Alignment and Morphing
for the Boundary Curves of Anatomical Organs

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Abstract. In this paper, we develop a tracking method for the deformable boundary curves of biological organs using variational registration method. We first define the relative distortion of a pair of curves using curvatures of curves. This minimum distortion aligns corresponding points of a pair of curves. Then, we derive the mean of curves as the curve which minimises the total distortion of a collection of shapes. We compute the intermediate boundary curve of a pair of curves as the mean of these curves.

1   Introduction

Morphing is a fundamental technique in computer graphics to interpolate and generate shapes and objects. In medical application, morphing is used for the description of deformation process of biological organs. This process predicates deformable motion of biological organs in human torso such as beating heart, deformation of lungs during blessing. Follow up analysis of tumors in censer diagnosis tracks and predicates deformation of censer. In this paper, we develop a tracking method for the deformable boundary curves of biological organs using variational registration method. This registration process between images clarifies the difference between images which is used for medical diagnosis. This registration process is mainly achieved by the matching process, which is an established fundamental idea in pattern recognition. In both structure pattern recognition [4, 5] and variational registration [1, 3], the mean shape of a collection of given shapes is interested.

Shape retrieval categorises and classifies shapes, and finds shapes from portions of shapes. In shape retrieval, the matching of shapes based on the defomorphism of shapes [8, 9, 16, 19] and descriptor of shape boundary contours [10] are used. In the matching process for discrete shapes, the string edit-distance [4, 6] computed by dynamic programming is a fundamental tool. Moreover, in the matching process of images, the variational registration strategy [1, 2] is a typical tool. Since, in registration of grey-valued images, the deformation is assumed to be relatively small, the point correspondences between the target and reference images are estimated as a local deformation of images [1–3]. For the matching of planar curves, we are required to estimate both alignments and local deformation of curves. In this paper, we separate this problem into alignment estimation [16, 17] in the normalised set of curves and deformation of curves.

2   Alignment of Curves

For a pair of planar curves $\boldsymbol{c}(s) = (c_1(s), c_2(s))^\top$ and $\bar{\boldsymbol{c}}(s) = (\bar{c}_1(s), \overline{c}_2(s))^\top$, whose lengths are C and $\bar{C}$, respectively, assuming $\boldsymbol{c}(s+C) = \boldsymbol{c}(s)$ and $\overline{\boldsymbol{c}}(s+\bar{C}) = \bar{\boldsymbol{c}}(s)$, the alignment of curves is obtained as

$$\text{Align}(c, \bar{c}) = \min_{t, \psi} \int_0^C |\bar{\boldsymbol{c}}(\psi(s) - t) - \boldsymbol{c}(s)|^2 ds, \tag{1}$$

where $\psi(\tau)$ is a monotone function from the interval $[0, C]$ to the interval $[0, \bar{C}]$ [13, 14, 16]. The function $\tau = \psi(s)$ and the displacement t define the correspondences of points on a pair of curves $\boldsymbol{c}(s)$ and $\bar{\boldsymbol{c}}(\tau)$.

The dynamic time warping (DTW) is a fundamental procedure to achieve curve alignment employing dynamic programming [13]. The time warping sometimes maps a point on a curve to a relatively long interval of another curve. The derivative dynamic time warping technique (DDTW) [14], which computes alignment of derivative curves, solves this pathological mapping. Therefore, we can also use

$$\text{Align}(c, \overline{c}) = \min_{t, \psi} \int_0^C |\dot{\bar{\boldsymbol{c}}}(\psi(s) - t) - \dot{\boldsymbol{c}}(s)|^2 ds, \tag{2}$$

for the derivative of curves [14] $\dot{\boldsymbol{c}}$ and $\dot{\bar{\boldsymbol{c}}}$.

For a planar curve S_i, the normal curve s_i is the curve whose length are normalised to unity is a normalised curve. For the normal curve $\boldsymbol{x}(s)$ the unit normal vector is $\boldsymbol{n}(s) = (-\sin\theta(s), \cos\theta(s))^\top$, if $\dot{\boldsymbol{x}}(s)/|\dot{\boldsymbol{x}}(s)| = (\cos\theta(s), \sin\theta(s))^\top$. We call $\theta(s)$ the p-expression of the curve. The p-expression is invariant for Euclidean motion, that is, for curve $\boldsymbol{x}(s)$, $\boldsymbol{y}(s) = \boldsymbol{x}(s) + \boldsymbol{a}$ derives the same p-expression.

We **define** the log measure between two normal curves as

$$H(\theta_1, \theta_2) = \int_0^1 \left| \ln \frac{\exp(i\theta_1(s - t_1))}{\exp(i\theta_2(s - t_2))} \right|^2 ds, \tag{3}$$

using p-expression of each curve of a pair. Using p-expression of each curve of a pair, the p-distance of a pair of simple polygonal curve S_i and S_j is defined by

$$D(S_i, S_j)^2 = \min H(\theta_i, \theta_j) = \min_t \int_0^1 |\theta_i(s-t) - \theta_j(s)|^2 ds. \tag{4}$$

where θ_i and θ_j are the p-expressions of the normalised curve s_i and s_j of S_i and S_j, respectively. Then, setting

$$t_{ij} = \arg\left(\min_t \left(\int_0^1 |\theta_i(s-t_{ij}) - \theta_j(s)|^2 ds\right)\right), \tag{5}$$

we define the alignment of s_i and s_j as $\theta_i(s-t_{ij})$ and $\theta_j(s)$.

The p-expression $\theta(k)$ of a normalised polygonal curve, whose vertices are $\{x_k\}_{k=1}^n$, is computed as

$$\frac{x_k - x_{k-1}}{|x_k - x_{k-1}|} = (\cos\theta(k), \sin\theta(k))^\top. \tag{6}$$

Furthermore, setting $\theta_i^k = \theta_i(k\Delta)$, the distance between a pair of normalised curves is approximately computed by

$$d_{ij} \approx \min_p \sum_{k=1}^m |\theta_i^k - \theta_j^{k-p}|^2 \tag{7}$$

for an appropriately large m such that $\Delta m = 1$.

Next, we define the mean $\phi_{ij}(s)$ of a pair of p-expressions $\theta_i(s)$ and $\theta_j(s)$ as the minimiser of the functional

$$\begin{aligned}
J(\phi_i, \phi_j, \phi_{ij}, t_i, t_j) = \int_0^1 & \Big\{ |(\theta_i(s-t_i) - \phi_i(s)) - \phi_{ij}(s)|^2 \\
& + |(\theta_j(s-t_2) - \phi_j(s)) - \phi_{ij}(s)|^2 \\
& + \lambda|\dot\theta_{ij}(s)|^2 + \mu|\dot\phi_i(s)|^2 + \mu|\dot\phi_j(s)|^2 \Big\} ds.
\end{aligned} \tag{8}$$

Equation (8) is converted to the problem,

$$\begin{aligned}
J_2(\phi_{ij}) = \int_0^1 & \Big\{ |(\theta_i(s-t_{ij}) - \phi_i(s) - \phi_{ij}(s))|^2 \\
& + |(\theta_j(s) - \phi_j(s) - \phi_{ij}(s)|^2 \\
& + \lambda|\dot\phi_{ij}(s)|^2 + \mu|\dot\phi_i(s)|^2 + \mu|\dot\phi_j(s)|^2 \Big\} ds
\end{aligned} \tag{9}$$

since t_{ij} aligns a pair of p-expressions, for a generalisation of eq. (9), the initial points of a collection of curves are required to be aligned.
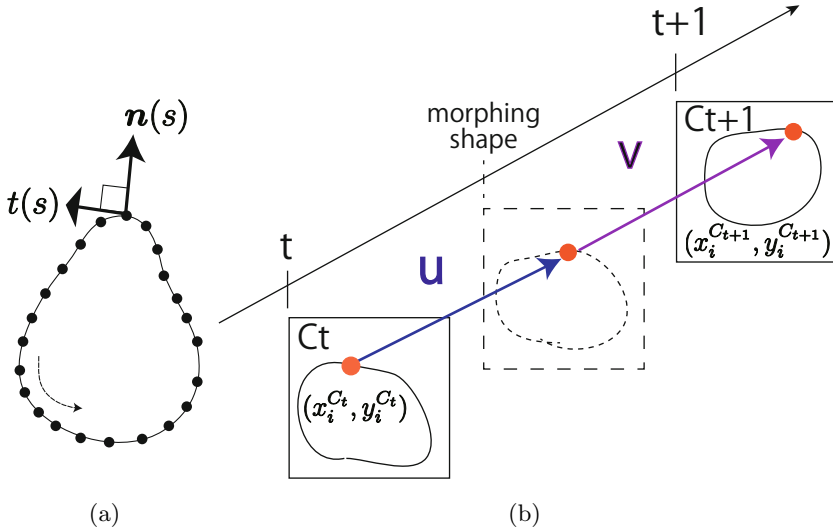
Fig. 1. Geometric property of the image boundary of curve. (a) Configuration of the normal and tangent vectors on a planar simple curve. (b) Morphing of temporal curves.

3   The Mean of Polygonal Curves

Assuming that the correspondence of the vertices of curves S_i and S_j in a collection of curves $\{S_\alpha\}_\alpha^n$ are established by minimising eq. (3), we define the distance between a pair of polygonal curves $S_i = \{\boldsymbol{f}_{ik}\}_{k=1}^n$ and $S_j = \{\boldsymbol{f}_{jk}\}_{k=1}^n$ where $\boldsymbol{f}_{\cdot k} = (x_{\cdot k}, y_{\cdot k})^\top$ is the vertex of the curve $S_{\cdot}$ with the condition $\boldsymbol{f}_{\cdot m+k} = \boldsymbol{f}_{\cdot k}$, as

$$
\begin{aligned}
d(S_i, S_j) &= \min_{\boldsymbol{u}_k^{ij}} \left\{ \sum_{k=1}^m \{(\boldsymbol{f}_{ik} - \boldsymbol{u}_k^{ij}) - \boldsymbol{f}_{jk}\}^2 + \mu \sum_{k=1}^m (\overline{\nabla} \boldsymbol{u}_k^{ij})^2 \right\} \\
&= \min_{\boldsymbol{u}_k^{ij}} \left\{ \sum_{k=1}^m \{\boldsymbol{f}_{ik} - (\boldsymbol{f}_{jk} - \boldsymbol{u}_k^{ji})\}^2 + \mu \sum_{k=1}^m (\overline{\nabla} \boldsymbol{u}_k^{ji})^2 \right\} \\
&= d(S_j, S_i),
\end{aligned}
\tag{10}
$$

where $\boldsymbol{u}_k^{ij} = -\boldsymbol{u}_k^{ji}$ is the displacement between $\boldsymbol{f}_{ik}$ and $\boldsymbol{f}_{jk}$ and $\overline{\nabla}$ stands for the discrete differential operation along a polygonal curve [1].

Definition 1. *Setting $\boldsymbol{u}_{ki}$ to be deformation of the vertex $\boldsymbol{f}_{ki}$ of the shape S_i, the vertices $\boldsymbol{g}$ of the mean curve S of S_i and S_j is the minimiser of the discrete variational problem*

$$
J(S_i(\boldsymbol{f}_i), S(\boldsymbol{g}), S_j(\boldsymbol{f}_j)) = d(S_i, S) + d(S, S_j) + \lambda P(S)
\tag{11}
$$

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[1] $\overline{\nabla} \boldsymbol{g}_k = \boldsymbol{g}_{k+\frac{1}{2}} - \boldsymbol{g}_{k-\frac{1}{2}}$ and $\overline{\nabla}^2 \boldsymbol{g}_k = \frac{1}{2}(\boldsymbol{g}_{k+1} - 2\boldsymbol{g}_k + \boldsymbol{g}_{k-1})$.

where

$$d(S_i, S) = \sum_{k=1}^{n} \{(f_{ik} - \boldsymbol{u}_{ik}) - \boldsymbol{g}_k\}^2 + \sum_{k=1}^{n} |\overline{\nabla}|\boldsymbol{u}_{ik}|^2,$$

$$d(S, S_j) = \sum_{k=1}^{n} \{(f_{jk} - \boldsymbol{u}_{jk}) - \boldsymbol{g}_k\}^2 + \sum_{k=1}^{n} |\overline{\nabla}|\boldsymbol{u}_{jk}|^2, \qquad (12)$$

$$P(S) = \sum_{k=1}^{n} |\overline{\nabla}\boldsymbol{g}_k|^2.$$

Setting

$$\boldsymbol{D} = \frac{1}{2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} \boldsymbol{I}_2 & \boldsymbol{e} \\ \boldsymbol{e}^\top & m \end{pmatrix}, \quad \boldsymbol{M} = Diag(\underbrace{\mu, \mu}, \lambda). \quad (13)$$

where $\boldsymbol{e} = (1, 1)^\top$, we have the Euler-Lagrange equation of eq. (13)

$$(\boldsymbol{I}_{m+1} \otimes \boldsymbol{D})\boldsymbol{s} = (\boldsymbol{M}^{-1}\boldsymbol{A} \otimes \boldsymbol{I}_2)\boldsymbol{s} - (\boldsymbol{M}^{-1} \otimes \boldsymbol{I}_n)\boldsymbol{c}, \qquad (14)$$

where $\boldsymbol{I}_k$ is the $k \times k$ identity matrix and

$$\boldsymbol{s} = (\boldsymbol{u}_i^\top, \boldsymbol{u}_j^\top, \boldsymbol{g}^\top)^\top, \quad \boldsymbol{c} = (\boldsymbol{f}_i^\top, \boldsymbol{f}_j^\top, \boldsymbol{f}_i^\top + \boldsymbol{f}_j^\top)^\top. \qquad (15)$$

Rewriting eq. (14) as

$$\boldsymbol{B}\boldsymbol{s} = \boldsymbol{K}\boldsymbol{s} - \boldsymbol{c}, \qquad (16)$$

the semi-implicit discretisation of the diffusion equation

$$\frac{\partial \boldsymbol{s}}{\partial t} = \boldsymbol{B}\boldsymbol{s} - \boldsymbol{K}\boldsymbol{s} + \boldsymbol{c} \qquad (17)$$

derives the iteration form

$$(\boldsymbol{I} + \tau\boldsymbol{K})\boldsymbol{s}^{(k+1)} = (\boldsymbol{I} + \tau\boldsymbol{B})\boldsymbol{s}^{(k)} + \tau\boldsymbol{c}. \qquad (18)$$

This iteration form implies that

$$\boldsymbol{g} = (\underbrace{\boldsymbol{0}_n, \cdots, \boldsymbol{0}_n}_{m}, \boldsymbol{I}_n)\boldsymbol{s}^{(\infty)}, \quad \boldsymbol{s}^{(\infty)} = \lim_{k \to \infty} \boldsymbol{s}^{(k)}. \qquad (19)$$

Setting $\boldsymbol{u}_{ti}$ and $\boldsymbol{u}_{(t+1)i}$ to be deformations from the vertices $\boldsymbol{f}_{ti}$ of the mean curve $S_{t+\frac{1}{2}}$ to vertices $\boldsymbol{f}_{ti}$ and $\boldsymbol{f}_{(t+1)i}$ of S_t and S_{t+1}, respectively, we define an interframe curve. Fig. 2 shows morphing and tracking of a temporal curve sequence using variational mean curve.

Definition 2. *The vertices $\boldsymbol{f}_{t+\frac{1}{2}}$ of the intermediate shape $S_{t+\frac{1}{2}}$ is the minimiser of the discrete variational problem*

$$J(S_t(\boldsymbol{f}_t), S_{t+\frac{1}{2}}(\boldsymbol{f}_{t+\frac{1}{2}}), S_{t+1}(\boldsymbol{f}_{t+1})) = D(S_t, S_{t+\frac{1}{2}}) + D(S_{t+\frac{1}{2}}, S_{t+1}) + \lambda P(S_{t+\frac{1}{2}}) \tag{20}$$

where

$$D(S_t, S_{t+\frac{1}{2}}) = \sum_{k=1}^{n}\{(f_{ik} - \boldsymbol{u}_{ik}) - \boldsymbol{g}_k\}^2 + \sum_{k=1}^{n}|\overline{\nabla}|\boldsymbol{u}_{ik}|^2,$$

$$D(S_{t+\frac{1}{2}}, S_{t+1}) = \sum_{k=1}^{n}\{(f_{jk} - \boldsymbol{u}_{jk}) - \boldsymbol{g}_k\}^2 + \sum_{k=1}^{n}|\overline{\nabla}|\boldsymbol{u}_{jk}|^2, \tag{21}$$

$$P(S_{t+\frac{1}{2}}) = \sum_{k=1}^{n}|\overline{\nabla}\boldsymbol{g}_k|^2.$$

4   Numerical Examples

Fig. 2 shows Images of the boundary curves of tumors observed in the years, 1996, 19980, 2000, 2002, and 2004.

Fig. 3 comparative results between the means and the linear averages of the years Y. Y^*, and $Y^\diamond$ express the images of the year Y computed by our method and linear average of corresponding vertices of the curve. In Fig. 4, we have evaluated the distances $D(Y, Y^*)$ and $D(Y, Y^\diamond)$. These results show our method derives smooth intermediate curves, by computing the alignment of corresponding vertices and by minimising the relative distortion of curves.
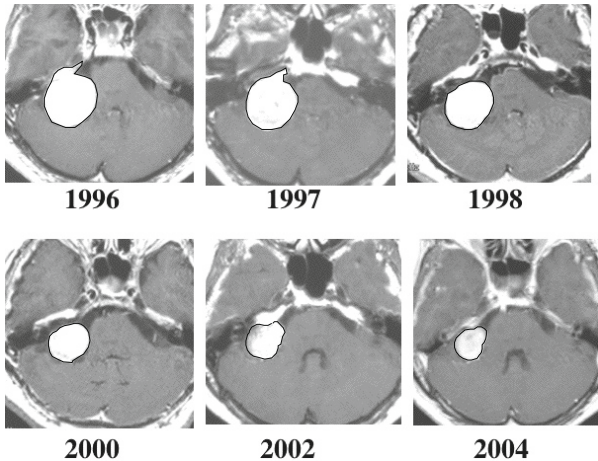

Fig. 2. Images of the boundary curves of tumors. The boundary curves observed in the years, 1996, 19980, 2000, 2002, and 2004.
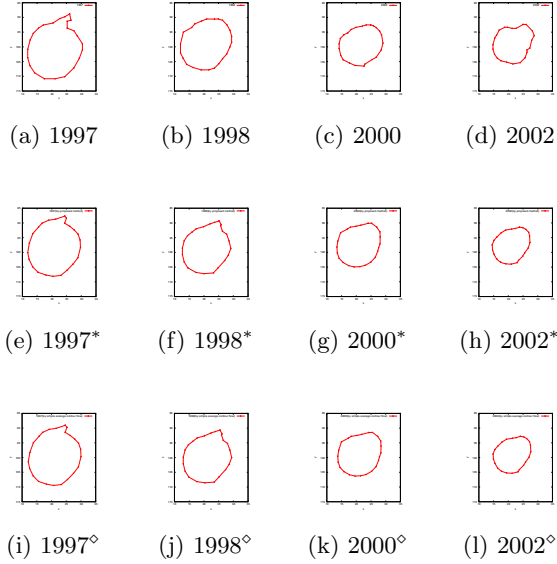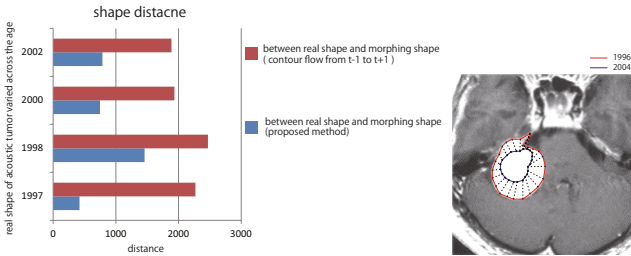
(a) 1997   (b) 1998   (c) 2000   (d) 2002

(e) 1997*   (f) 1998*   (g) 2000*   (h) 2002*

(i) 1997$^\diamond$ (j) 1998$^\diamond$ (k) 2000$^\diamond$ (l) 2002$^\diamond$

Fig. 3. Performance evaluation. from top to bottom the original images, the means, and the linear averages of the years Y. Y^*, and $Y^\diamond$ express the images of the year Y computed by our method and linear average of corresponding vertices of the curve.

Fig. 4. Evaluation. (a) For the results Fig. 3, we have evaluated the distances $D(Y, Y^*)$ and $D(Y, Y^\diamond)$. (b) Tracking of corresponding vertices between 1996 data and 2004 data.

Fig.5 shows shape morphing results. From the boundary curves observe by the years, 1996, 1998, 2000, 2002, and 2004, the boundary curves of the years $1996 + \frac{1}{2}$, $1997 + \frac{1}{2}$, $1998 + 1$, $2000 + 1$, and $2002 + 1$ are computed. Fig. 4 (b) shows the result for tracking of corresponding vertices between 1996 data and 2004 data. The results show that our method estimates intermediate tumor shapes during the therapy.
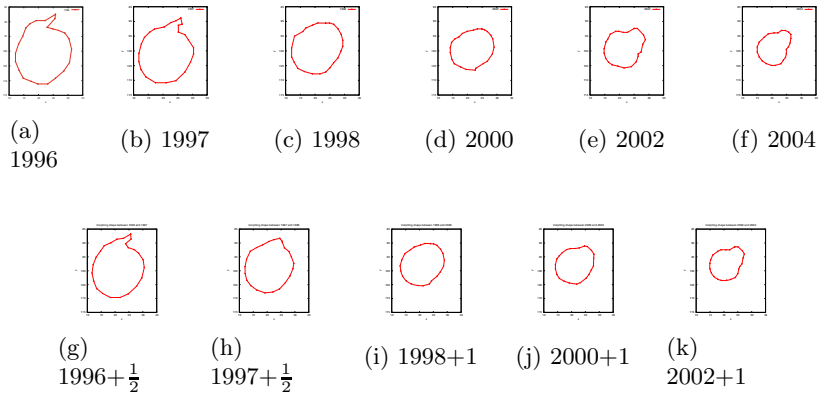
Fig. 5. Shape morphing results. from the boundary curves observe by the years, 1996, 19980, 2000, 2002, and 2004, the boundary curves of the years $1996 + \frac{1}{2}$, $1997 + \frac{1}{2}$, $1898 + 1$, $2000 + 1$, and $2002 + 1$ are computed.

5   Conclusions

We first define the relative distortion of a pair of curves using curvatures of curves. This minimum distortion aligns corresponding points of a pair of curves. Then, we derive the mean of curves as the curve which minimises the total distortion of a collection of shapes. We compute the intermediate boundary curve of a pair of curves as the mean of these curves.

Our method automatically detects and tracks corresponding vertices of the temporal-deformation curves.

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