Bicategories of Concurrent Games
(Invited Paper)

Glynn Winskel

University of Cambridge Computer Laboratory, England
Glynn.Winskel@cl.cam.ac.uk
http://www.cl.cam.ac.uk/users/gw104

Abstract. This paper summarises recent results on bicategories of concurrent games and strategies. Nondeterministic concurrent strategies, those nondeterministic plays of a game left essentially unchanged by composition with copy-cat strategies, have recently been characterized as certain maps of event structures. This leads to a bicategory of general concurrent games in which the maps are nondeterministic concurrent strategies. It is shown how the bicategory can be refined to a bicategory of winning strategies by adjoining winning conditions to games. Assigning "access levels" to moves addresses situations where Player or Opponent have imperfect information as to what has occurred in the game. Finally, a bicategory of deterministic "linear" strategies, a recently discovered model of MALL (multiplicative-additive linear logic), is described. All the bicategories become equivalent to simpler order-enriched categories when restricted to deterministic strategies.

Keywords: Games, strategies, concurrency, event structures, winning conditions, determinacy.

1 Introduction

Games and strategies are everywhere, in logic, philosophy, computer science, economics, leisure and in life. As abundant, but much less understood, are *concurrent* games in which a Player (or team of players) compete against an Opponent (or team of opponents) in a highly interactive and distributed fashion, especially when we recognize that the dichotomy Player vs. Opponent has several readings, as for example, process vs. environment, proof vs. refutation, or more ominously as ally vs. enemy. This paper summarises recent results on the mathematical foundations of concurrent games. It describes what it means to be a concurrent game, a concurrent strategy, a winning strategy, a concurrent game of imperfect information, and a linear strategy, and generally illustrates the rich mathematical structure concurrency brings to games.

Our primary motivation has come from the semantics of computation and the role of games in logic, although games are situated at a crossing point of several areas. In semantics it is becoming clear that we need an *intensional* theory to capture the *ways* of computing, to near operational and algorithmic

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concerns. Sometimes unexpected intensionality is forced through the demands of compositionality, *e.g.* in *nondeterministic dataflow* [1]. More to the point we need to repair the artificial division between *denotational* and *operational* semantics. But what is to be our fundamental model of processes? Game semantics provides a possible answer: *strategies.* (There are others, *e.g.* profunctors as maps between presheaf categories [2,3].) Meanwhile in logic the well-known Curry-Howard correspondence "propositions as types, proofs as programs" is being recast as "propositions as games, proofs as strategies."

However, in both semantics and logic, traditional definitions of strategies and games are not general enough: they do not adequately address the concurrent nature of computation and proof—see *e.g.* [4]. Game semantics has developed from simple sequential games, where only one move is allowed at a time and, for instance, it is often assumed that the moves of Player and Opponent alternate. Because of its history it is not obvious how to extend traditional game semantics to concurrent computation, or what relation it bears to other generalised domain theories such as those where domains are presheaf categories [2,3]. It is time to build game semantics on a broader foundation, one more squarely founded within a general model for concurrent processes. The standpoint of this paper is to base games and strategies on event structures, the analogue of trees but in a concurrent world; just as transition systems, an "interleaving" model, unfold to trees so do Petri nets, a "concurrent" model, unfold to event structures. In doing so we re-encounter earlier work of Abramsky and Melliès, first in their presentation of deterministic concurrent strategies as closure operators, and later in Melliès programme of *asynchronous games*, culminating in his definition with Mimram of ingenuous strategies; a consequence of the work described here is a characterization of Melliès and Mimram's *receptive* ingenuous strategies [5] as precisely those deterministic pre-strategies for which copy-cat strategies behave as identities.

Our slogan: processes are nondeterministic concurrent strategies. For methodology we adopt ideas of Joyal who recognized that there was a category of games underlying Conway's construction of the "surreal numbers" [6,7]. Like many 2-party games Conway's games support two important operations: a form of parallel composition $G\|H$; a dualizing operation $G^\perp$ which reverses the roles of Player and Opponent in G. Joyal defined a strategy σ *from* a game G *to* a game H, to be a strategy σ in $G^\perp\|H$. Following Conway's method of proof, Joyal showed that strategies compose, with identities given by copy-cat strategies.

We shall transport the pattern established by Joyal to a general model for concurrent computation: games will be represented by event structures and strategies as certain maps into them. The motivation is to obtain: forms of generalised domain theory in which domains are replaced by concurrent games and continuous functions by nondeterministic concurrent strategies; operations, including higher-order operations via "function spaces" $G^\perp\|H$, within a model for concurrency; techniques for logic (via proofs as concurrent strategies), and possibly verification and algorithmics. However, first things first, here we will concentrate on the rich algebra of concurrent strategies. Most proofs and background on stable families, on which proofs often rely, can be found in [8].

2 Event Structures

An *event structure* comprises $(E, \text{Con}, \leq)$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$\begin{aligned}
&\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\
&\{e\} \in \text{Con for all } e \in E,\\
&Y \subseteq X \in \text{Con} \implies Y \in \text{Con}, \quad \text{and}\\
&X \in \text{Con} \ \& \ e \leq e' \in X \implies X \cup \{e\} \in \text{Con}.
\end{aligned}$$

The *configurations*, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \text{Con}$, and
Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

Often we shall be concerned with just the finite configurations of an event structure. We write $\mathcal{C}(E)$ for the *finite* configurations of an event structure E.

Two events which are both consistent and incomparable w.r.t. causal dependency in an event structure are regarded as *concurrent*. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, will play a very important role. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X; note if $X \in \text{Con}$, then $[X] \in \text{Con}$.

Notation 1. Let E be an event structure. We use $x {-\!\!\subset} y$ to mean y covers x in $\mathcal{C}^\infty(E)$, *i.e.* $x \subset y$ in $\mathcal{C}^\infty(E)$ with nothing in between, and $x \stackrel{e}{-\!\!\!-\!\!\subset} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{C}^\infty(E)$ and event $e \notin x$. We sometimes use $x \stackrel{e}{-\!\!\!-\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \stackrel{e}{-\!\!\!-\!\!\subset} y$ for some y.

2.1 Maps of Event Structures

Let E and E' be event structures. A *(partial) map* of event structures $f : E \rightharpoonup E'$ is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$ its direct image $fx \in \mathcal{C}(E')$ and

$$\text{if } e_1, e_2 \in x \text{ and } f(e_1) = f(e_2) \text{ (with both defined), then } e_1 = e_2.$$

The map expresses how the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined. Partial maps of event structures compose as partial functions, with identity maps given by identity functions.

For any event e a map of event structures $f : E \rightharpoonup E'$ must send the configuration $[e]$ to the configuration $f[e]$. Partial maps preserve the concurrency relation, when defined.

We will say the map is *total* if the function f is total. Notice that for a total map f the condition on maps now says it is *locally injective*, in the sense that w.r.t. any configuration x of the domain the restriction of f to a function from x is injective; the restriction of f to a function from x to fx is thus bijective. A partial map of event structures which preserves causal dependency whenever it is defined, *i.e.* $e' \leq e$ implies $f(e') \leq f(e)$ whenever both $f(e')$ and $f(e)$ are defined, is called *partial rigid*. We reserve the term *rigid* for those total maps of event structures which preserve causal dependency.

2.2 Process Operations

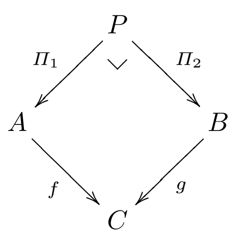
Products. The category of event structures with partial maps has *products* $A \times B$ with projections Π_1 to A and Π_2 to B. The effect is to introduce arbitrary synchronisations between events of A and events of B in the manner of process algebra.

Restriction. The restriction of an event structure E to a subset of events R, written $E \upharpoonright R$, is the event structure with events $E' = \{e \in E \mid [e] \subseteq R\}$ and causal dependency and consistency induced by E.

Synchronized Compositions and Pullbacks. Synchronized compositions play a central role in process algebra, with such seminal work as Milner's CCS and Hoare's CSP. Synchronized compositions of event structures A and B are obtained as restrictions $A \times B \upharpoonright R$. We obtain *pullbacks* as a special case. Let $f : A \to C$ and $g : B \to C$ be maps of event structures. Defining

$$P =_{\text{def}} A \times B \upharpoonright \{p \in A \times B \mid f\Pi_1(p) = g\Pi_2(p) \text{ with both defined}\}$$

we obtain a pullback square



in the category of event structures. When f and g are total the same construction gives the pullback in the category of event structures with *total* maps.

2.3 Projection

Let $(E, \leq, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$.

3 Event Structures with Polarities

Both a game and a strategy in a game are to be represented as an event structure with polarity, which comprises (E, pol) where E is an event structure with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity $+$ (Player) or $-$ (Opponent) to its events. The events correspond to (occurrences of) moves. Maps of event structures with polarity are maps of event structures which preserve polarity.

3.1 Operations

Dual. The *dual*, $E^\perp$, of an event structure with polarity E comprises a copy of the event structure E but with a reversal of polarities.

Simple Parallel Composition. The operation $A \| B$ simply forms the disjoint juxtaposition of A, B, two event structures with polarity; a finite subset of events is consistent if its intersection with each component is consistent.

4 Pre-strategies

Let A be an event structure with polarity, thought of as a game; its events stand for the possible occurrences of moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game. A pre-strategy represents a nondeterministic play of the game—all its moves are moves allowed by the game and obey the constraints of the game; the concept will later be refined to that of *strategy* (and *winning strategy* in Section 7). A *pre-strategy* in A is defined to be a total map $\sigma : S \to A$ from an event structure with polarity S. Two pre-strategies $\sigma : S \to A$ and $\tau : T \to A$ in A will be essentially the same when they are isomorphic, *i.e.* there is an isomorphism $\theta : S \cong T$ such that $\sigma = \tau\theta$; then we write $\sigma \cong \tau$.

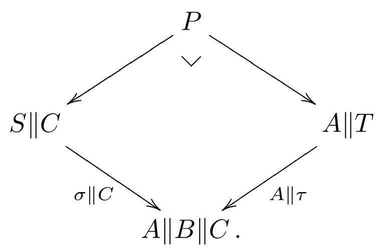
Let A and B be event structures with polarity. Following Joyal [7], a pre-strategy from A to B is a pre-strategy in $A^\perp \| B$, so a total map $\sigma : S \to A^\perp \| B$. It thus determines a span

$$A^\perp \xleftarrow{\ \sigma_1\ } S \xrightarrow{\ \sigma_2\ } B\,,$$

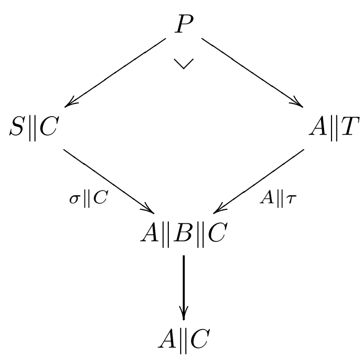
of event structures with polarity where σ_1, σ_2 are *partial* maps and for all $s \in S$ either, but not both, $\sigma_1(s)$ or $\sigma_2(s)$ is defined. Two pre-strategies from A to B will be isomorphic when they are isomorphic as pre-strategies in $A^\perp \| B$, or equivalently are isomorphic as spans. We write $\sigma : A \mathrel{-\!\!\!+\!\!\!\!\rightarrow} B$ to express that σ is a pre-strategy from A to B. Note a pre-strategy σ in a game A coincides with a pre-strategy from the empty game $\sigma : \emptyset \mathrel{-\!\!\!+\!\!\!\!\rightarrow} A$.

4.1 Composing Pre-strategies

We can present the composition of pre-strategies via pullbacks.[1] Given two pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$, ignoring polarities we can consider the maps on the underlying event structures, *viz.* $\sigma : S \to A \| B$ and $\tau : T \to B \| C$. Viewed this way we can form the pullback in the category of event structures



There is an obvious partial map of event structures $A \| B \| C \to A \| C$ undefined on B and acting as identity on A and C. The partial map from P to $A \| C$ given by following the diagram (either way round the pullback square)



factors as the composition of the partial map $P \to P \downarrow V$, where V is the set of events of P at which the map $P \to A \| C$ is defined, and a total map $P \downarrow V \to A \| C$. The resulting total map gives us the composition $\tau \odot \sigma : P \downarrow V \to A^\perp \| C$ once we reinstate polarities.

[1] The construction here gives the same result as that via synchronized composition in [9]— I'm grateful to Nathan Bowler for this observation. Notice the analogy with the composition of relations $S \subseteq A \times B$, $T \subseteq B \times C$ which can be defined as $T \circ S = (S \times C \cap A \times T) \downarrow A \times C$, the image of $S \times C \cap A \times T$ under the projection of $A \times B \times C$ to $A \times C$.

4.2 Concurrent Copy-Cat

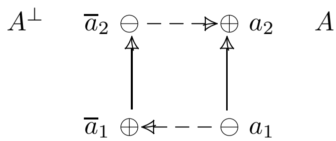
Identities w.r.t. composition are given by copy-cat strategies. Let A be an event structure with polarity. The copy-cat strategy from A to A is an instance of a pre-strategy, so a total map $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$. It describes a concurrent, or distributed, strategy based on the idea that Player moves, of +ve polarity, always copy previous corresponding moves of Opponent, of −ve polarity.

For $c \in A^\perp \| A$ we use $\overline{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component. Define $\mathbb{C}\mathbb{C}_A$ to comprise the event structure with polarity $A^\perp \| A$ together with extra causal dependencies $\overline{c} \leq_{\mathbb{C}\mathbb{C}_A} c$ for all events c with $pol_{A^\perp \| A}(c) = +$.

Proposition 1. *Let A be an event structure with polarity. Then event structure with polarity $\mathbb{C}\mathbb{C}_A$ is an event structure. Moreover, $x \in \mathcal{C}(\mathbb{C}\mathbb{C}_A)$ iff $x \in \mathcal{C}(A^\perp \| A)$ & $\forall c \in x.\ pol_{A^\perp \| A}(c) = + \implies \overline{c} \in x$.*

The *copy-cat* pre-strategy $\gamma_A : A \mathrel{+\!\!\!\!\rightarrow} A$ is defined to be the map $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$ where γ_A is the identity on the common set of events.

Example 1. We illustrate the construction of the copy-cat strategy for the event structure A comprising the single immediate dependency $a_1 \rightarrowtriangle a_2$ from an Opponent move a_1 to a Player move a_2. The event structure $\mathbb{C}\mathbb{C}_A$ is obtained from $A^\perp \| A$ by adjoining the additional immediate dependencies shown:



5 Strategies

The main result of [9] is that two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient for copy-cat to behave as identity w.r.t. the composition of pre-strategies. Receptivity ensures an openness to all possible moves of Opponent. Innocence restricts the behaviour of Player; Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$ beyond those imposed by the game.

Receptivity. A pre-strategy σ is *receptive* iff $\sigma x \overset{a}{-\!\!\!\subset}$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{-\!\!\!\subset}$ & $\sigma(s) = a$.

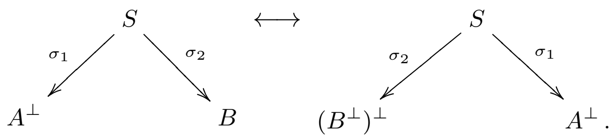
Innocence. A pre-strategy σ is *innocent* when it is both
$+$-*innocent:* if $s \rightarrowtriangle s'$ & $pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$, and
$-$-*innocent:* if $s \rightarrowtriangle s'$ & $pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

Theorem 1. *Let $\sigma : A \mathrel{+\!\!\!\!\rightarrow} B$ be pre-strategy. Copy-cat behaves as identity w.r.t. composition, i.e. $\sigma \circ \gamma_A \cong \sigma$ and $\gamma_B \circ \sigma \cong \sigma$, iff σ is receptive and innocent. Copy-cat pre-stategies $\gamma_A : A \mathrel{+\!\!\!\!\rightarrow} A$ are receptive and innocent.*

5.1 The Bicategory of Concurrent Games and Strategies

Theorem 1 motivates the definition of a *strategy* as a pre-strategy which is receptive and innocent. In fact, we obtain a bicategory, **Games**, in which the objects are event structures with polarity—the games, the arrows from A to B are strategies $\sigma : A \,+\!\!\!\!\rightarrow B$ and the 2-cells are maps of spans. The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the universality of pullback).

A strategy $\sigma : A \,+\!\!\!\!\rightarrow B$ corresponds to a dual strategy $\sigma^\perp : B^\perp \,+\!\!\!\!\rightarrow A^\perp$. This duality arises from the correspondence



The dual of copy-cat, $\gamma_A^\perp$, is isomorphic to the copy-cat of the dual, $\gamma_{A^\perp}$, for A an event structure with polarity. The dual of a composition of pre-strategies $(\tau\odot\sigma)^\perp$ is isomorphic to the composition $\sigma^\perp\odot\tau^\perp$. This duality is maintained in the major bicategories of games we shall consider.

One notable sub-bicategory of games, though one not maintaining duality, is obtained on restricting to objects which comprise purely +ve events: then we obtain the bicategory of stable spans, which have played a central role in the semantics of nondeterministic dataflow [1].

5.2 The Subcategory of Deterministic Strategies

Say an event structure with polarity S is *deterministic* iff

$$\forall X \subseteq_{\text{fin}} S.\ Neg[X] \in \text{Con}_S \implies X \in \text{Con}_S\,,$$

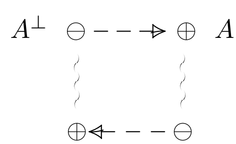
where $Neg[X] =_{\text{def}} \{s' \in S \mid pol(s') = - \ \&\ \exists s \in X.\ s' \le s\}$. In other words, S is deterministic iff any finite set of moves is consistent when it causally depends only on a consistent set of opponent moves. Say a strategy $\sigma : S \to A$ is deterministic if S is deterministic.

Lemma 1. *An event structure with polarity S is deterministic iff*

$$\forall s, s' \in S, x \in \mathcal{C}(S).\ \ x \overset{s}{\longrightarrow\!\!\!\!\subset}\ \&\ x \overset{s'}{\longrightarrow\!\!\!\!\subset}\ \&\ pol(s) = + \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

In general, a copy-cat strategy can fail to be deterministic, illustrated below.

Example 2. Take A to consist of two events, one +ve and one −ve event, inconsistent with each other (indicated by the wiggly line). The construction $\mathbb{C}\mathbb{C}_A$:



To see $\mathbb{C}\!\mathbb{C}_A$ is not deterministic, take x to be the singleton set consisting *e.g.* of the $-$ve event on the left and s, s' to be the $+$ve and $-$ve events on the right.

Copy-cat γ_A is deterministic iff immediate conflict in A respects polarity, or equivalently that there is no immediate conflict between $+$ve and $-$ve events, a condition we call *race-free.*

Lemma 2. *Let A be an event structure with polarity. The copy-cat strategy γ_A is deterministic iff A is race-free, i.e. for all $x \in \mathcal{C}(A)$,*

$$x \overset{a}{\longrightarrow\!\subset} \;\&\; x \overset{a'}{\longrightarrow\!\subset} \;\&\; pol(a) = + \;\&\; pol(a') = - \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,.$$

Lemma 3. *The composition of deterministic strategies is deterministic.*

Lemma 4. *A deterministic strategy $\sigma : S \to A$ is injective on configurations (equivalently, σ is mono in the category of event structures with polarity).*

We obtain a sub-bicategory **DGames** of **Games** by restricting objects to race-free games and strategies to being deterministic. Via Lemma 4, deterministic strategies in a game correspond to certain subfamilies of configurations of the game. A characterization of those subfamilies which correspond to deterministic strategies shows them to coincide with the *receptive* ingenuous strategies of Mimram and Melliès [5]. This work grew out of Abramsky and Melliès early work in which deterministic concurrent strategies are presented, essentially, as partial closure operators on the domain of configurations of an event structure [4]. Via the presentation of deterministic strategies as families **DGames** is equivalent to an order-enriched category. There are notable subcategories: when the objects are countable event structures with polarity which consist of purely $+$ve events we recover as a full subcategory the classical category of stable domain theory, *viz.* Berry's dI-domains and stable functions; this in turn has Girard's qualitative domains and coherence spaces, both with stable functions, as full subcategories [10]. The category of simple games [11,12], underlying both HO and AJM games, is a subcategory, though not full.

6 From Strategies to Profunctors

Let x and x' be configurations of an event structure with polarity. Write $x \subseteq^+ x'$ to mean $x \subseteq x'$ and $pol(x' \setminus x) \subseteq \{+\}$, *i.e.* the configuration x' extends the configuration x solely by events of $+$ve polarity. Similarly $x \subseteq^- x'$ means configuration x' extends x solely by events of $-$ve polarity. With this notation in place we can give an attractive characterization of concurrent strategies:

Lemma 5. *A strategy S in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that* *(i) whenever $y \subseteq^+ \sigma x$ in $\mathcal{C}(A)$ there is a (necessarily unique) $x' \in \mathcal{C}(S)$ so that $x' \subseteq x$ & $\sigma x' = y$, i.e.*

$$\begin{array}{ccc} x' & \cdots\subseteq\cdots & x \\ \sigma \, \vdots & & \downarrow \, \sigma \\ y & \subseteq^+ & \sigma x \,, \end{array}$$

and

(ii) whenever $\sigma x \subseteq^- y$ in $\mathcal{C}(A)$ there is a unique $x' \in \mathcal{C}(S)$ so that $x \subseteq x'$ & $\sigma x' = y$, i.e.

$$\begin{array}{ccc} x & \cdots\subseteq\cdots & x' \\ \sigma \, \downarrow & & \vdots \, \sigma \\ \sigma x & \subseteq^- & y \,. \end{array}$$

The above lemma tells us how to form a discrete fibration, so presheaf, from a strategy. For A, an event structure with polarity, we can define a new order, the *Scott* order, between configurations $x, y \in \mathcal{C}^\infty(A)$, by

$$x \sqsubseteq_A y \iff x \supseteq^- x \cap y \subseteq^+ y \,.$$

Proposition 2. *Let $\sigma : S \to A$ be a pre-strategy in game A. The map σ“ taking a finite configuration $x \in \mathcal{C}(S)$ to $\sigma x \in \mathcal{C}(A)$ is a discrete fibration from $(\mathcal{C}(S), \sqsubseteq_S)$ to $(\mathcal{C}(A), \sqsubseteq_A)$ iff σ is a strategy.*

As discrete fibrations correspond to presheaves, an alternative reading of Proposition 2 is that a pre-strategy $\sigma : S \to A$ is a strategy iff σ“ determines a presheaf over $(\mathcal{C}(A), \sqsubseteq_A)$.

Consequently, a strategy $\sigma : A \,{+\!\!\!\!\longrightarrow}\, B$ determines a discrete fibration over $(\mathcal{C}(A^\perp \| B), \sqsubseteq_{A^\perp \| B})$. But

$$(\mathcal{C}(A^\perp \| B), \sqsubseteq_{A^\perp \| B}) \cong (\mathcal{C}(A^\perp), \sqsubseteq_{A^\perp}) \times (\mathcal{C}(B), \sqsubseteq_B) \cong (\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B) \,,$$

so σ determines a presheaf over $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$, *i.e.* a *profunctor*

$$\sigma\text{“} : (\mathcal{C}(A), \sqsubseteq_A) \,{+\!\!\!\!\longrightarrow}\, (\mathcal{C}(B), \sqsubseteq_B) \,.$$

The operation σ“, on a strategy σ, forms a *lax* functor from **Games** to **Prof**, the bicategory of profunctors: whereas it preserves identities, it is *not* the case that $(\tau \odot \sigma)$“ and τ“$\circ \sigma$“ coincide up to isomorphism; the profunctor composition τ“ $\circ\, \sigma$“ will generally contain extra “unreachable” elements.

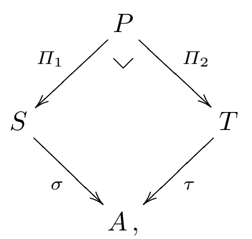
However, in special cases composition is preserved up to isomorphism. Say a strategy σ is *partial rigid* when the components σ_1, σ_2 are partial-rigid maps of event structures (with polarity). Partial-rigid strategies form a sub-bicategory of **Games**—see Section 9. For composable partial-rigid strategies σ and τ we do have $(\tau \odot \sigma)$“ $\cong \tau$“$\circ \sigma$“. Stable spans and simple games lie within the bicategory partial-rigid strategies.

7 Winning Strategies

A *game with winning conditions* comprises $G = (A, W)$ where A is an event structure with polarity and $W \subseteq \mathcal{C}^\infty(A)$ consists of the *winning configurations* for Player. We define the *losing conditions* to be $\mathcal{C}^\infty(A) \setminus W$.[2]

A strategy in G is a strategy in A. A strategy in G is regarded as *winning* if it always prescribes Player moves to end up in a winning configuration, no matter what the activity or inactivity of Opponent. Formally, a strategy $\sigma : S \to A$ in G is *winning (for Player)* if $\sigma x \in W$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$— a configuration x is +-maximal if whenever $x \overset{s}{\longrightarrow\!\!\!\!\subset}$ then the event s has −ve polarity. Any achievable position $z \in \mathcal{C}^\infty(S)$ of the game can be extended to a +-maximal, so winning, configuration (via Zorn's Lemma). So a strategy prescribes Player moves to reach a winning configuration whatever state of play is achieved following the strategy. Note that for a game A, if winning conditions $W = \mathcal{C}^\infty(A)$, *i.e.* every configuration is winning, then any strategy in A is a winning strategy.

Informally, we can also understand a strategy as winning for Player if when played against any counter-strategy of Opponent, the final result is a win for Player. Suppose $\sigma : S \to A$ is a strategy in a game (A, W). A counter-strategy is strategy of Opponent, so a strategy $\tau : T \to A^\perp$ in the dual game. We can view σ as a strategy $\sigma : \emptyset \mathrel{-\!\!\!+\!\!\!\!\!\rightarrow} A$ and τ as a strategy $\tau : A \mathrel{-\!\!\!+\!\!\!\!\!\rightarrow} \emptyset$. Their composition $\tau \odot \sigma : \emptyset \mathrel{-\!\!\!+\!\!\!\!\!\rightarrow} \emptyset$ is not in itself so informative. Rather it is the status of the configurations in $\mathcal{C}^\infty(A)$ their full interaction induces which decides which of Player or Opponent wins. Ignoring polarities, we have total maps of event structures $\sigma : S \to A$ and $\tau : T \to A$. Form their pullback,



to obtain the event structure P resulting from the interaction of σ and τ. Because σ or τ may be nondeterministic there can be more than one maximal configuration z in $\mathcal{C}^\infty(P)$. A maximal configuration z in $\mathcal{C}^\infty(P)$ images to a configuration $\sigma \Pi_1 z = \tau \Pi_2 z$ in $\mathcal{C}^\infty(A)$. Define the set of *results* of the interaction of σ and τ to be

$$\langle \sigma, \tau \rangle =_{\text{def}} \{\sigma \Pi_1 z \mid z \text{ is maximal in } \mathcal{C}^\infty(P)\}\,.$$

It can be shown that a strategy σ is a winning for Player iff all the results of the interaction $\langle \sigma, \tau \rangle$ lie within the winning configurations W, for any counter-strategy $\tau : T \to A^\perp$ of Opponent.

[2] It is fairly straightforward to generalize to the situation where configurations may be neutral, neither winning nor losing [13,8].

7.1 Operations

There is an obvious *dual* of a game with winning conditions $G = (A, W_G)$:

$$G^{\perp} = (A^{\perp}, \mathcal{C}^{\infty}(A) \setminus W_G)\,,$$

reversing the role of Player and Opponent, and consequently that of winning and losing conditions.

The parallel composition of two games with winning conditions $G = (A, W_G)$, $H = (B, W_H)$ is

$$G \parr H =_{\rm def} (A \| B,\ W_{G \parr H})$$

where, for $x \in \mathcal{C}^{\infty}(A\|B)$,

$$x \in W_{G \parr H} \ \text{ iff } \ x_1 \in W_G \text{ or } x_2 \in W_H$$

—a configuration x of $A\|B$ comprises the disjoint union of a configuration x_1 of A and a configuration x_2 of B. To win in $G \parr H$ is to win in either game. The unit of $\|$ is $(\emptyset, \emptyset)$. Defining $G \otimes H =_{\rm def} (G^{\perp} \| H^{\perp})^{\perp}$ we obtain a game where to win is to win in both games G and H. The unit of $\otimes$ is $(\emptyset, \{\emptyset\})$.

Defining $G \multimap H =_{\rm def} G^{\perp} \parr H$, a win in $G \multimap H$ is a win in H conditional on a win in G: For $x \in \mathcal{C}^{\infty}(A^{\perp}\|B)$,

$$x \in W_{G \multimap H} \ \text{ iff } \ \ x_1 \in W_G \implies x_2 \in W_H\,.$$

7.2 The Bicategory of Winning Strategies

We can again follow Joyal and define strategies between games now with winning conditions: a (winning) strategy from G, a game with winning conditions, to another H is a (winning) strategy in $G \multimap H$. We compose strategies as before. The composition of winning strategies is winning. However, for a general game with winning conditions (A, W) the copy-cat strategy need not be winning, as shown in the following example.

Example 3. Let A be the event structure with polarity of Example 2. Take as winning conditions the set $\{\{\oplus\}\}$. To see $\mathrm{C\!\!C}_A$ is not winning consider the configuration x consisting of the two $-$ve events in $\mathrm{C\!\!C}_A$. Then x is $+$-maximal as any $+$ve event is inconsistent with x. However, $x_1 \in W$ while $x_2 \notin W$, failing the winning condition of $(A, W) \multimap (A, W)$.

Recall from Section 6, that each event structure with polarity$A$ possesses a Scott order on its configurations $\mathcal{C}^{\infty}(A)$: $x' \sqsubseteq x$ iff $x' \supseteq^{-} x \cap x' \subseteq^{+} x$. With it we can express a necessary and sufficient for copy-cat to be winning w.r.t. a game (A, W):

$$\forall x, x' \in \mathcal{C}^{\infty}(A).\ \text{ if } x' \sqsubseteq x \ \&\ x' \text{ is } +\text{-maximal } \&\ x \text{ is } -\text{-maximal, then } \ x \in W \implies x' \in W\,. \qquad \textbf{(Cwins)}$$

The condition **(Cwins)** is assured for event structures with polarity which are race-free.

We can now refine the bicategory of strategies **Games** to the bicategory **WGames** with objects games with winning conditions $G, H, \cdots$ satisfying (**Cwins**) and arrows winning strategies $G \rightarrow\!\!\!\!+\!\!\!\!\rightarrow H$; 2-cells, their vertical and horizontal composition are as before. Its restriction to deterministic strategies yields a bicategory equivalent to a simpler order-enriched category.

7.3 Applications

As an application of winning conditions we apply them to pick out a subcategory of "total strategies," informally strategies in which Player can always answer a move of Opponent [14,11]—see [8] for details. Often problems can be reduced to whether Player or Opponent has a winning strategy, for which it is important to know when concurrent games are *determined*, *i.e.* either Player or Opponent has a winning strategy. As a first step, well-founded, race-free concurrent games have now been shown to be determined and have been applied to give a concurrent game semantics to predicate logic [8,15]. (A game A is well-founded if all configurations in $\mathcal{C}^\infty(A)$ are finite.) The game semantics extends to Hintikka's "independence-friendly" logic, using ideas of the next section to associate 'levels' with quantified variables.

8 Imperfect Information

Consider the game "rock, scissors, paper" in which the two participants Player and Opponent independently sign one of r ("rock"), s ("scissors") or p ("paper"). The participant with the dominant sign w.r.t. the relation

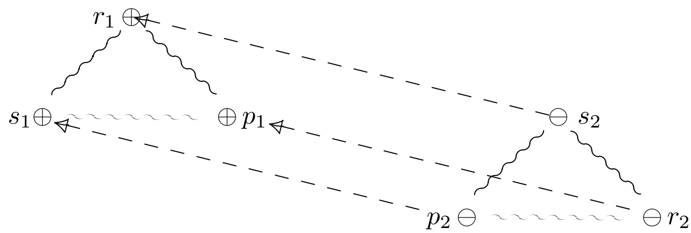
$$r \textit{ beats } s, \;\; s \textit{ beats } p \text{ and } p \textit{ beats } r$$

wins. It seems sensible to represent this game by RSP, the event structure with polarity

comprising the three mutually inconsistent possible signings of Player in parallel with the three mutually inconsistent signings of Opponent. In the absence of neutral configurations, a reasonable choice is to take the *losing* configurations (for Player) to be

$$\{s_1, r_2\}, \;\; \{p_1, s_2\}, \;\; \{r_1, p_2\}$$

and all other configurations as winning for Player. In this case there is a winning strategy for Player, *viz.* await the move of Opponent and then beat it with a dominant move. Explicitly, the winning strategy $\sigma : S \to RSP$ is given as the obvious map from S, the following event structure with polarity:



But this strategy cheats. In "rock, scissors, paper" participants are intended to make their moves *independently*. The problem with the game RSP as it stands is that it is a game of *perfect information* in the sense that all moves are visible to both participants. This permits the winning strategy above with its unwanted dependencies on moves which should be unseen by Player. To adequately model "rock, scissors, paper" requires a game of *imperfect information* where some moves are masked, or inaccessible, and strategies with dependencies on unseen moves are ruled out.

We can extend concurrent games to games with imperfect information. To do so in way that respects the operations of the bicategory of games we suppose a fixed preorder of *levels* $(\Lambda, \preceq)$. The levels are to be thought of as levels of access, or permission. Moves in games and strategies are to respect levels: moves will be assigned levels in such a way that a move is only permitted to causally depend on moves at equal or lower levels; it is as if from a level only moves of equal or lower level can be seen.

A Λ-game (G, l) comprises a game $G = (A, W)$ with winning conditions together with a *level function* $l : A \to \Lambda$ such that

$$a \leq_A a' \implies l(a) \preceq l(a')$$

for all $a, a' \in A$. A Λ-strategy in the Λ-game (G, l) is a strategy $\sigma : S \to A$ for which

$$s \leq_S s' \implies l\sigma(s) \preceq l\sigma(s')$$

for all $s, s' \in S$.

For example, for "rock, scissors, paper" we can take Λ to be the discrete preorder consisting of levels 1 and 2 unrelated to each other under $\preceq$. To make RSP into a suitable Λ-game the level function l takes +ve events in RSP to level 1 and −ve events to level 2. The strategy above, where Player awaits the move of Opponent then beats it with a dominant move, is now disallowed because it is not a Λ-strategy—it introduces causal dependencies which do not respect levels. If instead we took Λ to be the unique preorder on a single level the Λ-strategies would coincide with all the strategies.

Fortunately the introduction of levels meshes smoothly with the bicategorical structure on games. For Λ-games (G, l_G) and (H, l_H), define the dual $(G, l_G)^\perp$ to be $(G^\perp, l_{G^\perp})$ where $l_{G^\perp} = l_G$, and define the parallel composition $(G, l_G) \parr$

(H, l_H) to be $(G \parr H, l_{G \parr H})$ where $l_{G \parr H}(a) = l_G(a)$ for $a \in G$, $l_{G \parr H}(b) = l_H(b)$ for $b \in H$.

A Λ-strategy between Λ-games from (G, l_G) to (H, l_H) is a strategy in $(G, l_G)^\perp \parr (H, l_H)$. Let (G, l_G) be a Λ-game where G satisfies **(Cwins)**. The copy-cat strategy on G is a Λ-strategy. The composition of Λ-strategies is a Λ-strategy.

9 Linear Strategies

It has recently become clear that concurrent strategies support several refinements. For example, define a *partial-rigid strategy* to be a strategy σ in which both components σ_1 and σ_2 are partial rigid. Copy-cat strategies are partial rigid, and the composition of partial-rigid strategies is partial-rigid, so partial-rigid strategies form a sub-bicategory of **Games**. We can refine partial-rigid strategies further to *linear* strategies, where each +ve output event depends on a maximum +ve event of input, and dually, a −ve event of input depends on a maximum −ve event of output. By introducing this extra relevance, of input to output and output to input, we can recover coproducts and products lacking in **Games**.

Formally, a (nondeterministic) *linear* strategy is a strategy

$$A^\perp \xleftarrow{\ \sigma_1\ } S \xrightarrow{\ \sigma_2\ } B\,,$$

where σ_1 and σ_2 are partial rigid maps such that

$$\begin{array}{l} \forall s \in S.\ pol_S(s) = + \ \&\ \sigma_2(s) \text{ is defined} \\ \quad \Longrightarrow \\ \exists s_0 \in S.\ pol_S(s_0) = - \ \&\ \sigma_1(s_0) \text{ is defined } \&\ s_0 \leq_S s\ \& \\ \forall s_1 \in S.\ pol_S(s_1) = - \ \&\ \sigma_1(s_1) \text{ is defined } \&\ s_1 \leq_S s \implies s_1 \leq_S s_0 \end{array}$$

and

$$\begin{array}{l} \forall s \in S.\ pol_S(s) = + \ \&\ \sigma_1(s) \text{ is defined} \\ \quad \Longrightarrow \\ \exists s_0 \in S.\ pol_S(s_0) = - \ \&\ \sigma_2(s_0) \text{ is defined } \&\ s_0 \leq_S s\ \& \\ \forall s_1 \in S.\ pol_S(s_1) = - \ \&\ \sigma_2(s_1) \text{ is defined } \&\ s_1 \leq_S s \implies s_1 \leq_S s_0\,. \end{array}$$

Copy-cat strategies are linear and linear strategies are closed under composition. Linear strategies form a sub-bicategory **Games**. Its sub-bicategory **Lin** of deterministic subcategories is a model of MALL (multiplicative-additive linear logic) and a promising candidate in which to establish full-completeness—work in progress.

10 Conclusion

We have summarised the main results on concurrent strategies to date (December 2011). Two current research directions: One current is the development of an intensional semantics of processes and proofs. But games and concurrent strategies form a generalized *affine* domain theory. Does the bicategory **Lin** of deterministic linear strategies provide a fully-complete model of MALL? A next step is to extend concurrent games to allow *back-tracking* via "copying" monads in event structures with symmetry [16]. Another direction concerns the possible application of concurrent games for which we seek stronger determinacy results.

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