Small Point Sets for Simply-Nested Planar Graphs*

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Abstract. A point set $P \subseteq \mathbb{R}^2$ is universal for a class $\mathcal{G}$ if every graph of $\mathcal{G}$ has a planar straight-line embedding into P. We prove that there exists a $O(n(\frac{\log n}{\log \log n})^2)$ size universal point set for the class of simply-nested n-vertex planar graphs. This is a step towards a full answer for the well-known open problem on the size of the smallest universal point sets for planar graphs [1,5,9].

1 Introduction

A *planar straight-line embedding* of a graph G *into a point set* P is a mapping of each vertex of G to a distinct point of P and of each edge of G to the straight-line segment between the corresponding endpoints so that no two edges cross. Let $\mathcal{G}$ be a class of n-vertex planar graphs and P be a point set of size m, with $m \geq n$. Point set P is *universal* for the class $\mathcal{G}$ if for every $G \in \mathcal{G}$, G has a planar straight-line embedding into P.

Asymptotically, the smallest universal point set for general planar graphs is known to have size at least $1.235n$ [6,12], while the best known upper bound is $O(n^2)$ [7,10,13]. Characterizing the asymptotic size of the smallest universal point set is a well-known open problem also referred in [1,5,9].

A subclass of planar graphs for which a "small" universal point set is known is the class of outerplanar graphs, that is, the graphs that admit a straight-line planar embedding with all vertices incident to the outer face. Gritzmann et al. [11] and Bose [4] proved that any point set of size n is universal for outerplanar graphs. In [11] it is noticed that outerplanar graphs are the largest class of graphs for which any arbitrary point set is universal.

A generalization of outerplanar graphs are k-outerplanar graphs, $k \geq 2$. A planar embedding of a graph is *k-outerplanar* if removing the vertices of the outer face yields a $(k-1)$-outerplanar embedding, where 1-outerplanar is an outerplanar embedding. Vertices removed at the i-th step are at level i. A graph is *k-outerplanar* if it admits a

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k-outerplanar embedding. Note that no (arbitrarily large) convex point set is universal for k-outerplanar graphs, $k \geq 2$.

The decision question of whether a given planar graph admits a planar straight-line embedding into a given point set of the same size was proved to be $\mathcal{NP}$-hard, even for 2-outerplanar graphs and 3-level point sets [5].

A k-outerplanar graph is *simply-nested* [8] if levels 1 to $k-1$ are chordless cycles and level k is either a cycle or a tree. A planar graph is *simply-nested* if it is k-outerplanar simply-nested for some $k \leq n$. Simply-nested graphs turned out to be useful to derive some properties of planar graphs. Cimikowski [8] proved hamiltonicity of simply-nested planar triangulations. Baker [3] used these graphs to derive approximation algorithms for various NP-complete problems on planar graphs. A variant of nested triangulations was explored by Yannakakis in his celebrated result on book embeddings of planar graphs [14].

In this paper we show a $O(n(\frac{\log n}{\log \log n})^2)$-size universal point set for simply-nested n-vertex graphs (Sect. 3). Such result is based on the construction of a $8n+8$-size universal point set for simply-nested n-vertex graphs for which the number of vertices on each of level is known in advance (Sect. 2).

Our results find applications to another class of graphs, quite popular in Graph Drawing. In [2] Bachmaier *et al.* defined a graph to be (*proper*) *k-radial planar* if given a partition of its vertices into k concentric circles, its edges can be drawn as monotonic curves between (consecutive) circles without crossings and showed that radial planarity is decidable in linear time. Our results give a small universal point set for proper k-radial planar graphs, since they can be easily proved to be a subclass of simply-nested planar graphs.

2 A Universal Point Set for Simply-Nested Planar Graphs with n_i Vertices on Level i

In this section we describe a universal point set P of size $8(\sum_{i=1}^{k} n_i + k) = O(n)$ for simply-nested planar graphs in which the number n_i of vertices at each level i is known in advance. Note that, when this strong assumption is not possible, the same construction yields a point set with a quadratic number of points, namely $8(\sum_{i=1}^{k} n + k) = O(n^2)$, as $k = O(n)$. However, constructing the point set under this assumption is the basis of a construction, described in Sect. 3, that leads to subquadratic size in the general case.

We aim at placing the vertices of level i on a circle with a number of available points proportional to n_i. Then, we would like to place the vertices of level $i+1$ greedily on a circle internal to the previous one. This is difficult for the following reason. If a vertex of level $i+1$ is connected to many vertices of level i, the angle spanned by its connections gets close to 2π, and an arbitrary number of points of the internal circle become "unusable". See Fig. 1(a). Hence, we use a technique that places the vertices of each level on two concentric circles.

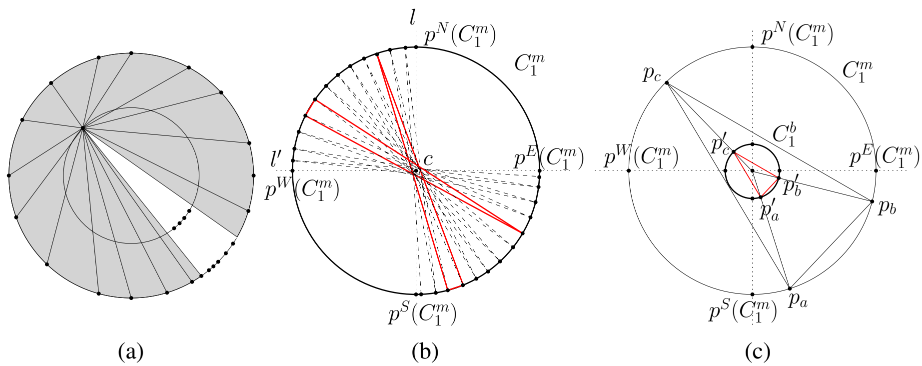


Fig. 1. (a) Problems in using one circle per level. (b) Construction of circle C_1^m. (c) Construction of circle C_1^b.

2.1 Construction of the Point Set

The points of P are on $2k$ concentric circles $C_1^m, C_1^b, \ldots, C_k^m, C_k^b$. For each level $i = 1, \ldots, k$, circles C_i^m and C_i^b are the *main circle* and the *back-up circle* of level i, respectively. Both have $4n_i + 4$ points. In the following we describe how to choose the radius and the distribution of the points for each circle.

Let l and l' be two orthogonal lines crossing at a point c, that is the center of circles C_i^m and C_i^b ($i = 1, \ldots, k$). The parts of the plane delimited by l and l' are the *quadrants*. For each circle C, denote by $p^N(C)$ and $p^S(C)$ the intersections between C and l, and by $p^W(C)$ and $p^E(C)$ the intersections between C and l'. Points $p^N(C)$, $p^S(C)$, $p^E(C)$, and $p^W(C)$ are the *cardinal* points of C.

Let C_1^m be a circle centered at c with any radius r_1^m. Place a point of P on each of $p^N(C_1^m)$, $p^S(C_1^m)$, $p^W(C_1^m)$, and $p^E(C_1^m)$. Then, place n_1 points of P in each arc of C_1^m determined by lines l and l', in such a way that for any two consecutive points p_a and p_b that are internal to a quadrant there exists a point p_c in the opposite quadrant, that is, its unique non-adjacent quadrant, such that triangle (p_a, p_b, p_c) contains c. Such a placement of points is always realizable. Namely, consider two opposite quadrants Q and Q'. Place a point p_a on C_1^m in Q and a point p'_a on C_1^m in Q' such that the center c is to the left of the oriented segment $\overrightarrow{p_a p'_a}$. Then place a point p_b on C_1^m in Q such that c is to the left of the oriented segment $\overrightarrow{p'_a p_b}$. Keeping on placing points in this way yields a point set with the desired property. See Fig. 1(b).

Let C_1^b be a circle centered at c with a radius $r_1^b < r_1^m$ such that, for every triangle (p_a, p_b, p_c) composed of three points of C_1^m, if (p_a, p_b, p_c) contains c, then it also contains C_1^b. Then, place $4n_1 + 4$ points on C_1^b in such a way that, for each point $p \in C_1^m$ there exists a point p' on the intersection between C_1^b and the radius of C_1^m to p. Note that, this implies that for any two consecutive points p'_a and p'_b of C_1^b that are internal to a quadrant there exists a point p'_c of C_1^b in its opposite quadrant such that (p'_a, p'_b, p'_c) contains c. See Fig. 1(c).

Then, for each level i, with $i = 2, \ldots, k$, construct the main circle C_i^m and the back-up circle C_i^b as follows.

Circle C_i^m is centered at c, has radius $r_i^m < r_{i-1}^b$, and for any triangle composed of two consecutive points p'_a and p'_b of C_{i-1}^b and a point p'_c in the opposite quadrant of C_{i-1}^b, if (p'_a, p'_b, p'_c) contains c, then it also contains C_i^m.

Place a point of P on each cardinal point of C_i^m. Then, place n_i points in each arc of C_i^m determined by l and l' in such a way that: (a) for any two consecutive points p_a and p_b of C_i^m that are internal to a quadrant there exists a point p_c of C_i^m in the opposite quadrant such that (p_a, p_b, p_c) contains c; (b) for any two points p_1, p_2 of C_{i-1}^m that are in opposite quadrants, consider the quadrant Q that is completely contained in the wedge delimited by the half-lines from c to p_1 and from c to p_2 whose angle is smaller than π. Then, there exists a point p_3 of C_i^m in Q such that triangle (p_1, p_2, p_3) contains no point of C_i^m (see Fig. 2(a)); (c) the quadrilateral composed of points $p^N(C_{i-1}^m)$, $p^S(C_{i-1}^m)$, $p^W(C_i^m)$, and $p^E(C_i^m)$ contains all the points of C_i^m (see Fig. 2(b)); (d) the quadrilateral composed of points $p^E(C_{i-1}^m)$, $p^W(C_{i-1}^m)$, $p^N(C_i^m)$, and $p^S(C_i^m)$ contains all the points of C_i^m (see Fig. 2(b)). Note that a point set with these properties can always be constructed. Namely, a point set satisfying property (a) can be constructed analogously as for C_1^m (see Fig. 1(b)), while properties (b)–(d) can be easily satisfied by making the radius of C_i^m small enough.

Circle C_i^b is centered at c, has radius $r_i^b < r_i^m$, and is such that for every triangle (p_a, p_b, p_c) composed of three points placed on C_i^m, if (p_a, p_b, p_c) contains c, then it also contains C_i^b. Then, place $4n_i + 4$ points of P on C_i^b in such a way that, for each point $p \in C_i^m$ there exists a point p' on the intersection between C_i^b and the radius of C_i^m to p. Note that, this implies that for any two consecutive points p'_a and p'_b of C_i^b that are internal to a quadrant there exists a point p'_c of C_i^b in its opposite quadrant such that triangle (p'_a, p'_b, p'_c) contains c.

2.2 Embedding a Simply-Nested Planar Graph on Point Set P

Let G be any simply-nested planar graph. We assume that G has only triangular faces; if it is not the case, we add dummy edges.

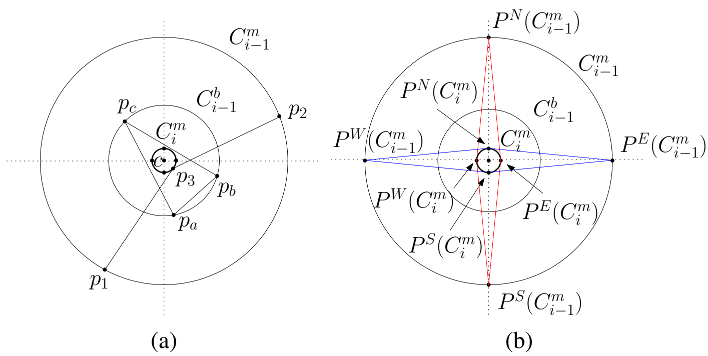


Fig. 2. Construction of circle C_i^m: (a) Triangle (p_1, p_2, p_3) contains no point of C_i^m; (b) quadrilaterals $(P^N(C_{i-1}^m), P^E(C_i^m), P^S(C_{i-1}^m), P^W(C_i^m))$ and $(P^N(C_i^m), P^E(C_{i-1}^m), P^S(C_i^m), P^W(C_{i-1}^m))$ contain all the points of C_i^m.

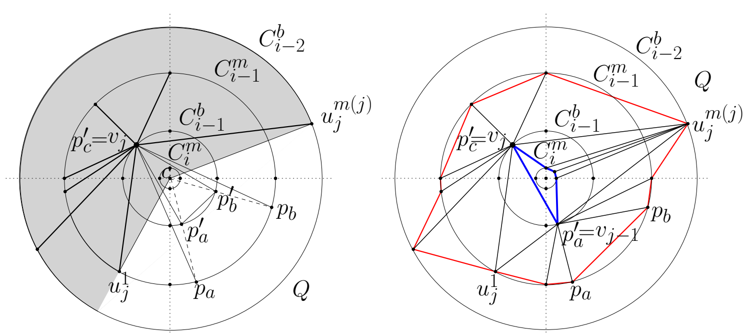
The drawing of G on P is constructed iteratively, starting by placing the vertices of level 1 on any n_1 points of circle C_1^m in such a way that the polygon representing the cycle composed of such vertices contains the center c. Note that, as any triangle composed of three points of C_1^m and containing c also contains C_1^b, the constructed polygon contains C_1^b, as well.

In order to describe how to embed the vertices of level $i = 2, \ldots, k$, we first give a further definition. We say that the drawing of the vertices of level i is 2-*radial* if it satisfies the following properties: (a) all the vertices of level i are on circle C_i^m, except for at most two vertices v'_* and v''_*, that are possibly drawn on two points of circle C_{i-1}^b. (b) Given the two lines tangent to C_i^b through v'_* (through v''_*), the triangle composed of their tangent points to C_i^b and v'_* (v''_*) does not contain any vertex of level i placed on a point of C_i^m.

Then, for each level $i = 2, \ldots, k$, we assume that a 2-radial drawing of level $i-1$ is given, and we greedily construct a 2-radial drawing of level i, as follows.

Consider the vertices $v_1, \ldots, v_h$ of level i that have more than one neighbor in level $i-1$. Observe that, the set of vertices that is the union of the neighbors of $v_1, \ldots, v_h$ coincides with the set of vertices of level $i-1$. As the vertices of level $i-1$ are already drawn, it is possible to determine, for each vertex v_j ($j = 1, \ldots, h$) of level i, the angle α_j of the smallest wedge W_j centered at c and containing all the neighbors $u_j^1, \ldots, u_j^{m(j)}$ of v_j. The wedge W_j of a vertex v_j is depicted as a shaded region in Fig. 3(a). Note that, $\sum_j \alpha_j = 2\pi$, and hence at most one angle α_j, with $1 \leq j \leq h$, can be greater than or equal to π.

First, we study the case (Case 1) when there exists one angle $\alpha_j \geq \pi$. Note that, there exists at least one quadrant Q such that Q is not completely contained into W_j, while the opposite quadrant of Q is. Refer to Fig. 3(a). Then, by construction, there exist two consecutive points p'_a and p'_b of C_{i-1}^b in Q that are not in W_j (they might be on the two delimiting half-lines of W_j) and a point p'_c of C_{i-1}^b in the opposite quadrant of Q such that triangle (p'_a, p'_b, p'_c) contains circle C_i^m. This implies that triangle (p_a, p_b, p'_c) contains C_i^m, as well, where p_a and p_b are the points of C_{i-1}^m on the same radius as p'_a and p'_b, respectively.



(a) (b)

Fig. 3. (a) Case 1. Placement of a vertex v_j such that $\alpha_j \geq \pi$. (b) Case 1.1.1. There exists one angle $\alpha_j \geq \pi$, $v_{j-1} = v_{j+1}$, and $v' = v_{j-1}$.

Place vertex v_j on point p'_c and draw the edges between v_j and its neighbors $u_j^1, \ldots, u_j^{m(j)}$. As (p_a, p_b, p'_c) contains C_i^m, none of such edges crosses C_i^m.

Note that, vertex u_j^1 (vertex $u_j^{m(j)}$) has at least one neighbor v' (one neighbor v'') of level i different from v_j, possibly $v' = v_{j-1}$ (possibly $v'' = v_{j+1}$).

First (Case 1.1), suppose that $v_{j-1} = v_{j+1}$. We distinguish three cases, based on whether $v' = v_{j-1}$ (Case 1.1.1), $v'' = v_{j+1}$ (Case 1.1.2), or none of the two cases holds (Case 1.1.3). Cases 1.1.1 and 1.1.2 are mutually exclusive.

If $v' = v_{j-1}$ (Case 1.1.1), place v_{j-1} on p'_a. By construction, triangle (p'_a, p'_b, p'_c) contains C_i^m, which implies that edges (v_j, v_{j-1}), $(u_j^{m(j)}, v_{j-1})$, and (u_j^1, v_{j-1}) do not cross C_i^m. Also, all the vertices of level i that remain to be drawn are adjacent to $u_j^{m(j)}$. As such vertex, which lies in a quadrant Q on circle either C_{i-1}^m or C_{i-1}^b, has complete visibility to all the n_i points of circle C_i^m in the same quadrant Q, it is possible to draw all its neighbors on such points so that the polygon composed of vertices of level i contains C_i^m. See Fig. 3(b).

If $v'' = v_{j+1}$ (Case 1.1.2), then place v_{j+1} on p'_b and place the other vertices analogously to the previous case.

If none of the two cases holds (Case 1.1.3), we further distinguish three cases, based on whether u_j^1 and $u_j^{m(j)}$ lie in opposite quadrants, in adjacent quadrants, or in the same quadrant. In the first case (see Fig. 4(a)), place v_{j+1} on either p'_a or p'_b and apply the same drawing algorithm as in the previous cases. If they lie in adjacent quadrants Q and Q' (see Fig. 4(b)), place v_{j-1} on the cardinal point, say $p^E(C_i^m)$, that is between Q and Q'. Note that, the wedge W centered at $p^E(C_i^m)$, delimited by the half-lines from $p^E(C_i^m)$ to u_j^1 and from $p^E(C_i^m)$ to $u_j^{m(j)}$, and whose angle is smaller than π is external to quadrilateral $(p^N(C_{i-1}^m)$,$p^E(C_i^m)$, $p^S(C_{i-1}^m)$, $p^W(C_i^m))$. As, by construction, such a quadrilateral contains all the points of C_i^m, W does not contain any of these points. Hence, both u_j^1 and $u_j^{m(j)}$ have complete visibility to all the n_i points of quadrants Q and Q' of circle C_i^m, respectively, and it is possible to draw all their neighbors on such points. Finally, if u_j^1 and $u_j^{m(j)}$ lie in the same quadrant (see Fig. 4(c)), they both have

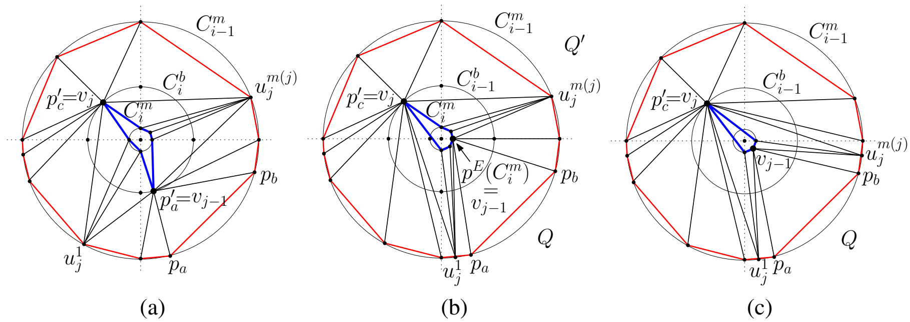


Fig. 4. Case 1.1.3. There exists one angle $\alpha_j \geq \pi$, $v_{j-1} = v_{j+1}$, $v' \neq v_{j-1}$, and $v'' \neq v_{j+1}$. Illustrations of the cases in which u_j^1 and $u_j^{m(j)}$ lie (a) in opposite quadrants, (b) in adjacent quadrants, and (c) in the same quadrant.

visibility to all the points of C_i^m in such quadrant, and all their neighbors, including v_{j+1}, can be drawn on such points.

In each of the cases, all the vertices of level i are on the main circle C_i^m of level i, except for vertex v_j and, in one case, for vertex v_{j-1}, which are on the back-up circle C_{i-1}^b of level $i-1$. Also, no vertex is drawn on C_i^m in the same quadrant as the vertex (v_j or v_{j-1}) that is on C_{i-1}^b. Hence, given the two lines through v_j (through v_{j-1}) tangent to C_i^b, the triangle composed of v_j (of v_{j-1}) and of the two tangent points does not contain any vertex of level i placed on a point of C_i^m. It follows that the constructed drawings are 2-radial drawings.

Suppose (Case 1.2) that $v_{j-1} \neq v_{j+1}$. Let $u_{j-1}^1, \dots, u_{j-1}^{m(j-1)}$ be the neighbors of v_{j-1} of level $i-1$. Note that $u_{j-1}^{m(j-1)} = u_j^1$. If u_{j-1}^1 is in the same quadrant as u_j^1 (Fig. 5(a)), place the first neighbor v_j^1 of u_j^1 on the first cardinal point of C_i^m encountered when rotating clockwise the radius to u_j^1. If it is in the adjacent quadrant (Fig. 5(b)), place v_{j-1} on the cardinal point of C_i^m between such two quadrants. Finally, if it is in the opposite quadrant (Fig. 5(c)), place v_{j-1} on a point p^* of C_i^m in its adjacent quadrant such that triangle (u_j^1, u_{j-1}^1, p^*) does not contain any point of C_i^m, which exists by construction. Then, place the first neighbor v_{j-1}^1 of u_{j-1}^1 different from v_{j-1} on the first cardinal point encountered when rotating clockwise the radius to u_{j-1}^1.

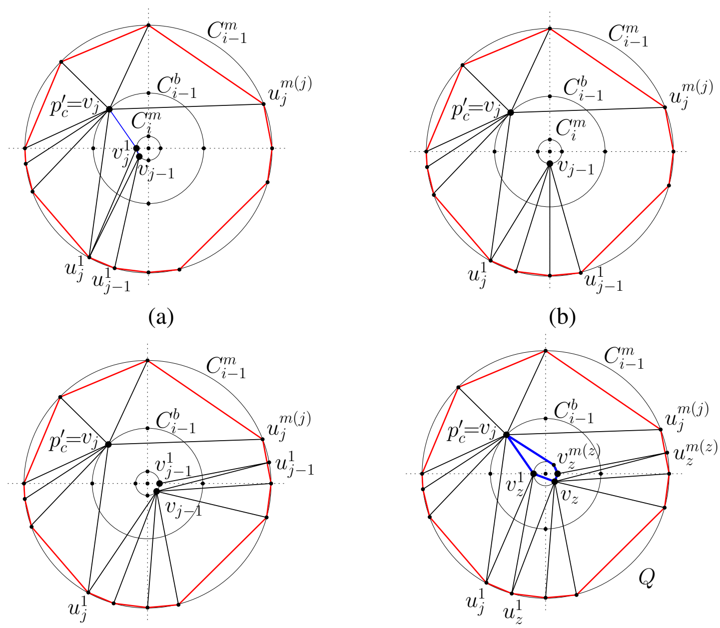


Fig. 5. Case 1.2. There exists an angle $\alpha_j \geq \pi$, $v_{j-1} \neq v_{j+1}$ and $v' \neq v_{j-1}$. Illustrations of the cases in which u_j^1 and u_{j-1}^1 lie (a) in the same quadrant, (b) in adjacent quadrants, and (c) in opposite quadrants. In (a), the placement of v_{j-1} is depicted, but it is not decided at this step. (d) Placement of vertices v_z such that u_z^1 and $u_z^{m(z)}$ are in opposite quadrants. Note that the first neighbor v_z^1 of u_z^1 coincides with v_{j-1}, while $v_z^{m(z)}$ does not coincide with v_{j+1}.

Then, consider each vertex v_z such that u_z^1 and $u_z^{m(z)}$ are in different quadrants. If such two quadrants are adjacent, place v_z on the cardinal point of C_i^m between them. If such two quadrants are opposite, then place v_z on a point p^* of C_i^m in the quadrant Q between them such that triangle $(u_z^1, u_z^{m(z)}, p^*)$ does not contain any point of C_i^m, and place the first neighbor v_z^1 of u_z^1 and the first neighbor $v_z^{m(z)}$ of $u_z^{m(z)}$ on the extremal points of Q, if such two vertices do not coincide with v_{j-1} and v_{j+1}, respectively. Note that, if they coincide with either v_{j-1} or v_{j+1}, the point where they had been placed in the previous step of the algorithm still allows for a planar drawing (see Fig. 5(d)).

Observe that, in each of the described cases all the vertices of level $i-1$ whose neighbors of level i still remain to be placed have complete visibility to all the n_i points of a quadrant of circle C_i^m, and hence it is possible to draw all their neighbors on such points. Further, no vertex is drawn on C_i^m in the same quadrant as v_j. Hence, given the two lines through v_j (through v_{j-1}) tangent to C_i^b, the triangle composed of v_j (of v_{j-1}) and of the two tangent points does not contain any vertex of level i placed on a point of C_i^m. It follows that the constructed drawings are 2-radial drawings.

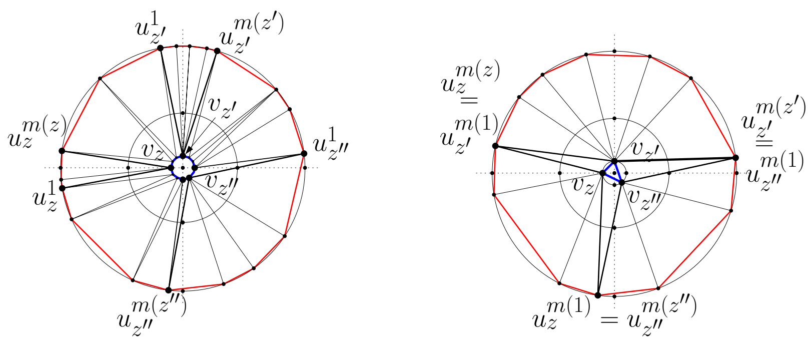


Fig. 6. Case 2. There exists no angle $\alpha_j \geq \pi$. Illustration for the cases when (a) $v_{z''}^1 \neq v_{z'}$ and $v_{z''}^{m(z'')} \neq v_z$, and (b) $v_{z''}^1 = v_{z'}$ and $v_{z''}^{m(z'')} = v_z$.

Suppose (Case 2) that there exists no angle $\alpha_j \geq \pi$. For each vertex v_z such that u_z^1 and $u_z^{m(z)}$ are in adjacent quadrants, place v_z on the cardinal point between them (see Fig. 6(a)). Then, for each vertex v_j such that u_j^1 and $u_j^{m(j)}$ are in opposite quadrants, place v_j on a point p^* of the quadrant Q between them such that triangle $(u_j^1, u_j^{m(j)}, p^*)$ does not contain any point of C_i^m, and place the first neighbors of u_j^1 and of $u_j^{m(j)}$ on the extremal points of Q, if such two vertices have not been already placed. Again, if this is the case, the point where they had been placed still allows for a planar drawing (see Fig. 6(a) and (b)).

Observe that, in each of the described cases all the vertices of level $i-1$ whose neighbors of level i still remain to be placed have complete visibility to all the n_i points of a quadrant of circle C_i^m, and hence it is possible to draw all their neighbors on such points. The above discussion leads to the following.

Theorem 1. *Let $\mathcal{G}$ be the class of simply-nested planar graphs with k levels and such that each level i has n_i vertices. There exists a universal point set for $\mathcal{G}$ of size $8(\sum_{i=1}^{k} n_i + k)$.*

3 A Universal Point Set for Simply-Nested Planar Graphs

Let G be a simply-nested n-vertex planar graph. In Sect. 2 we described a universal point set of linear size provided that the number of levels of G and the number of vertices in each level is known. In this section we show how to limit the size even if such information is not known in advance.

3.1 A Simple Point Set of Size $O(n^{3/2})$

We group the levels of the graph into *dense levels* and *sparse levels*, depending on whether the level contains at least $\sqrt{n}$ vertices or not. Clearly, G contains at most $\sqrt{n}$ dense levels and at most n sparse levels.

Point set P is composed of $\sqrt{n}$ *dense levels*, each containing $8n + 8$ points, and n *sparse levels*, each containing $8\sqrt{n} + 8$ points. As in the point set of Sect. 2, levels of P are composed of a main and a backup circle. We start placing $\sqrt{n}$ outermost sparse levels. Then we place inside them a single dense level. Then again $\sqrt{n}$ sparse levels, followed by a dense level, and so on, until the total number of sparse levels reaches n and the number of dense levels reaches $\sqrt{n}$. This gives a point set of $n + \sqrt{n}$ levels and a total size of $O(n^{3/2})$ points.

Levels of G are assigned to levels of P as follows. Consider the levels of G starting from level 1 and the levels of P starting from the outermost one, proceeding inwards. Let i be the current level of G. If i is sparse, then assign it to the next available sparse level of P. Otherwise (i is dense), assign it to the next available dense level of P. Clearly, a dense level is skipped only if all the $\sqrt{n}$ sparse levels before it were already used. Hence, these previous sparse levels can account for the missing dense level. Summarizing, after scanning all n sparse and $\sqrt{n}$ dense levels of the graph, all its levels are assigned to the levels of the point set according to their size. We conclude with the following:

Lemma 1. *There is a universal point set of size $O(n^{3/2})$ for the class of simply-nested n-vertex planar graphs.*

3.2 Further Refinement

We refine now the classes of dense and sparse levels both of G and of P into m different classes $\mathcal{K}_i, 1 \leq i \leq m$. We say that level j of G, with n_j vertices, belongs to class $\mathcal{K}_i$, with $1 \leq i \leq m$, if $n^{(i-1)/m} \leq n_j < n^{i/m}$. Hence the number of levels in class $\mathcal{K}_i$ is at most $n^{(m-i+1)/m}$, as G has n vertices. As discussed in Sect. 2, if the j-th level of the graph belongs to class $\mathcal{K}_i$, we can accommodate it in a level of P of size $8n^{i/m} + 8$. Hence, in what follows, a level of P containing $8n^{i/m} + 8$ points is called a level of the class $\mathcal{K}_i$.

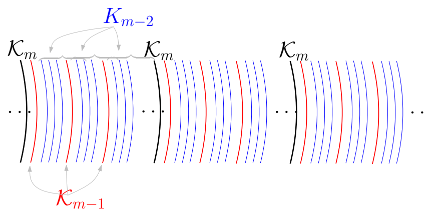


Fig. 7. Constructing the order of the levels

Now we discuss the number of levels and the size of P. The levels of P are first ordered and then placed on the plane one into the other according to the computed order. In order to construct such an order, we first place contiguously the $n^{1/m}$ levels of class $\mathcal{K}_m$ (each having $8n + 8$ points). Then, to the right of each level of class $\mathcal{K}_m$, we insert $n^{1/m}$ levels of class $\mathcal{K}_{m-1}$ (each having $8n^{m-1/m} + 8$ points), in total $n^{2/m}$ levels. We iterate this construction with increasing $i \leq m - 1$: to the right of each of the $n^{(i+1)/m}$ levels of class $\mathcal{K}_{m-i}$, we insert $n^{1/m}$ levels of class $\mathcal{K}_{m-i-1}$ (each having $8n^{m-i-1/m} + 8$ points), which gives in total $n^{(i+1)/m}$ levels. See Fig. 7. Finally, we scan the constructed order from right to left and construct the circles as in Sect. 2.

Summarizing, the total number of points for class $\mathcal{K}_i$ is $\Theta(n^{(m+1)/m})$. Thus, the overall number of points in P is $\Theta(mn^{(m+1)/m}) = \Theta(nmn^{1/m})$. Choosing m such that $mn^{1/m}$ is minimal, we get $m = \Theta(\frac{\log n}{\log \log n})$. Thus the total size of the constructed point set is $O(n(\frac{\log n}{\log \log n})^2)$.

Next we assign the levels of G of class $\mathcal{K}_i$ to the levels of P of class $\mathcal{K}_i$, $i = 1, \ldots, m$, by proceeding from the outside to the center. Intuitively we assign the next graph level of class $\mathcal{K}_i$ to the next unused point set level of class $\mathcal{K}_i$. To show the correctness we give a more formal description.

Let R_m be the minimal sequence of consecutive levels of G, starting from the outer level, that contains in total at least $n^{(m-1)/m}$ and at most n vertices. Note that sequence R_m ends latest at the outermost level of class $\mathcal{K}_m$. For the point set P, we similarly define a block of levels B_m to be the sequence of outer levels of P ending and including the outermost level of the class $\mathcal{K}_m$. We will describe below how to map the graph levels of R_m to the point set levels of B_m. Then, we shrink G by $G \setminus R_m$ and P by $P \setminus B_m$ and iterate. Note that by the structure of the graph and the point set we do this at most $n^{1/m}$ times.

If R_m contains a level of class $\mathcal{K}_m$, we map it to the single level of B_m of class $\mathcal{K}_m$, which is also the last level of B_m, by construction. The other levels of R_m have at most $n^{m-1/m}$ vertices. We repeat the above procedure: we identify a minimal initial sequence R_{m-1} of R_m that contains at least $n^{(m-2)/m}$ and at most $n^{(m-1)/m}$ vertices in total. Note that if $R_m = R_m \setminus R_{m-1}$ then this can be done at most $n^{1/m}$ times, as otherwise R_m would not be minimal. Concerning the point set, we set B_{m-1} to be the minimal sequence of outer levels of B_m that contains a single level of the class $\mathcal{K}_{m-1}$. Putting $B_m = B_m \setminus B_{m-1}$ this procedure can be applied exactly $n^{1/m}$ times, because of the structure of the point set. Finally, the graph levels B_{m-1} are mapped to the point-set levels R_{m-1} recursively. Summarizing the above we have the following theorem.

Theorem 2. *There is a universal point set of size* $O(n(\frac{\log n}{\log \log n})^2)$ *for the class of simply-nested* n*-vertex planar graphs.*

4 Concluding Remarks

In this paper we described a $O(n(\frac{\log n}{\log \log n})^2)$-size universal point set for simply-nested n-vertex planar graphs, doing a step towards answering the well-known open problem on the size of the smallest universal point set for planar graphs.

Several problems remain open in this field: (a) We use points with real coordinates. Is it possible to find a small point set for simply-nested planar graphs with points at integer coordinates and with an overall polynomial area? (b) Simply-nested planar graphs do not have chords between vertices of the same level. Is it possible to find a small point set if such chords are allowed? (c) Is there a small point set for k-outerplanar graphs if k is equal to 2 or 3?

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