_i$. We highlight such a subgraph for x_i in Fig. 7. The constructed graph is then 4-connected with a nontriangle outerface where each internal face is a triangle or quadrangle.

We define the weight function as follows. Define $m = min_{i=1}^{n} x_n$. For each vertex X_i, we define $w(X_i) = x_i$. We give a very small weight, say $\delta = m/20$ to each vertex a_i, b_i, c_i, p_i, q_i for $1 \le i \le n$ and to each vertex L, R, T and B. We give a very large weight $W(M)$ to the vertex M such that $\sqrt{W(M) + 2A + (n+1)\delta} - \sqrt{W(M)} < \sqrt{m}$. Finally we give weights to the vertices t_1 and t_2 such that $w(t_1) : w(t_2) = \sqrt{W(M) + 2A + (n+1)\delta} - \sqrt{W(M)} : \sqrt{W(M)}$.

We now have the following lemma and we only sketch its proof here.

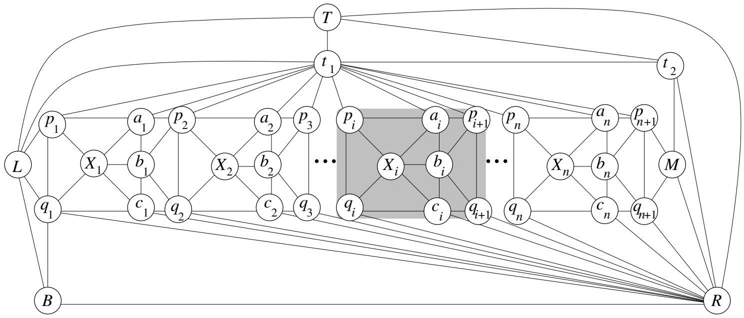

Fig. 7. The graph constructed from an instance of a **Partition** problem

Lemma 6. *There exists a subset I of S such that $\sum_{x_i \in I} x_i = A$ if and only if there is a rectangular cartogram of G with respect to the weight function w.*

Proof Sketch. Suppose first that there exists a cartogram Γ of G with respect to w. Then the rectangles for L, T, R, B, t_1 and t_2 define a rectangular frame F inside which lie the remaining rectangles. The left and bottom of the rectangle for M define two lines l_v and l_h and each subgraph for x_i has a drawing in one of two configurations; see Fig. 8. Then the vertices corresponding to the rectangles X_i lying to the left of l_v form the subset I of S. Conversely if we are given a subset I of S such that $\sum_{x_i \in I} x_i = A$, then

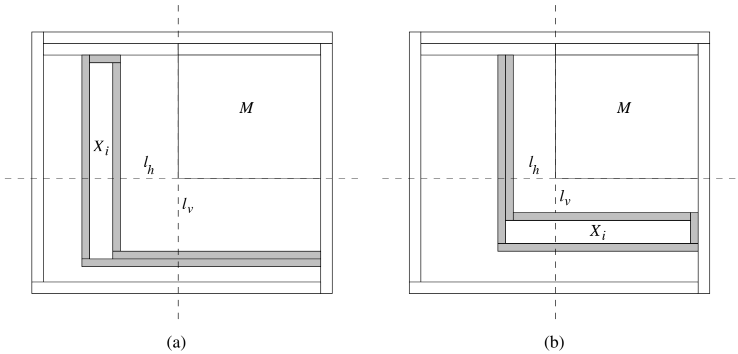

Fig. 8. Illustration of the proof of Lemma 6

we first draw the rectangles for L, T, R, B, t_1 and t_2 such that they enclose a square frame F of size $W(M) + 2A + (n+1)\delta$. We then draw the subgraphs for x_i as in Fig. 8(a) if $x_i \in I$; otherwise we draw this subgraph as in Fig. 8(b). □

We can thus reduce an instance S of Problem **Partition** to an instance (G, w) of Problem **RectangleCartogram** such that S has a solution if and only if G has a rectilinear cartogram with respect to w. This yields the following theorem.

Theorem 4. *Problem* ***RectangleCartogram*** *is* **NP***-hard.*

5 Conclusions and Open Problems

We addressed the problem of proportional contact representation of 4-connected internally triangulated planar graphs and showed that non-rectilinear polygons with complexity 6 are sufficient. Narrowing the gap between this upper bound and the currently best known lower bound of 4 remains open. With the additional restriction of using rectilinear polygons, there are instances where complexity 6 is required and here we showed that it is **NP**-hard to decide whether a 4-connected planar graph with triangle and quadrangle internal faces admits a representation with rectangles. It will be interesting to investigate the complexity of the same problem for internally triangulated graphs. On the other hand, the currently best known upper and lower bounds for rectilinear cartograms of 4-connected internally triangulated planar graphs are 8 and 6, respectively. Thus to find out whether the true bound in this case is 6 or 8 is open.

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