Path-Based Distance with Varying Weights and Neighborhood Sequences

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Abstract. This paper presents a path-based distance where local displacement costs vary both according to the displacement vector and with the travelled distance. The corresponding distance transform algorithm is similar in its form to classical propagation-based algorithms, but the more variable distance increments are either stored in look-up-tables or computed on-the-fly. These distances and distance transform extend neighborhood-sequence distances, chamfer distances and generalized distances based on Minkowski sums. We introduce algorithms to compute, in $\mathbb{Z}^2$, a translated version of a neighborhood sequence distance map with a limited number of neighbors, both for periodic and aperiodic sequences. A method to recover the centered distance map from the translated one is also introduced. Overall, the distance transform can be computed with minimal delay, without the need to wait for the whole input image before beginning to provide the result image.

1 Introduction

In [8] discrete distances were introduced along with sequential algorithms to compute the distance transform (DT) of a binary image, where each point is mapped to its distance to the background. These discrete distances are built from adjacency and connected paths (path-based distances): the distance between two points is equal to the cost of the shortest path that joins them. For distance d_4 ("d" in [8]), defined in the square grid $\mathbb{Z}^2$, each point has four neighbors located at its top, left, bottom and right edges. Similarly, for distance d_8 ("d^*" in [8]), each point has four extra diagonally located neighbors. In both cases, d_4 and d_8, the cost of a path is defined as the number of displacements. These simple distances have been extended in different ways, by changing the neighborhood depending on the travelled distance ([9,2]), by weighting displacements [5,2], or even by a mixed approach of weighted neighborhood sequence paths [10].

Section 2 presents definitions of distances, disks and some properties of non-decreasing integer sequences that will be used later. Section 3 introduces a new generalization of path-based distances where displacement costs vary both on the displacement vector and on the travelled distance. An application is presented in section 4 for the efficient computation of neighborhood-sequence DT in 2D.

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2 Preliminaries

Lambek-Moser inverse of a integer sequence [4]. Let the function f define a non-decreasing sequences of integers $(f(1), f(2), \ldots)$ For the sake of simplicity, we call f a sequence. The inverse sequence of f, denoted by $f^\dagger$, is a non-decreasing sequence of integers defined by:

$$f(m) < n \;\Leftrightarrow\; f^\dagger(n) \nless m\,. \tag{1}$$

An interesting property of a sequence f and its inverse $f^\dagger$ is that, by adding the rank of each term to these two sequences, we obtain two complementary sequences $f(m) + m$ and $f^\dagger(n) + n$ [4]. This property extends the results given by Ostrowski *et al.* [7] about Beatty sequences [1]. From [4], we deduce that the inverse of the sequence $f(m) = \lfloor \tau m \rfloor$ with a scalar τ, is $f^\dagger(n) = \lceil \frac{n}{\tau} - 1 \rceil$ so $f(m)+m = \lfloor (1+\tau)m \rfloor$ and $f^\dagger(n)+n = \lceil (1+\frac{1}{\tau})n - 1 \rceil$ are two complementary sequences. If τ is irrational, these sequences are Beatty sequences and, for any positive n, $\lceil (1+\frac{1}{\tau})n - 1 \rceil$ is equal to $\lfloor (1+\frac{1}{\tau})n \rfloor$ as given in [1].

Proposition 1. *$f^\dagger(f(m)+1)+1$ is the rank of the smallest term greater than m where f increases.*

Proof.

$$\begin{aligned}
f^\dagger(f(m)+1)+1 = m' \;&\Leftrightarrow\; \begin{cases} f^\dagger(f(m)+1) < m' \\ f^\dagger(f(m)+1) \geq m'-1 \end{cases} \\
&\Leftrightarrow\; \begin{cases} f(m') \geq f(m)+1 \\ f(m'-1) < f(m)+1 \end{cases} \\
&\Leftrightarrow\; f(m') > f(m) \text{ and } f(m'-1) \leq f(m)\,.
\end{aligned}$$

If we extend f with $f(0) = 0$, and define g by $g(0) = 0$, $g(n+1) = f^\dagger(f(g(n)) + 1) + 1$, then $f(g(n))$ takes, in increasing order, all the values of f, each one appearing once.

Definition 1 (Discrete distance). *A function $d : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{N}$ is a* translation-invariant distance *if the following conditions holds $\forall x, y, z \in \mathbb{Z}^n$, $\forall \lambda \in \mathbb{Z}$:*

1. ***translation invariance*** $d(x+z, y+z) = d(x, y)$,
2. ***positive definiteness*** $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$,
3. ***symmetry*** $d(x, y) = d(y, x)$,

In the following sections, we will drop definiteness and symmetry to define "asymmetric pseudo-distances".

Definition 2 (Disk). *The disk $D(p, r)$ of center p and radius r and the symmetrical disk $\check{D}(p, r)$ are the sets:*

$$\begin{aligned} D(p, r) &= \{q \,:\, d(p, q) \leq r\}\,, \\ \check{D}(p, r) &= \{q \,:\, d(q, p) \leq r\}\,. \end{aligned} \tag{2}$$

Table 1. Example of a non-decreasing sequence f and its Lambek-Moser inverse. f is the cumulative sequence of the periodic sequence $(1, 2, 0, 3)$, $f^{\dagger}$ its inverse. $f^{\dagger}(f(n) + 1)+1$ locates the rank of the next f increase. For instance, $f(6) = 9$, $f^{\dagger}(f(6)+1)+1 = 8$ is the rank of appearance of the first value greater than 9, which is 12 in this case.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$f(n)$	1	3	3	6	7	9	9	12	13	15	15	18	19	21
$f^{\dagger}(n)$	0	1	1	3	3	3	4	5	5	7	7	7	8	9
$f^{\dagger}(f(n) + 1) + 1$	2	4	4	5	6	8	8	9	10	12	12	13	14	16

By definition, any disk of negative radius is empty and the disk of radius 0 only contains its center ($D(p, 0) = \{p\}$).

Definition 3 (Distance transform). *The* distance transform DT_X *of the binary image* X *is a function that maps each point* p *to its distance from the closest background point:*

$$\begin{aligned} \mathrm{DT}_X &: \mathbb{Z}^n \to \mathbb{N} \\ \mathrm{DT}_X(p) &= \min\left\{d(q,p) \,:\, q \in \mathbb{Z}^n \setminus X\right\}. \end{aligned} \tag{3}$$

Alternatively, since all points at a distance less than $\mathrm{DT}_X(p)$ to p belong to X ($\check{D}(p, \mathrm{DT}_X(p) - 1) \subset X$) and at least one point at a distance to p equal to $\mathrm{DT}_X(p)$ is not in X ($\check{D}(p, \mathrm{DT}_X(p)) \not\subset X$) then:

$$\mathrm{DT}_X(p) \geq r \;\Leftrightarrow\; \check{D}(p, r-1) \subset X\,. \tag{4}$$

The DT is usually defined as the distance *to* the background which is equivalent to the distance *from* the background by symmetry. The equivalence is lost with asymmetric distances, and this definition better reflects the fact that DT algorithms always propagate paths from the background points.

In this paper, we consider path-based distances, *i.e.* distance functions that associate to each couple of points (p, q), the minimal cost of a path from p to q. For a simple distance, a path is a sequence of points where the difference between two successive points is a displacement vector taken in a fixed neighborhood $\mathcal{N}$, and the cost (or length) of a path is the number of its displacements. The cost of the path $(p_0, \ldots, p_n, p_n + \boldsymbol{v})$ derives from the cost of the path $(p_0, \ldots, p_n)$:

$$\mathcal{L}(p_0, \ldots, p_n) = r \;\Rightarrow\; \forall \boldsymbol{v} \in \mathcal{N}, \mathcal{L}(p_0, \ldots, p_n, p_n + \boldsymbol{v}) = r + 1\,. \tag{5}$$

Rosenfeld and Pfaltz specifically forbid paths where a point appears more than once [8]. This restriction has no effect on the distance because a path where a point appears more than once can not be minimal. In a similar manner, they exclude the null vector from the neighborhood, forbidding a point to appear several times consecutively. As before, it has no effect on the distance. Notice that, in terms of distance, forbidding a path is equivalent to giving it an infinite cost, so that it can not be minimal. (5) can be rewritten as:

$$\mathcal{L}(p_0, \ldots, p_n) = r \;\Rightarrow\; \forall \boldsymbol{v}, \mathcal{L}(p_0, \ldots, p_n, p_n + \boldsymbol{v}) = r + c_{\boldsymbol{v}}\,,$$

where

$$c_{\boldsymbol{v}} = \begin{cases} 1 & \text{if } \boldsymbol{v} \in \mathcal{N} \\ \infty & \text{otherwise} \end{cases} .$$

For a NS-distance characterized by the sequence B:

$$\mathcal{L}(p_0, \ldots, p_n) = r \Rightarrow \forall \boldsymbol{v}, \mathcal{L}(p_0, \ldots, p_n, p_n + \boldsymbol{v}) = r + c^B_{\boldsymbol{v}}(r) , \tag{6}$$

where the displacement cost $c^B_{\boldsymbol{v}}(r)$ is 1 for a displacement vector in the neighborhood $B(r+1)$ and infinite otherwise:

$$c^B_{\boldsymbol{v}}(r) = \begin{cases} 1 & \text{if } \boldsymbol{v} \in \mathcal{N}_{B(r+1)} \\ \infty & \text{otherwise} \end{cases} \tag{7}$$

For a weighted distance with mask $\mathcal{M} = \{(\boldsymbol{v_k}; w_k) \in \mathbb{Z}^n \times \mathbb{N}^*\}_{1 \le k \le m}$, the distance increment only depends on the displacement vector, but not on the distance already travelled:

$$\mathcal{L}(p_0, \ldots, p_n) = r \Rightarrow \forall \boldsymbol{v}, \mathcal{L}(p_0, \ldots, p_n, p_n + \boldsymbol{v}) = r + c_{\boldsymbol{v}} , \tag{8}$$

$$c_{\boldsymbol{v}} = \begin{cases} w & \text{if } (\boldsymbol{v}; w) \in \mathcal{M} \\ \infty & \text{otherwise} \end{cases}$$

Briefly, the displacement cost for a vector $\boldsymbol{v}$ and the travelled distance r, is 1 or ∞, independently of r for simple distances, is equal to 1 or ∞ whether $\boldsymbol{v}$ belongs or not to $\mathcal{N}_{B(r)}$ for a NS-distance, is in $\mathbb{N}^* \cup \{\infty\}$ according to the chamfer mask and independently of r for a weighted distance.

In the following, we propose to use a displacement cost, denoted by $c_{\boldsymbol{v}}(r)$, with values in $\mathbb{N}^* \cup \{\infty\}$, that depends both on the displacement vector $\boldsymbol{v}$ and on the travelled distance r. According to the previous remarks, the cost associated to the null displacement will always be unitary:

$$\forall r \in \mathbb{N}, c_{\mathbf{0}}(r) = 1 . \tag{9}$$

3 Path-Based Distance with Varying Weights

Definition 4 (Path). *We call* path from p to q, *any finite sequence of points* $P = (p = p_0, p_1, \ldots, p_n = q)$ *with at least one point, and denote by* $\mathcal{P}(p, q)$, *the set of these paths.*

Notice that this definition of a path is not related to any adjacency relation. The sequence $P = (p)$ is allowed as a path from p to itself. It is distinct from $P = (p, p)$, the path from p to itself with a null displacement.

Definition 5 (Partial and total costs of a path). *Let* $\mathcal{N}$ *be a set of vectors containing the null vector* $\mathbf{0}$ *and the positive displacement costs* $c_{\boldsymbol{v}}$ *(with* $c_{\mathbf{0}}(r) = 1$ *and* $c_{\boldsymbol{v} \notin \mathcal{N}}(r) = \infty$*). The total cost of the path* $P = (p_0, p_1, \ldots, p_n)$ *is:*

$$\mathcal{L}(P) = \mathcal{L}_n(P) , \tag{10}$$

where $\mathcal{L}_i(P)$ is the partial cost of the path truncated to its $i+1$ first points (i.e., to its i first displacements):

$$\mathcal{L}_0(P) = \mathcal{L}(p_0) = 0\,, \tag{11}$$

$$\mathcal{L}_{i+1}(P) = \mathcal{L}(p_0, \ldots, p_{i+1}) = \mathcal{L}_i(P) + c_{\boldsymbol{p_i p_{i+1}}}(\mathcal{L}_i(P))\,. \tag{12}$$

Definition 6. *We use the notation $C_{\boldsymbol{v_k}}(r) = r + c_{\boldsymbol{v_k}}(r)$. $c_{\boldsymbol{v_k}}(r)$ is the* relative cost *of the displacement $\boldsymbol{v_k}$ when the distance travelled so forth is r. $C_{\boldsymbol{v_k}}(r)$ represents the partial cost of the path after this displacement (the* absolute cost *of this displacement):*

$$\mathcal{L}_{i+1}(P) = \mathcal{L}_i(P) + c_{\boldsymbol{p_i p_{i+1}}}(\mathcal{L}_i(P)) = C_{\boldsymbol{p_i p_{i+1}}}(\mathcal{L}_i(P))\,. \tag{13}$$

Definition 7. *The pseudo-distance induced by $\left(\left\{\boldsymbol{v_k}\right\}, c_{\boldsymbol{v_k}}\right)$ is defined by:*

$$d(p,q) = 0 \;\Leftrightarrow\; p = q$$
$$d(p,q) = \min_{P \in \mathcal{P}(p,q)} \left\{\mathcal{L}(P)\right\}.$$

Definition 8. *We call* minimal relative *(resp.* absolute*)* cost of displacement, *denoted by $\hat{c}$ (resp. $\hat{C}$), the quantity $\hat{c}_{\boldsymbol{v}}(r) = \min\left\{c_{\boldsymbol{v}}(s) + s - r, \forall s \geq r\right\}$ (resp. $\hat{C}_{\boldsymbol{v}}(r) = \min\left\{C_{\boldsymbol{v}}(s), \forall s \geq r\right\}$).*

Proposition 2 (Preservation of cost order by concatenation). *Appending the same displacement to existing paths preserves the relation order of their costs. Let $P = (p_1, \cdots, p_{n_P})$ and $Q = (q_1, \cdots, q_{n_Q})$ be two paths with costs $\mathcal{L}(P)$ and $\mathcal{L}(Q)$, $\boldsymbol{v}$ a vector and $P' = (p_1, \cdots, p_{n_P}, p_{n_P}+\boldsymbol{v})$, $Q' = (q_1, \cdots, q_{n_Q}, q_{n_Q}+\boldsymbol{v})$ the extended paths with costs $\mathcal{L}(P')$ and $\mathcal{L}(Q')$ measured with minimal displacement costs. Then:*

$$\mathcal{L}(P) \leq \mathcal{L}(Q) \;\Rightarrow\; \mathcal{L}(P') \leq \mathcal{L}(Q')\,. \tag{14}$$

Proof. *From (13), $\mathcal{L}(P') = \hat{C}_{\boldsymbol{v}}(\mathcal{L}(P))$ and $\mathcal{L}(Q') = \hat{C}_{\boldsymbol{v}}(\mathcal{L}(Q))$. By definition of $\hat{C}_{\boldsymbol{v}}$, $s \leq r \;\Rightarrow\; \hat{C}_{\boldsymbol{v}}(s) \leq \hat{C}_{\boldsymbol{v}}(r)$, which gives (14).*

Proposition 3. *Let $\mathcal{N} = \left\{\boldsymbol{v_k}\right\}$ be a set of vectors and, $c_{\boldsymbol{v}}(r)$, the displacement costs for these vectors. There exists a path P from p to q of cost $\mathcal{L}(P) = r$ measured with costs $c_{\boldsymbol{v}}(r)$ if and only if there exists a path P' from p to q of cost $\mathcal{L}'(P') = r$ measured with the minimal displacement costs $\hat{c}_{\boldsymbol{v}}(r)$.*

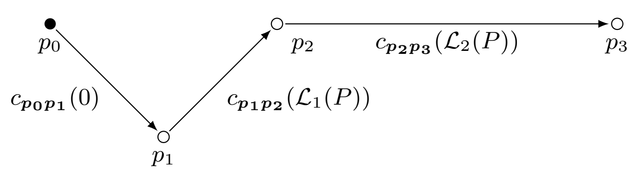


Fig. 1. Total cost of a path $P = (p_0, p_1, p_2)$. Costs of displacements $\boldsymbol{p_0p_1}$, $\boldsymbol{p_1p_2}$ and $\boldsymbol{p_2p_3}$ depend on the partial costs $\mathcal{L}_0(P) = 0$, $\mathcal{L}_1(P) = c_{\boldsymbol{p_0p_1}}(0) + 0$ and $\mathcal{L}_2(P) = c_{\boldsymbol{p_1p_2}}(\mathcal{L}_1(P)) + \mathcal{L}_1(P)$. The total cost of P is $c_{\boldsymbol{p_2p_3}}(\mathcal{L}_2(P)) + \mathcal{L}_2(P)$.

Proof. *Consider the cost of P after i displacements, $\mathcal{L}_i(P) = \mathcal{L}_i(p_0, p_1, \ldots, p_i)$, we note $m_0 = 1, m_{0<i\leq n} = 1 + \mathcal{L}_i(P) - \mathcal{L}_{i-1}(P) - \hat{c}_{\boldsymbol{p_{i-1}p_i}}(\mathcal{L}_{i-1}(P)) = 1 + c_{\boldsymbol{p_{i-1}p_i}}(\mathcal{L}_{i-1}(P)) - \hat{c}_{\boldsymbol{p_{i-1}p_i}}(\mathcal{L}_{i-1}(P))$ and $M_i = \sum_{j=0}^{i} m_i$ the cumulated sum of m_i. Clearly, if $\mathcal{L}(P)$ is finite then each m_i is finite and positive because $\hat{c}_{\boldsymbol{v}}(r)$ is less than or equal to $c_{\boldsymbol{v}}(r)$ by construction. Let P' be the (finite) path obtained by m_i occurrences of each point p_i:*

$$P' = (p_0, \underbrace{p_1 \ldots p_1}_{m_1}, \ldots, \underbrace{p_i \ldots p_i}_{m_i}, \ldots, \underbrace{p_n \ldots p_n}_{m_n}) .$$

We take as an induction hypothesis that the partial cost of P' after m_i occurrences of p_i, $\mathcal{L}'_{M_i-1}(P')$, is equal to $\mathcal{L}_i(P)$. It holds for $i = 0$ because $\mathcal{L}'_{M_0-1}(P') = \mathcal{L}'_{m_0-1}(P') = \mathcal{L}'_0(P') = 0 = \mathcal{L}_0(P)$. If the hypothesis holds for $i-1$, then the partial cost of P' after the first occurrence of p_i is $\mathcal{L}'_{M_{i-1}}(P') = \mathcal{L}_{i-1}(P) + \hat{c}_{\boldsymbol{p_{i-1}p_i}}(\mathcal{L}_{i-1}(P))$, and after $m_i - 1$ repeats of p_i, equals: $\mathcal{L}'_{M_{i-1}+m_i-1}(P') = \mathcal{L}'_{M_i-1}(P') = \mathcal{L}_{i-1}(P) + \hat{c}_{\boldsymbol{p_{i-1}p_i}}(\mathcal{L}_{i-1}(P)) + m_i - 1 = \mathcal{L}_{i-1}(P) + c_{\boldsymbol{p_{i-1}p_i}}(\mathcal{L}_{i-1}(P))$ $= \mathcal{L}_i(P)$ and the hypothesis is true at rank i. Therefore, for every path of finite cost r measured with $\mathcal{L}$, there exists a path with the same cost measured with $\mathcal{L}'$. This is shown in Fig. 2a.

Conversely, let P' be a path with finite cost measured by $\mathcal{L}'$. We build a path P where each point of P' appears m'_i times consecutively with m'_i such that $m'_i - 1 + c_{\boldsymbol{p_ip_{i+1}}}(\mathcal{L}'_i(P') + m'_i - 1) = \hat{c}_{\boldsymbol{p_ip_{i+1}}}(\mathcal{L}'_i(P'))$. By definition of $\hat{c}$, $\forall r, \exists s$: $\hat{c}_{\boldsymbol{v}}(r) = c_{\boldsymbol{v}}(s) + s - r$, so m'_i exists. Let $M'_0 = 0$ and $M'_{0<i\leq n} = \sum_{j=0}^{i-1} m_j$, be the cumulated sum of the previous terms of m'_i.

The induction hypothesis is that the partial cost of P, measured with $\mathcal{L}$, at the first occurence of p_i, $\mathcal{L}_{M'_i}(P)$, is equal to $\mathcal{L}'_i(P')$. It holds for $i = 0$ with a null partial cost $\mathcal{L}_{M'_0}(P) = \mathcal{L}_0(P) = 0 = \mathcal{L}'_0(P')$. If the hypothesis holds at rank i, the partial cost of P, after $m'_i - 1$ repetitions of p_i, if $\mathcal{L}_{M'_i+m'_i-1}(P) = \mathcal{L}_{M'_i}(P) + m'_i - 1 = \mathcal{L}'_i(P') + m'_i - 1$, and at the first occurence of p_{i+1}, equals $\mathcal{L}'_i(P') + m'_i - 1 + c_{\boldsymbol{p_ip_{i+1}}}(\mathcal{L}'_i(P') + m'_i - 1) = \mathcal{L}'_i(P') + \hat{c}_{\boldsymbol{p_ip_{i+1}}}(\mathcal{L}'_i(P')) = \mathcal{L}'_{i+1}(P')$ and the hypothesis also holds at rank $i+1$. An example of such a path is shown on Fig. 2b.

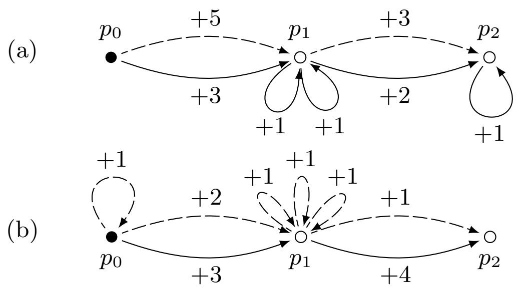
Corollary 1. *Displacement costs $c_{\boldsymbol{v}}$ and $\hat{c}_{\boldsymbol{v}}$ induce the same pseudo-distance.*

According to (9), any path from p to q of cost less than r can be extended with null displacements to reach cost r:

$$\mathcal{L}(p_0, \ldots, p_n = q) = s < r \;\Rightarrow\; \mathcal{L}(p_0, \ldots, \underbrace{p_n = q, \ldots, q}_{1+r-s}) = r \tag{15}$$

Proposition 4. *There exists a path of cost r from p to q if and only if $d(p, q) \leq r$.*

Proof. *If a path of cost r from p to q exists then by definition of the distance, $d(p, q) = r$ if P cost is minimal, $d(p, q) < r$ otherwise. Conversely, if $d(p, q) = s$ then there exists a path of cost s from p to q that, according to (15), can be extended to cost $r \geq s$.*



r	0	1	2	3	4	5	6
$c_{\boldsymbol{v}}(r)$	5	2	∞	∞	∞	3	1
$\hat{c}_{\boldsymbol{v}}(r)$	3	2	5	4	3	2	1

r	0	1	2	3	4	5	6
$c_{\boldsymbol{v}}(r)$	5	2	∞	∞	∞	3	1
$\hat{c}_{\boldsymbol{v}}(r)$	3	2	5	4	3	2	1

Fig. 2. (a) Given $P = (p_0, p_1, p_2)$, shown with dashed lines, has a total cost $\mathcal{L}(P) = 8$ measured with displacement costs $c_{\boldsymbol{v}}$. $P' = (p_0, p_1, p_1, p_1, p_2, p_2)$, solid lines, is built in such a way that its cost $\mathcal{L}'(P')$ measured with minimal displacement costs $\hat{c}_{\boldsymbol{v}}$, is equal to $\mathcal{L}(P) = 8$. (b) Given $P' = (p_0, p_1, p_2)$, shown with solid lines, has a total cost $\mathcal{L}'(P') = 7$ measured with displacement costs $\hat{c}_{\boldsymbol{v}}$. $P = (p_0, p_0, p_1, p_1, p_1, p_1, p_2)$, dashed lines, is built in such a way that $\mathcal{L}(P) = \mathcal{L}'(P') = 7$.

Corollary 2. *For any value of r greater than or equal to $d(p, q)$, there exists a path from p to q which cost is exactly r. The closed disk centered in p with radius r is the set of points for which a path from p of cost equal to r exists:*

$$q \in D(p, r) \;\Leftrightarrow\; \exists P \in \mathcal{P}(p, q), \mathcal{L}(P) = r \,. \tag{16}$$

An iterative construction rule of disks is deduced from (16):

$$\begin{aligned} \forall r > 0, D(p, r) &= \bigcup_{\boldsymbol{v} \in \mathcal{N}} \left\{ q \,:\, \exists P \in \mathcal{P}(p, q - \boldsymbol{v}) \text{ and } C_{\boldsymbol{v}}(\mathcal{L}(P)) = r \right\} \\ &= \bigcup_{\substack{\boldsymbol{v} \in \mathcal{N} \\ s \,:\, C_{\boldsymbol{v}}(s) = r}} D(p + \boldsymbol{v}, s) \end{aligned} \tag{17}$$

4 Minimal Delay Distance Transform

In [12], Wang and Bertrand, proposed a single scan asymmetric generalized DT based on a neighborhood for which there exists a scanning order such that when a point p in the image is scanned, all neighbors of p have already been scanned (forward scan condition). Then, they extended this result to a sequence where two neighborhoods with forward scan condition are alternated (*i.e.*, $B = (1, 2)$) [13]. In the following we propose a method to compute an asymmetric generalized DT based on any number of neighborhoods having forward scan condition used in an arbitrary order defined by a sequence B, either periodic or not. For our purpose, we will use translated versions of regular NS-distances neighborhoods, in order to meet the forward scan condition for each of them. The resulting translated distance map can easily be transformed back into a regular, symmetrical, NS-distance map.

Proposition 5. *The DT of an image X with the distance induced by the neighborhood $\mathcal{N}$ and the displacement costs $C_{\boldsymbol{v}}$ is such that:*

$$\mathrm{DT}_X(p) = \begin{cases} 0 & \text{if } p \notin X \\ \min\left\{\hat{C}_{\boldsymbol{v}}(\mathrm{DT}_X(p-\boldsymbol{v})), \boldsymbol{v} \in \mathcal{N}^*\right\} & \text{otherwise} \end{cases} \quad (18)$$

where $\hat{C}_{\boldsymbol{v}}$ represents the minimal absolute displacement costs corresponding to $C_{\boldsymbol{v}}$ (definition 8).

Proof. *Case $p \notin X$ directly results from definitions 3 and 7. Suppose now that $p \in X$ so any path from $q \notin X$ to p has at least one displacement. Prop. 3 states that distances induced by $\left(\{\boldsymbol{v_k}\}, C_{\boldsymbol{v_k}}\right)$ and $\left(\{\boldsymbol{v_k}\}, \hat{C}_{\boldsymbol{v_k}}\right)$ are equal so we consider the latter cost increments for which prop. 2 holds. According to prop. 2, if $P = (q = p_0, \ldots, p_n = p - \boldsymbol{v})$ is a minimal path from q to $p - \boldsymbol{v}$ then $P' = (q = p_0, \ldots, p_n, p + \boldsymbol{v})$ has a minimal cost — among paths from q to p with second last point $p - \boldsymbol{v}$ — equal to $\hat{C}_{\boldsymbol{v}}(\mathcal{L}(P))$. So $\hat{C}_{\boldsymbol{v}}(\mathrm{DT}_X(p-\boldsymbol{v}))$ is the shortest distance from a point $q \notin X$ to p via $p-\boldsymbol{v}$. Since all paths which last displacement $\boldsymbol{v}$ does not belong to $\mathcal{N}$ have an infinite cost and can not be minimal, (18) holds.*

4.1 Generalized Distance Transform

When all vectors in $\mathcal{N}^*$ are directed forward relatively to the scan order, (18) propagates paths from background pixels in a single scan. As a consequence, a generalized DT using any number of neighborhoods $\mathcal{N}_1 \ldots \mathcal{N}_n$, selected by a sequence $B, B(i) \in [1, n]$, derives directly from (7, 18) and minimal costs given by:

$$\hat{C}_{\boldsymbol{v}}(r) = \min\left\{s \,:\, s > r \text{ and } \boldsymbol{v} \in \mathcal{N}_{B(s)}\right\}. \quad (19)$$

let $\chi_{\boldsymbol{v}}(r)$ denote the characteristic function of the set $\mathcal{N}_{B(r)}$ (*i.e.*, $\chi_{\boldsymbol{v}}(r) = 1$ if $\boldsymbol{v} \in \mathcal{N}_{B(r)}$; 0 otherwise) and $\chi_{\boldsymbol{v}}^{\Sigma}(r)$ its cumulative sum ($\chi_{\boldsymbol{v}}^{\Sigma}(r) = \sum_{s \le r} \chi_{\boldsymbol{v}}(r)$).

Then according to prop. 1:

$$\hat{C}_{\boldsymbol{v}}(r) = [\chi_{\boldsymbol{v}}^{\Sigma}]^{\dagger}(\chi_{\boldsymbol{v}}^{\Sigma}(r) + 1) + 1\,. \quad (20)$$

Algorithm 1 produces a generalized DT using any sequence of neighborhoods ($\mathcal{N}$ represents their union) in forward scan condition, using displacement costs given by (20). A similar algorithm was already presented for the decomposition of convex structuring polygons [6].

4.2 Translated NS-Distance Transform

The sequence of disks for a NS-distance induced by a sequence B is produced by iterative Minkowski sums of neighborhoods:

$$D(p, 0) = \{p\}, \;\; D(p, r) = D(p, r-1) \oplus \mathcal{N}_{B(r)}\,.$$

For each neighborhood $\mathcal{N}_j$, we apply a translation vector $\boldsymbol{t}_j$ such that the translated neighborhood $\mathcal{N}'_j = \mathcal{N}_j \oplus \{\boldsymbol{t}_j\}$ is in forward scan condition. In a translation

Data: X: a set of points
Data: $\mathcal{N}$: neighborhood in forward scan condition
Data: $\hat{C}_{\boldsymbol{v}}$: minimal absolute displacement costs
Result: DT_X: generalized distance transform of X

```
foreach p in DT domain, in raster scan do
    if p ∉ X then
        DT_X(p) ← 0
    else
        l ← ∞
        foreach v in N do
            l ← min{l; Ĉ_v(DT_X(p − v))}
        end
        DT_X(p) ← l
    end
end
```

Algorithm 1. Single scan asymmetric distance transform

preserved scan order, $\boldsymbol{t}_j$ translates the first visited point in $\mathcal{N}_j$ to the origin. Assuming a nD standard raster scan order:

$$\boldsymbol{t}_j = (\underbrace{0, \ldots, 0}_{n-j}, \underbrace{1, \ldots, 1}_{j}) \tag{21}$$

The translated neighborhoods $\mathcal{N}'_1$ and $\mathcal{N}'_2$ obtained with $\boldsymbol{t}_1 = (0,1)$ and $\boldsymbol{t}_2 = (1,1)$ are depicted in Fig. 3a and Fig. 3b. Characteristic functions for vectors in $\mathcal{N}'_1 \setminus \mathcal{N}'_2$, $\mathcal{N}'_2 \setminus \mathcal{N}'_1$ and $\mathcal{N}'_1 \cap \mathcal{N}'_2$ (see Fig. 3c-e) are respectively $\mathbf{1}_B$, $\mathbf{2}_B$ and the constant value 1 resulting in the following minimal displacement costs:

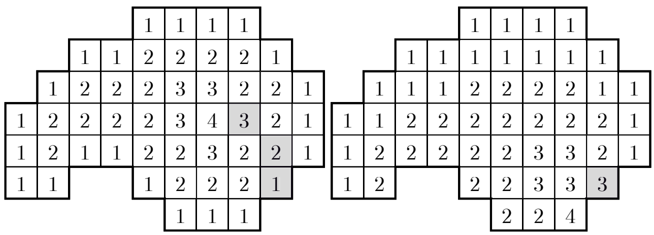
$$\hat{C}_{\boldsymbol{v}}(r) = \begin{cases} \hat{C}^1_{\boldsymbol{v}}(r) = \mathbf{1}^{\dagger}_B(\mathbf{1}_B(r)+1)+1 & \text{if } \boldsymbol{v} \in \mathcal{N}'_1 \text{ and } \boldsymbol{v} \notin \mathcal{N}'_2 \\ \hat{C}^2_{\boldsymbol{v}}(r) = \mathbf{2}^{\dagger}_B(\mathbf{2}_B(r)+1)+1 & \text{if } \boldsymbol{v} \notin \mathcal{N}'_1 \text{ and } \boldsymbol{v} \in \mathcal{N}'_2 \\ \hat{C}^{12}_{\boldsymbol{v}}(r) = r+1 & \text{if } \boldsymbol{v} \in \mathcal{N}'_1 \text{ and } \boldsymbol{v} \in \mathcal{N}'_2 \end{cases}$$

Periodic sequence. When B is a periodic sequence, minimal relative costs $\hat{c}_{\boldsymbol{v}}$ are also periodic sequences. Take the periodic sequence of the octagonal distance $B = (\overline{1,2})$, then $\mathbf{1}_B(r)_{r\geq 0} = (0,1,1,2,\ldots)$, $\mathbf{1}^{\dagger}_B(r)_{r>0} = (0,2,4,\ldots)$, $\hat{C}^1_{\boldsymbol{v}}(r)_{r\geq 0} = (1,3,3,5\ldots)$ and $\hat{c}^1_{\boldsymbol{v}}(r)_{r\geq 0} = (1,2,1,2\ldots)$. Similarly, $\mathbf{2}_B(r)_{r\geq 0} = (0,0,1,1,2,\ldots)$, $\mathbf{2}^{\dagger}_B(r)_{r>0} = (1,3,\ldots)$, $\hat{C}^2_{\boldsymbol{v}}(r)_{r\geq 0} = (2,2,4\ldots)$ and $\hat{c}^2_{\boldsymbol{v}}(r)_{r\geq 0} = (2,1,2,1\ldots)$.

Rate-based sequence. Suppose now that the sequence of neighborhoods is defined as a Beatty sequence (as in [3]): $B(r) = \lfloor \tau r \rfloor - \lfloor \tau(r-1) \rfloor$, with $\tau \in [1,2]$ so that $B(r) \in \{1,2\}$. $\mathbf{1}_B$ and $\mathbf{2}_B$ are respectively the cumulative sums of $2 - B(r) = \lceil (2-\tau)r \rceil - \lceil (2-\tau)(r-1) \rceil$ and $B(r) - 1 = \lfloor (\tau-1)r \rfloor - \lfloor (\tau-1)(r-1) \rfloor$. Then $\mathbf{1}_B(r) = \lceil (2-\tau)r \rceil$, $\mathbf{2}_B(r) = \lfloor (\tau-1)r \rfloor$, $\mathbf{1}^{\dagger}_B(r) = \lfloor \frac{r-1}{2-\tau} \rfloor$ and $\mathbf{2}^{\dagger}_B(r) = \lceil \frac{r}{\tau-1} - 1 \rceil$. This allows to compute $\hat{C}^1_{\boldsymbol{v}}$ and $\hat{C}^2_{\boldsymbol{v}}$ on the fly. For the octagonal distance, $\tau = \frac{3}{2}$, $\mathbf{1}_B(r) = \lceil \frac{r}{2} \rceil$, $\mathbf{2}_B(r) = \lfloor \frac{r}{2} \rfloor$, $\mathbf{1}^{\dagger}_B(r) = 2r - 2$ and $\mathbf{2}^{\dagger}_B(r) = 2r - 1$.

(a) (b) (c) (d) (e) (f)

Fig. 3. Neighborhoods used for the translated NS-distance transform. (a), and (b) are respectively the type 1 and 2 translated neighborhoods, $\mathcal{N}'_1$ and $\mathcal{N}'_2$. (c) and (d) and (e) are respectively $\mathcal{N}'_1 \setminus \mathcal{N}'_2$, $\mathcal{N}'_2 \setminus \mathcal{N}'_1$ and $\mathcal{N}'_1 \cap \mathcal{N}'_2$, each set associated to a different sequence of displacement costs. (f) is the whole set of neighbors, $\mathcal{N}'_1 \cup \mathcal{N}'_2$, used for the translated NS-DT.



(a) (b)

Fig. 4. (a) Octagonal DT of a binary image. (b) Translated octagonal DT. Highlighted centers of disks (a) are translated to the same location, highlighted (b) with value 3.

An example result of algorithm 1 for the translated octagonal distance (with displacement costs obtained either from sequence $B = (1, 2)$ either from $\tau = \frac{3}{2}$) is shown in Fig. 4b.

4.3 Symmetric DT from Asymmetric DT

Let $\{\boldsymbol{t}(r), r \in \mathbb{N}^*\}$ be a sequence of translation vectors such that the translated disks $D'(p, r) = D(p + \boldsymbol{t}(r), r)$ and $\check{D}'(p, r) = \check{D}(p - \boldsymbol{t}(r), r)$ are increasing according to the set inclusion. For a sequence of disks produced by translated neighborhoods defined in (21), the translation vectors are:

$$\begin{aligned} \boldsymbol{t}(r) &= \boldsymbol{t}(r-1) + \boldsymbol{t}_{B(r)} \\ &= \sum_j \mathbf{j}_B(r)\boldsymbol{t}_j \\ &= \left(\sum_{j=n}^{n} \mathbf{j}_B(r), \ldots, \sum_{j=1}^{n} \mathbf{j}_B(r) \right) \end{aligned}$$

In particular, for the 2D case:

$$\boldsymbol{t}(r) = (\mathbf{2}_B(r), \mathbf{1}_B(r) + \mathbf{2}_B(r)) = (\mathbf{2}_B(r), r)\,. \qquad (22)$$

```
Data: DT'_X: translated distance map of X
Result: DT_X: centered distance map of X
foreach p in DT' domain do
    if DT'(p) = 0 then
        DT(p) ← 0
    else
        foreach j do
            r ← max {1; DT'(p + t_j)}
            r ← j†_B(j_B(r)) + 1;  // First r ≥ DT'(p − t_j) such that B(r) = j
            while r ≤ DT'(p) do
                DT(p − t(r − 1)) ← r
                r ← j†_B(j_B(r) + 1) + 1;           // Next r such that B(r) = j
            end
        end
    end
end
```

Algorithm 2. Obtention of a regular (centered) DT from a translated DT

DT'_X has equivalence with values of DT_X:

$$\begin{aligned}\mathrm{DT}_X(p) \geq r \;&\Leftrightarrow\; \check{D}(p, r-1) \subseteq X \\ &\Leftrightarrow\; \check{D}'(p + \boldsymbol{t}(r-1), r-1) \subseteq X \\ &\Leftrightarrow\; \mathrm{DT}'_X(p + \boldsymbol{t}(r-1)) \geq r\,. \end{aligned} \tag{23}$$

Consequently:

$$\begin{aligned}\mathrm{DT}_X(p) = r \;&\Leftrightarrow\; \mathrm{DT}_X(p) \geq r \text{ and } \mathrm{DT}_X(p) < r+1 \\ &\Leftrightarrow\; \mathrm{DT}'_X(p + \boldsymbol{t}(r)) \leq r \leq \mathrm{DT}'_X(p + \boldsymbol{t}(r-1))\,. \end{aligned} \tag{24}$$

Knowing $\mathrm{DT}'_X(p)$ and $\mathrm{DT}'_X(p+\boldsymbol{t})$, we can deduce the values of $\mathrm{DT}_X(p-\boldsymbol{t}(r-1))$ for all values of r between $\mathrm{DT}'_X(p+\boldsymbol{t})$ and $\mathrm{DT}'_X(p)$ for which $\boldsymbol{t}(r) = \boldsymbol{t}(r-1)+\boldsymbol{t}$, *i.e.*, $\boldsymbol{t} = \boldsymbol{t}_{B(r)}$. Algorithm 2 recovers the values r of the centered DT by selecting all r in the interval $[\mathrm{DT}'_X(p+\boldsymbol{t}_j), \mathrm{DT}'_X(p)]$ such that $B(r) = j$. Iterating through values r with $B(r) = j$ is achieved using prop. 1. Values of DT'_X become available before the whole image is computed. For instance, in a standard raster scan, as soon as line y is processed, all lines of DT'_X above $y - r_{max}$ are fully recovered (where $r_{\max}$ denotes the maximal value of DT' in that line).

5 Conclusion

In this paper, a path-based pseudo-distance scheme where displacement costs vary both with the displacement vector and with the travelled distance was presented. This scheme is generic enough to describe neighborhood-sequence distances, weighted distances as well as generalized distances produced by Minkowski sums. It was shown that a set of displacement costs can be provided in a minimal form, where each displacement vector is assigned a non-decreasing sequence of costs, without altering the distance function. These non-decreasing sequences are directly applied in the distance transform algorithm to keep track of the costs of minimal paths from the background. An application to a translated neighborhood-sequence distance transform in a single scan was presented along with a method to recover the proper, centered, distance transform. Combined methods provide partial result with a minimal delay, before the input image is fully processed. Their efficiency can benefit all applications where neighborhood-sequence distances are involved, particularly when pipelined processing architectures are involved, or when the size of objects in the source image is limited.

The pseudo-distance presented here is strongly linked to the properties of non-decreasing integer sequences studied by Lambek and Moser. An implementation in C language is publicly available at `http://www.irccyn.ec-nantes.fr/~normand/LUTBasedNSDistanceTransform`.

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