Chapter 18
Well-Posedness

Having studied the asymptotic stability in previous chapters we are now in a position to rigorously state the problem of well-posedness, that is, the dependence of the solution of a damped system on the right hand side as well as on the initial conditions.

We will say that the system (1.1) or, equivalently its phase-space formulation (3.2) is *well posed*, if there is a constant $\mathcal{C}$ such that

$$\sup_{t\geq 0}\|y(t)\| \leq \mathcal{C}(\sup_{t\geq 0}\|g(t)\| + \|y_0\|) \tag{18.1}$$

for all initial data y_0.

Theorem 18.1 *The system (3.2) is well-posed, if and only if it is asymptotically stable.*

Proof. Let the system be asymptotically stable. From (3.1) and (13.8) as well as the contractivity of e^{At} we have

$$\|y(t)\| \leq \|y_0\| + Fe^{-\nu t}\sup_{t\geq 0}\|g(t)\|\int_0^t e^{\nu\tau}d\tau$$

$$= \|y_0\| + F\sup_{t\geq 0}\|g(t)\|\frac{1-e^{\nu t}}{\nu},$$

so (18.1) holds with $\mathcal{C} = 1 + F/\nu$. Conversely, if the system is not asymptotically stable then Proposition 17.4 provides an unbounded solution. Q.E.D.

Of course, in applications the mere existence of the constant $\mathcal{C} = 1 + F/\nu$ is of little value, if one does not know its size. To this end the bounds (13.10) and (16.3) may prove useful.

If the system is not asymptotically stable we may modify the definition of well-posedness so that *only the homogeneous system* is considered in which case no $g(t)$ appears in (18.1). Then any damped system is trivially well-posed with $\mathcal{C} = 1$.

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