Improved Lower Bounds on the Area Requirements of Series-Parallel Graphs*

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Abstract. We show that there exist series-parallel graphs requiring $\Omega(n2^{\sqrt{\log n}})$ area in any straight-line or poly-line grid drawing, improving the previously best known $\Omega(n \log n)$ lower bound.

1 Introduction

Determining asymptotic bounds for the area requirements of straight-line and poly-line drawings of planar graphs is a classical topic in the Graph Drawing literature. Groundbreaking works of the beginning of the nineties [5,10] have shown that every n-vertex planar graph admits a planar straight-line drawing in an $O(n^2)$ grid. Such a bound is worst-case optimal, even for poly-line drawings [7,5]. Hence, it is natural to search for interesting sub-classes of planar graphs admitting sub-quadratic area drawings.

In this paper we deal with series-parallel graphs, a class of planar graphs that has been widely investigated in Graph Theory and Graph Drawing (see, e.g., [11,8,1,6]). Series-parallel graphs can be equivalently defined as the graphs excluding K_4 as a minor or, inductively, by series and parallel compositions of smaller series-parallel graphs.

Biedl [2,3] proved that a series-parallel graph with n vertices admits a poly-line grid drawing in $O(n^{3/2})$ area. She achieved such a bound by first constructing *visibility representations* of series-parallel graphs in $O(n^{3/2})$ area and by then turning such representations into poly-line drawings with asymptotically the same area. No sub-quadratic area upper bound is known for straight-line grid drawings of series-parallel graphs.

The author proved [9] that there exist series-parallel graphs requiring $\Omega(n \log n)$ area in any straight-line or poly-line grid drawing. To achieve such a bound, the following theorem[1] was proved in [9], improving upon previous results of Biedl et al. [4].

Theorem 1. *Every planar straight-line or poly-line grid drawing of* $K_{2,n}$ *in a* $W \times H$ *grid satisfies* $\max\{W, H\} \geq c \cdot n$*, for some constant* $c \leq 1/2$.

In this paper, we prove the following.

Theorem 2. *There exist series-parallel graphs with* n *vertices requiring* $\Omega(2^{\sqrt{\log n}})$ *width and* $\Omega(2^{\sqrt{\log n}})$ *height in any straight-line or poly-line grid drawing.*

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[1] Theorem 1 is stated in [9] in an equivalent form using the $\Omega(n)$ notation.

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Such a result is achieved by carefully constructing a graph out of several copies of $K_{2,n}$ and by then exploiting Theorem 1 and some further geometric considerations. Theorem 1, together with Theorem 2, directly implies the following.

Theorem 3. *There exist series-parallel graphs with* n *vertices requiring* $\Omega(n2^{\sqrt{\log n}})$ *area in any straight-line or poly-line grid drawing.*

We remark that the function $2^{\sqrt{\log n}}$ is asymptotically greater than any polylogarithmic function of n and smaller than any polynomial function of n.

2 Preliminaries

A *planar grid drawing* of a graph is a mapping of each vertex to a distinct point of the plane with integer coordinates and of each edge to a Jordan curve between the endpoints of the edge, so that no two edges intersect except, possibly, at common endpoints. In the following we always refer to planar grid drawings. A *straight-line* drawing is such that all edges are rectilinear segments. A *poly-line* drawing is such that the edges are sequences of rectilinear segments. In a poly-line drawing a *bend* is a point in which an edge changes its slope, i.e., a point common to two consecutive segments in the sequence of segments representing the edge. In a grid drawing bends have integer coordinates. A *polygonal path* is a poly-line grid drawing of a path. The *bounding box* of a drawing Γ is the smallest rectangle with sides parallel to the axes that covers Γ completely. The *height* (*width*) of Γ is the height (resp. width) of its bounding box. The *area* of Γ is the height of Γ times its width.

A drawing of the complete bipartite graph $K_{2,n}$ can be thought as a drawing of n paths that start and end at the same two vertices and that do not share any other vertex. In the following we will refer to such paths as to the *paths of* $K_{2,n}$.

In the next section, we will use the following lemmata [9].

Lemma 1. *Consider any poly-line grid drawing of* $K_{2,n}$*, any path* π *of* $K_{2,n}$*, and any vector* $\boldsymbol{v}$*. There exists a grid point* $p \in \pi$ *such that* $\boldsymbol{v} \cdot p \geq \boldsymbol{v} \cdot p'$*, for any point* $p' \in \pi$*.*

Lemma 2. *Let* a *and* b *be the endvertices of the paths of* $K_{2,n}$*. Consider any planar drawing of* $K_{2,n}$*. Let* l *be any line that does not intersect nor contain the open segment* $\overline{ab}$*. No three paths* $\pi_1, \pi_2,$ *and* π_3 *of* $K_{2,n}$ *exist such that: (i)* $\pi_1, \pi_2,$ *and* π_3 *do not intersect each other; (ii)* $\pi_1, \pi_2,$ *and* π_3 *are contained in the closed half-plane delimited by* l *and containing* a *and* b*; (iii) each of* $\pi_1, \pi_2,$ *and* π_3 *touches* l *at least once.*

3 Proof of Theorem 2

As straight-line drawings are also poly-line drawings, it suffices to prove Theorem 2 for poly-line drawings. Let $f(n)$ be a function to be computed later and let $d = c/4$, where c is the constant of Theorem 1. Observe that $d \leq 1/8$.

Graph G_1 is $K_{2,f(n)-2}$. Graph G_{i+1} is defined as follows. Consider $f(n)$ copies $G_{i,1,1}, G_{i,1,2}, G_{i,2,1}, G_{i,2,2}, \dots, G_{i,j,1}, G_{i,j,2}, \dots, G_{i,f(n)/2,1}, G_{i,f(n)/2,2}$ of G_i; construct $f(n)/2$ series-parallel graphs $G_{i,1}, G_{i,2}, \dots, G_{i,j}, \dots, G_{i,f(n)/2}$, where $G_{i,j}$ is

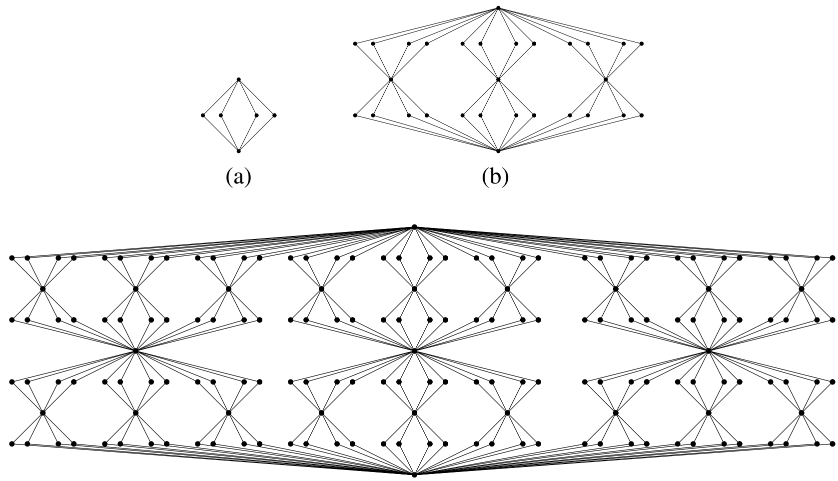

Fig. 1. Graphs G_i, with $f(n) = 6$. (a) G_1. (b) G_2. (c) G_3

the series composition of $G_{i,j,1}$ and $G_{i,j,2}$; then, G_{i+1} is the parallel composition of graphs $G_{i,1}, G_{i,2}, \ldots, G_{i,f(n)/2}$. See Fig. 1.

First, we prove Theorem 2 for sufficiently large graphs, that is, for graphs having a number of vertices that is at least some constant n_0 to be determined later. From now till it is otherwise specified, assume that $n \geq n_0$.

Suppose that $f(n) \geq 8$, $\forall n \geq n_0$. Let n be the number of vertices of graph G_k. We have the following main lemma.

Lemma 3. *Let Γ_i be any poly-line grid drawing of G_i and let a_i and b_i be the poles of G_i, for each $1 \leq i \leq k$. Then, one of the following holds:*

- *Condition 1: The height and the width of Γ_i are both greater than or equal to $d \cdot f(n)$.*
- *Condition 2: The width of Γ_i is greater than or equal to $d \cdot f(n)$ and Γ_i contains a polygonal path l_i connecting a_i to b_i that has height greater than or equal to 2^i and such that, for every point $p \in l_i$, $\min\{y(a_i), y(b_i)\} \leq y(p) \leq \max\{y(a_i), y(b_i)\}$; or the height of Γ_i is greater than or equal to $d \cdot f(n)$ and Γ_i contains a polygonal path l_i connecting a_i to b_i that has width greater than or equal to 2^i and such that, for every point $p \in l_i$, $\min\{x(a_i), x(b_i)\} \leq x(p) \leq \max\{x(a_i), x(b_i)\}$.*

Proof: We prove the statement by induction on i. In the base case, consider any poly-line grid drawing Γ_1 of G_1. By Theorem 1, one of the height and the width of Γ_1, say the width of Γ_1, is at least $c \cdot f(n)$, hence it is at least $d \cdot f(n)$.

Assume, without loss of generality, that $y(a_1) \leq y(b_1)$. Suppose that at least $2d \cdot f(n)$ paths of $G_1 = K_{2,f(n)-2}$ intersect the open half-plane $H^-(a_1)$ defined as $y < y(a_1)$ or the open half-plane $H^+(b_1)$ defined as $y > y(b_1)$. By Lemma 1 with $\boldsymbol{v} = (0, -1)$, for each path π of G_1 intersecting $H^-(a_1)$, a grid point $p \in \pi$ exists whose y-coordinate is minimum among the points of π. Clearly, p belongs to $H^-(a_1)$. Hence, p belongs to a horizontal grid line h not intersecting nor containing segment $\overline{a_1b_1}$. By Lemma 2, at most two paths of G_1 have their points with smallest y-coordinate in h. Analogously, by Lemma 1 with $\boldsymbol{v} = (0,1)$, for each path π of G_1 that intersects $H^+(b_1)$, a grid point $p \in \pi$ exists whose y-coordinate is maximum among the points of π. Clearly, p belongs to $H^+(b_1)$. Hence, p belongs to a horizontal grid line h not intersecting nor containing segment $\overline{a_1b_1}$. By Lemma 2, at most two paths of G_1 have their points with greatest y-coordinate in h. Hence, as $2d \cdot f(n)$ paths of G_1 intersect $H^-(a_1)$ or $H^+(b_1)$, it follows that Γ_1 has height at least $d \cdot f(n)$.

Now suppose that no $2d \cdot f(n)$ paths of G_1 intersect $H^-(a_1)$ or $H^+(b_1)$. Then, since $d \leq 1/8$, at least $f(n) - 2 - 2d \cdot f(n) + 1 \geq 3f(n)/4 - 1$ paths of G_1 are such that, for every point p of any such a path, $y(a_1) \leq y(p) \leq y(b_1)$. By planarity of Γ_1 at most one path of G_1 contains a point p such that $y(p) = y(a_1)$ and $x(p) < x(a_1)$. Analogously, at most one path of G_1 contains a point p such that $y(p) = y(a_1)$ and $x(p) > x(a_1)$, at most one path of G_1 contains a point p such that $y(p) = y(b_1)$ and $x(p) < x(b_1)$, and at most one path of G_1 contains a point p such that $y(p) = y(b_1)$ and $x(p) > x(b_1)$. Since $f(n) \geq 8$, it follows that $3f(n)/4 - 1 \geq 5$, hence there is at least one path of G_1 whose only vertex $v \neq a_1, b_1$ is such that $y(v) > y(a_1)$ and $y(v) < y(b_1)$. Then, the polygonal path (a_1, v, b_1) has height at least two and is such that, for every point $p \in (a_1, v, b_1)$, $y(a_1) \leq y(p) \leq y(b_1)$, thus proving the base case.

In the inductive case, consider any poly-line grid drawing Γ_{i+1} of G_{i+1}, containing drawings $\Gamma_{i,1,1}, \Gamma_{i,1,2}, \Gamma_{i,2,1}, \Gamma_{i,2,2}, \dots, \Gamma_{i,j,1}, \Gamma_{i,j,2}, \dots, \Gamma_{i,f(n)/2,1}, \Gamma_{i,f(n)/2,2}$ of $G_{i,1,1}, G_{i,1,2}, G_{i,2,1}, G_{i,2,2}, \dots, G_{i,j,1}, G_{i,j,2}, \dots, G_{i,f(n)/2,1}, G_{i,f(n)/2,2}$, respectively. By induction, for $1 \leq j \leq f(n)/2$ and $1 \leq k \leq 2$, $\Gamma_{i,j,k}$ satisfies Condition 1 or 2.

If two indices $1 \leq j \leq f(n)/2$ and $1 \leq k \leq 2$ exist such that $\Gamma_{i,j,k}$ satisfies Condition 1, then the width and the height of $\Gamma_{i,j,k}$ are both greater than or equal to $d \cdot f(n)$, hence so are the width and the height of Γ_{i+1}.

Hence, we can assume that, for every $1 \leq j \leq f(n)/2$ and $1 \leq k \leq 2$, $\Gamma_{i,j,k}$ satisfies Condition 2. If indices $1 \leq j', j'' \leq f(n)/2$ and $1 \leq k', k'' \leq 2$ exist, where $j' = j''$ and $k' = k''$ do not hold simultaneously, such that the width of $\Gamma_{i,j',k'}$ is greater than or equal to $d \cdot f(n)$ and the height of $\Gamma_{i,j'',k''}$ is greater than or equal to $d \cdot f(n)$, then the width and the height of Γ_{i+1} are both greater than or equal to $d \cdot f(n)$.

Hence, we can assume that, for every $1 \leq j \leq f(n)/2$ and $1 \leq k \leq 2$, the width of $\Gamma_{i,j,k}$ is greater than or equal to $d \cdot f(n)$ and $\Gamma_{i,j,k}$ contains a polygonal path $l_{i,j,k}$ connecting a_i to b_i that has height greater than or equal to 2^i and such that, for every point $p \in l_{i,j,k}$, $\min\{y(a_i), y(b_i)\} \leq y(p) \leq \max\{y(a_i), y(b_i)\}$; the case in which, for every $1 \leq j \leq f(n)/2$ and $1 \leq k \leq 2$, the height of $\Gamma_{i,j,k}$ is greater than or equal to $d \cdot f(n)$ and $\Gamma_{i,j,k}$ contains a polygonal path $l_{i,j,k}$ connecting a_i to b_i that has width greater than or equal to 2^i and such that, for every point $p \in l_{i,j,k}$, $\min\{x(a_i), x(b_i)\} \leq x(p) \leq \max\{x(a_i), x(b_i)\}$ can be treated analogously.

Denote by $l_{i,j}$ the path connecting a_{i+1} and b_{i+1} composed of $l_{i,j,1}$ and $l_{i,j,2}$. Assume, without loss of generality, that $y(a_{i+1}) \leq y(b_{i+1})$. Suppose that at least $2d \cdot f(n)$ paths $l_{i,j}$ intersect the open half-plane $H^-(a_{i+1})$ defined as $y < y(a_{i+1})$ or the open half-plane $H^+(b_{i+1})$ defined as $y > y(b_{i+1})$. By Lemma 1 with $\boldsymbol{v} = (0, -1)$, for each path $l_{i,j}$ that intersects $H^-(a_{i+1})$, a grid point $p \in l_{i,j}$ exists whose y-coordinate is minimum among the points of $l_{i,j}$. Clearly, p belongs to $H^-(a_{i+1})$. Hence, p belongs to a horizontal grid line h not intersecting nor containing segment $\overline{a_{i+1}b_{i+1}}$. By Lemma 2, at most two paths $l_{i,j}$ have their points with smallest y-coordinate in h. Analogously, by Lemma 1 with $\boldsymbol{v} = (0, 1)$, for each path $l_{i,j}$ that intersects $H^+(b_{i+1})$, a grid point $p \in l_{i,j}$ exists whose y-coordinate is maximum among the points of $l_{i,j}$. Clearly, p belongs to $H^+(b_{i+1})$. Hence, p belongs to a horizontal grid line h not intersecting nor contain segment $\overline{a_{i+1}b_{i+1}}$. By Lemma 2, at most two paths $l_{i,j}$ have their points with greatest y-coordinate in h. Hence, as $2d \cdot f(n)$ paths $l_{i,j}$ intersect $H^-(a_{i+1})$ or $H^+(b_{i+1})$, it follows that Γ_{i+1} has height at least $d \cdot f(n)$.

Now suppose that no $2d \cdot f(n)$ paths $l_{i,j}$ intersect $H^-(a_{i+1})$ or $H^+(b_{i+1})$. Then, since $d \leq 1/8$, at least $f(n) - 2d \cdot f(n) + 1 \geq 3f(n)/4 + 1$ paths $l_{i,j}$ are such that, for every point p of any such a path, $y(a_{i+1}) \leq y(p) \leq y(b_{i+1})$. By planarity of Γ_{i+1} at most one path $l_{i,j}$ contains a point p such that $y(p) = y(a_{i+1})$ and $x(p) < x(a_{i+1})$. Analogously, at most one path $l_{i,j}$ contains a point p such that $y(p) = y(a_{i+1})$ and $x(p) > x(a_{i+1})$, at most one path $l_{i,j}$ contains a point p such that $y(p) = y(b_{i+1})$ and $x(p) < x(b_{i+1})$, and at most one path $l_{i,j}$ contains a point p such that $y(p) = y(b_{i+1})$ and $x(p) > x(b_{i+1})$. Since $f(n) \geq 8$, it follows that $3f(n)/4 + 1 \geq 5$, hence there is at least one path $l_{i,j}$ composed of path $l_{i,j,1}$, that connects the poles a_{i+1} and v of $G_{i,j,1}$, and of path $l_{i,j,2}$, that connects the poles b_{i+1} and v of $G_{i,j,2}$, such that $y(v) > y(a_{i+1})$ and $y(v) < y(b_{i+1})$. By inductive hypothesis, $l_{i,j,1}$ has height greater than or equal to 2^i and, for every point $p \in l_{i,j,1}$, $y(a_{i+1}) \leq y(p) \leq y(v)$; further, $l_{i,j,2}$ has height greater than or equal to 2^i and, for every point $p \in l_{i,j,2}$, $y(v) \leq y(p) \leq y(b_{i+1})$; hence, $l_{i,j}$ has height greater than or equal to 2^{i+1} and, for every point $p \in l_{i,j}$, $y(a_{i+1}) \leq y(p) \leq y(b_{i+1})$, thus completing the induction. □

Corollary 1. *Any poly-line grid drawing of G_k has height and width that are both greater than or equal to* $\min\{d \cdot f(n), 2^k\}$.

Let $f(n) = n^{x(k)}$. By construction $|G_1| = n^{x(k)}$; since G_i is composed of $f(n) = n^{x(k)}$ copies of G_{i-1}, $|G_i| \leq n^{x(k)} \cdot |G_{i-1}|$; inductively, we obtain $|G_k| \leq n^{k \cdot x(k)}$. Assuming $|G_k| = n$, we get $x(k) \geq 1/k$ and $f(n) \geq n^{1/k}$.

We now choose k in such a way that $n^{1/k}$ and 2^k are equal. This is done as follows. $2^k = n^{1/k} \Rightarrow \log_2(2^k) = \log_2(n^{1/k}) \Rightarrow k \log_2(2) = 1/k \log_2(n) \Rightarrow k^2 = \log_2(n) \Rightarrow k = \sqrt{\log_2(n)}$. By Corollary 1, both the height and the width of Γ_k, with $k = \sqrt{\log_2(n)}$, are greater than or equal to $\min\{d \cdot n^{1/\sqrt{\log_2(n)}}, 2^{\sqrt{\log_2(n)}}\} = d \cdot 2^{\sqrt{\log_2(n)}} = \Omega(2^{\sqrt{\log_2(n)}})$, and Theorem 2 follows if $n \geq n_0$.

As we need $f(n) = 2^{\sqrt{\log_2(n)}} \geq 8, \forall n \geq n_0$, then $n_0 = 512$. However, $d \cdot 2^{\sqrt{\log_2(n)}} < 1$, for all $n < 512$, as $d \leq 1/8$. Since every drawing of a graph that is not a collection of paths has height and width at least one, the $d \cdot 2^{\sqrt{\log_2(n)}}$ lower bound holds also for graphs with less than 512 nodes, thus completing the proof of Theorem 2.

4 Conclusions and Open Problems

In this paper we have shown that there exist series-parallel graphs with n vertices requiring $\Omega(n2^{\sqrt{\log n}})$ area in any straight-line or poly-line grid drawing.

The best known area upper bound for poly-line grid drawings of series-parallel graphs is $O(n^{3/2})$ [2,3], while no sub-quadratic area upper bound is known in the case of straight-line grid drawings. Hence, in both cases, closing the gap between upper and lower bound is an intriguing challenge.

Concerning straight-line drawings, David Wood [12] conjectures the following: Let $p_1, \ldots, p_k$ be positive integers. Let $G(p_1)$ be the graph obtained from K_3 by adding p_1 new vertices adjacent to v and w for each edge (v, w) of K_3. For $k \geq 2$, let $G(p_1, p_2, \ldots, p_k)$ be the graph obtained from $G(p_1, p_2, \ldots, p_{k-1})$ by adding p_k new vertices adjacent to v and w for each edge (v, w) of $G(p_1, p_2, \ldots, p_{k-1})$. Observe that $G(p_1, p_2, \ldots, p_k)$ is a series-parallel graph.

Conjecture 1. (D. R. Wood) Every straight-line grid drawing of $G(p_1, p_2, \ldots, p_k)$ requires $\Omega(n^2)$ area for some choice of k and $p_1, p_2, \ldots, p_k$.

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