Monotone Drawings of Graphs*

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Abstract. We study a new standard for visualizing graphs: A monotone drawing is a straight-line drawing such that, for every pair of vertices, there exists a path that monotonically increases with respect to some direction. We show algorithms for constructing monotone planar drawings of trees and biconnected planar graphs, we study the interplay between monotonicity, planarity, and convexity, and we outline a number of open problems and future research directions.

1 Introduction

A traveler that consults a road map to find a route from a site u to a site v would like to easily spot at least one path connecting u and v. Such a task is harder if each path from u to v on the map has legs moving away from v. Travelers rotate maps to better perceive their content. Hence, even if in the original orientation of the map all the paths from u to v have annoying back and forth legs, the traveler might be happy to find at least one orientation where a path from u to v smoothly flows from left to right.

Leaving the road map metaphora for the Graph Drawing terminology, we say that a path P in a straight-line drawing of a graph is *monotone* if there exists a line l such that the orthogonal projections of the vertices of P on l appear along l in the order induced by P. A straight-line drawing of a graph is *monotone* if it contains at least one monotone path for each pair of vertices. Having at disposal a monotone drawing (map), for each pair of vertices u and v a user (traveler) can find a rotation of the drawing such that there exists a path from u to v always increasing in the x-coordinate.

In a monotone drawing each monotone path is monotone with respect to a different line. *Upward drawings* [7,10] are related to monotone drawings, as in an upward drawing every directed path is monotone. Even more related to monotone drawings are *greedy drawings* [13,12,2]. Namely, in a greedy drawing, between any two vertices a path exists such that the Euclidean distance from an intermediate vertex to the destination decreases at every step, while, in a monotone drawing, between any two vertices a path and a line l exist such that the Euclidean distance from the projection of an intermediate vertex on l to the projection of the destination on l decreases at every step.

Monotone drawings have a strict correlation with an important problem in Computational Geometry: Arkin, Connelly, and Mitchell [3] studied how to find monotone trajectories connecting two given points in the plane avoiding convex obstacles. As a

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corollary, they proved that every planar convex drawing is monotone. Hence, the graphs admitting a convex drawing [6] have a planar monotone drawing. Such a class of graphs is a super-class (sub-class) of the triconnected (biconnected) planar graphs.

In this paper we first deal with trees (Sect. 4). We prove several properties relating the monotonicity of a tree drawing to its planarity and "convexity" [5]. Moreover, we show two algorithms for constructing monotone planar grid drawings of trees. The first one constructs drawings lying on a grid of size $O(n^{1.6}) \times O(n^{1.6})$. The second one has a better area requirement, namely $O(n^3)$, but a worse $\Omega(n)$ aspect ratio.

The existence of monotone drawings of trees allows to construct a monotone drawing of any graph G by drawing any of its spanning trees. However, the obtained monotone drawing could be non-planar even if G is a planar graph. Motivated by this and since every triconnected planar graph admits a planar monotone drawing, we devise an algorithm to construct planar monotone drawings of biconnected planar graphs (Sect. 5). Such an algorithm exploits the SPQR-tree decomposition of a biconnected planar graph. We conclude the paper with several open problems (Sect. 6). Due to space reasons several proofs are omitted and can be found in [1].

2 Definitions and Preliminaries

A *straight-line drawing* of a graph is a mapping of each vertex to a distinct point of the plane and of each edge to a segment connecting its endpoints. A drawing is *planar* if its edges do not cross but, possibly, at common endpoints. A graph is *planar* if it admits a planar drawing. A planar drawing partitions the plane into topologically connected regions, called *faces*. The unbounded face is the *outer face*. A *strictly convex drawing* (resp. a *(non-strictly) convex drawing*) is a straight-line planar drawing in which each face is delimited by a strictly (resp. non-strictly) convex polygon.

We denote by $P(v_1, v_m)$ a path between vertices v_1 and v_m. A graph G is *connected* if every pair of vertices is connected by a path and is *biconnected* (resp. *triconnected*) if removing any vertex (resp. any two vertices) leaves G connected. A *subdivision* of G is obtained by replacing each edge of G with a path. If each path has at most one internal vertex we have a 1-*subdivision*. A *subdivision of a drawing* Γ of G is a drawing Γ' of a subdivision G' of G such that, for every edge (u, v) of G that has been replaced by a path $P(u, v)$ in G', u and v are drawn at the same point in Γ and in Γ', and all the vertices of $P(u, v)$ lie on the segment between u and v.

Let p be a point in the plane and l an half-line starting at p. The *slope* of l, denoted by $slope(l)$, is the angle spanned by a counter-clockwise rotation that brings a horizontal half-line starting at p and directed towards increasing x-coordinates to coincide with l. We consider slopes that are equivalent modulo 2π as the same slope (e.g., $\frac{3}{2}\pi$ is regarded as the same slope as -$\frac{\pi}{2}$). Let Γ be a drawing of a graph G and (u, v) an edge of G. The half-line starting at u and passing through v, denoted by $d(u, v)$, is the *direction* of (u, v). The *slope* of (u, v), denoted by $slope(u, v)$, is the slope of $d(u, v)$. Observe that $slope(u, v) = slope(v, u) - \pi$. When comparing directions and their slopes, we assume that they are applied at the origin of the axes. An edge (u, v) is *monotone* with respect to a half-line l if it has a "positive projection" on l, i.e., if $slope(l) - \frac{\pi}{2} < slope(u, v) < slope(l) + \frac{\pi}{2}$. A path $P(u_1, u_n) = (u_1, \ldots, u_n)$ is *monotone with respect to a half-line* l if (u_i, u_{i+1}) is monotone with respect to l, for $i = 1, \ldots, n-1$; $P(u_1, u_n)$ is *monotone* if there exists a half-line l such that $P(u_1, u_n)$ is monotone with respect to l. Observe that if a path $P(u_1, u_n) = (u_1, \ldots, u_n)$ is monotone with respect to l, then the orthogonal projections on l of $u_1, \ldots, u_n$ appear in this order along l. A drawing Γ of a graph G is *monotone* if, for each pair of vertices u and v in G, there exists a monotone path $P(u, v)$ in Γ. Observe that monotonicity implies connectivity.

The *Stern-Brocot tree* [14,4] is an infinite tree whose nodes are in bijective mapping with the irreducible positive rational numbers. The Stern-Brocot tree $\mathcal{SB}$ has two nodes $0/1$ and $1/0$ that are connected to the same node $1/1$, where $1/1$ is the right child of $0/1$ and $1/1$ is the left child of $1/0$. An ordered binary tree is then rooted at $1/1$ as follows. Consider a node y/x of the tree. The left child of y/x is the node $(y + y')/(x + x')$, where y'/x' is the ancestor of y/x that is closer to y/x (in terms of graph-theoretic distance in $\mathcal{SB}$) and that has y/x in its right subtree. The right child of y/x is the node $(y+y'')/(x+x'')$, where y''/x'' is the ancestor of y/x that is closer to y/x and that has y/x in its left subtree. The *first level* of $\mathcal{SB}$ is composed of node $1/1$. The *i-th level* of $\mathcal{SB}$ is composed of the children of the nodes of the $(i-1)$-th level of $\mathcal{SB}$. The following property of the Stern-Brocot tree is well-known and easy to observe:

Property 1. The sum of the numerators of the elements of the i-th level of $\mathcal{SB}$ is 3^{i-1} and the sum of the denominators of the elements of the i-th level of $\mathcal{SB}$ is 3^{i-1}.

To decompose a biconnected graph into its triconnected components, we use the *SPQR-tree*, a data structure introduced by Di Battista and Tamassia [8,9]. Definitions about SPQR-trees can be found in [8,9,11] and in [1]. Here we give some notation. Let $\mathcal{T}$ be the SPQR-tree of a graph G. We denote by $pert(\mu)$ the *pertinent* of a node μ of $\mathcal{T}$, that is, the subgraph of G induced by the vertices of G in μ. We denote by $skel(\mu)$ the *skeleton* of a node μ of $\mathcal{T}$, that is, the graph representing the arrangement of the triconnected components composing $pert(\mu)$. The edges of $skel(\mu)$ are called *virtual edges*. The nodes shared by $pert(\mu)$ and the rest of the graph are called *poles* of μ.

3 Properties of Monotone Drawings

Property 2. Any sub-path of a monotone path is monotone.

Property 3. A path $P(u_1, u_n) = (u_1, u_2, \ldots, u_n)$ is monotone if and only if it contains two edges e_1 and e_2 such that the closed wedge centered at the origin of the axes, delimited by the two half-lines $d(e_1)$ and $d(e_2)$, and having an angle smaller than π, contains all the half-lines $d(u_i, u_{i+1})$, for $i = 1, \ldots, n-1$.

Edges e_1 and e_2 as in Prop. 3 are the *extremal edges* of $P(u_1, u_n)$. The closed wedge delimited by $d(e_1)$ and $d(e_2)$ and containing all the half-lines $d(u_i, u_{i+1})$, for $i = 1, \ldots, n-1$, is the *range* of $P(u_1, u_n)$ and is denoted by $range(P(u_1, u_n))$, while the closed wedge delimited by $d(e_1) - \pi$ and $d(e_2) - \pi$, and not containing $d(e_1)$ and $d(e_2)$, is the *opposite range* of $P(u_1, u_n)$ and is denoted by $opp(P(u_1, u_n))$.

Property 4. The range of a monotone path $P(u_1, u_n)$ contains the half-line from u_1 through u_n.

Lemma 1. *Let* $P(u_1, u_n) = (u_1, u_2, \ldots, u_n)$ *be a monotone path and let* (u_n, u_{n+1}) *be an edge. Then, path* $P(u_1, u_{n+1}) = (u_1, u_2, \ldots, u_n, u_{n+1})$ *is monotone if and only if* $d(u_n, u_{n+1})$ *is not contained in opp*$(P(u_1, u_n))$*. Further, if* $P(u_1, u_{n+1})$ *is monotone,* $range(P(u_1, u_n)) \subseteq range(P(u_1, u_{n+1}))$.

Corollary 1. *Let* $P(u_1, u_n) = (u_1, \ldots, u_n)$ *and* $P(u_n, u_{n+k}) = (u_n, \ldots, u_{n+k})$ *be monotone paths. Then, path* $P(u_1, u_{n+k}) = (u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+k})$ *is monotone if and only if* $range(P(u_1, u_n)) \cap opp(P(u_n, u_{n+k})) = \emptyset$*. Further, if* $P(u_1, u_{n+k})$ *is monotone,* $range(P(u_1, u_n)) \cup range(P(u_n, u_{n+k})) \subseteq range(P(u_1, u_{n+k}))$.

The following properties relate monotonicity to planarity and convexity.

Property 5. A monotone path is planar.

Lemma 2. *[3] Any strictly convex drawing of a planar graph is monotone.*

In the following we will construct non-strictly convex drawings of graphs. Observe that any graph containing a degree-2 vertex does not admit a strictly convex drawing, but it might admit a non-strictly convex drawing. While not every non-strictly convex drawing is monotone, we can relate non-strict convexity and monotonicity:

Lemma 3. *Any non-strictly convex drawing of a graph such that each set of parallel edges forms a collinear path is monotone.*

On the relationship between convexity and monotonicity, we also have:

Lemma 4. *Consider a strictly convex drawing* Γ *of a graph* G*. Let* u*,* v*, and* w *be three consecutive vertices incident to the outer face. Let* d *be any half-line that splits the angle* $\widehat{uvw}$ *into two angles smaller than* $\frac{\pi}{2}$*. Then, for each vertex* t *of* G*, there exists a path from* v *to* t *in* Γ *that is monotone with respect to* d*.*

Next, we provide a powerful tool for "transforming" monotone drawings.

Lemma 5. *An affine transformation of a monotone drawing gives a monotone drawing.*

4 Monotone Drawings of Trees

The first property we present is on the relationship between monotonicity and planarity, and descends from the fact that every monotone path is planar (by Prop. 5) and that in a tree there exists exactly one path between every pair of vertices.

Property 6. Every monotone drawing of a tree is planar.

The second property relates monotonicity and convexity. A *convex drawing* of a tree T [5] is a straight-line planar drawing such that replacing each edge between an internal vertex u and a leaf v with a half-line starting at u through v yields a partition of the plane into convex unbounded polygons. A convex drawing of a tree might not be monotone, because of the presence of two parallel edges. However, if such parallel lines are not used, then a convex drawing is also monotone. Define a *strictly convex drawing* of a tree T as a straight-line planar drawing such that each set of parallel edges forms a collinear path and such that replacing every edge of T between an internal vertex u and a leaf v with a half-line starting at u through v yields a partition of the plane into convex unbounded polygons. We have the following:

Property 7. Every strictly convex drawing of a tree is monotone.

A simple modification of the algorithm presented in [5] constructs strictly convex drawings of trees. Hence, monotone drawings exist for all trees.

We introduce *slope-disjoint drawings* of trees and show that they are monotone. Let T be a tree rooted at a node r. Denote by $T(u)$ the subtree of T rooted at a node u. A *slope-disjoint* drawing of T is such that: (P1) For every node $u \in T$, there exist two angles $\alpha_1(u)$ and $\alpha_2(u)$, with $0 < \alpha_1(u) < \alpha_2(u) < \pi$, such that, for every edge e that is either in $T(u)$ or that connects u with its parent, it holds that $\alpha_1(u) < slope(e) < \alpha_2(u)$; (P2) for every two nodes $u, v \in T$ with v child of u, it holds that $\alpha_1(u) < \alpha_1(v) < \alpha_2(v) < \alpha_2(u)$; (P3) for every two nodes v_1, v_2 with the same parent, it holds that $\alpha_1(v_1) < \alpha_2(v_1) < \alpha_1(v_2) < \alpha_2(v_2)$. We have the following:

Theorem 1. *Every slope-disjoint drawing of a tree is monotone.*

Proof: Let T be a tree and let Γ be a slope-disjoint drawing of T. We show that, for every two vertices $u, v \in T$, a monotone path between u and v exists in Γ. Let w be the lowest common ancestor of u and v in T.

If $w = u$, then, by P1, for every edge e in the path $P(u, v)$, $0 < slope(e) < \pi$. Hence, $P(u, v)$ is monotone with respect to a half-line with slope $\pi/2$. Analogously, if $w = v$ then $P(u, v)$ is monotone with respect to a half-line with slope $-\pi/2$. If $w \neq u, v$, let u' and v' be the children of w in T such that $u \in T(u')$ and $v \in T(v')$. Path $P(u, v)$ is composed of path $P(u, w)$ and of path $P(w, v)$. As before, $P(u, w)$ is monotone with respect to a half-line with slope $-\pi/2$. By P1, for every edge $e \in P(u, w)$, $\alpha_1(u') - \pi < slope(e) < \alpha_2(u') - \pi$. Hence, $\alpha_1(u') < slope(l) < \alpha_2(u')$, for each half-line l contained into the closed wedge $opp(P(u, w))$. Analogously, $P(w, v)$ is monotone with respect to a half-line with slope $\pi/2$ and, by P1, for every edge $e \in P(w, v)$, $\alpha_1(v') < slope(e) < \alpha_2(v')$. Hence, $\alpha_1(v') < slope(l) < \alpha_2(v')$, for each half-line l contained into the closed wedge $range(P(w, v))$. Finally, since u' and v' are children of the same node, by P3 $\alpha_1(u') < \alpha_2(u') < \alpha_1(v') < \alpha_2(v')$ (the case in which $\alpha_1(v') < \alpha_2(v') < \alpha_1(u') < \alpha_2(u')$ being symmetric). Since $\alpha_1(u') < slope(l) < \alpha_2(u')$ for each half-line l contained into the closed wedge $opp(P(u, w))$ and since $\alpha_1(v') < slope(l) < \alpha_2(v')$ for each half-line l contained into the closed wedge $range(P(w, v))$, we have $opp(P(u, w)) \cap range(P(w, v)) = \emptyset$. By Corollary 1, $P(u, v)$ is monotone. □

By Theorem 1, as long as the slopes of the edges in a drawing of a tree T guarantee the slope-disjoint property, one can *arbitrarily* assign lengths to such edges always obtaining a monotone drawing of T. In the following we present two algorithms for constructing slope-disjoint drawings of any tree T. In both algorithms, we individuate a suitable set of elements of the Stern-Brocot tree $\mathcal{SB}$. Each of such elements, say $s = y/x$, is then used as a slope of an edge of T in the drawing.

Algorithm BFS-based: Consider the first $\lceil \log_2(n) \rceil$ levels of the Stern-Brocot tree $\mathcal{SB}$. Such levels contain a total number of at least $n - 1$ elements y/x of $\mathcal{SB}$. Order such elements by increasing value of the ratio y/x and consider the first $n - 1$ elements in such an order S, say $s_1 = y_1/x_1, s_2 = y_2/x_2, \ldots, s_{n-1} = y_{n-1}/x_{n-1}$. Consider the

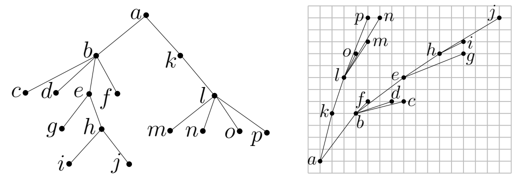


Fig. 1. (a) A tree T. (b) The drawing of T constructed by Algorithm BFS-based.

subtrees of r, say $T_1(r), T_2(r), \dots, T_{k(r)}(r)$. Assign to $T_i(r)$ the $|T_i(r)|$ elements of S from the $(1 + \sum_{j=1}^{i-1} |T_j(r)|)$-th to the $(\sum_{j=1}^{i} |T_j(r)|)$-th. Consider a node u of T and suppose that a sub-sequence $S(u) = s_a, s_{a+1}, \dots, s_b$ of S has been assigned to $T(u)$, where $|T(u)| = b - a$. Consider the subtrees $T_1(u), T_2(u), \dots, T_{k(u)}(u)$ of u and assign to $T_i(u)$ the $|T_i(u)|$ elements of $S(u)$ from the $(1 + \sum_{j=1}^{i-1} |T_j(u)|)$-th to the $(\sum_{j=1}^{i} |T_j(u)|)$-th. Now we construct a grid drawing of T. Place r at $(0,0)$. Consider a node u of T, suppose that a sequence $S(u) = s_a, s_{a+1}, \dots, s_b$ of S has been assigned to $T(u)$ and suppose that the parent $p(u)$ of u has been already placed at the grid point $(p_x(u), p_y(u))$. Place u at grid point $(p_x(u) + x_b, p_y(u) + y_b)$, where $s_b = y_b/x_b$. See Fig. 1 for an example of application. We have the following:

Theorem 2. *Let T be a tree. Then, Algorithm BFS-based constructs a monotone drawing of T on a grid of area $O(n^{1.6}) \times O(n^{1.6})$.*

Algorithm DFS-based: Consider the sequence S composed of the first $n-1$ elements $1/1, 2/1, \dots, n-1/1$ of the rightmost path of $\mathcal{SB}$. Assign sub-sequences of S to the subtrees of T and construct a grid drawing in the same way as in Algorithm BFS-based. We have the following.

Theorem 3. *Let T be a tree. Then, Algorithm DFS-based constructs a monotone drawing of T on a grid of area $O(n^2) \times O(n)$.*

As a further consequence of Theorem 1, we have the following:

Corollary 2. *Every (even non-planar) graph admits a monotone drawing.*

Namely, for any graph G, construct a monotone drawing of a spanning tree T of G with vertices in general position. Draw the other edges of G as segments, obtaining a straight-line drawing of G in which, for any pair of vertices, there exists a monotone path (the one whose edges belong to T). Such a drawing is a monotone drawing of G.

5 Planar Monotone Drawings of Biconnected Graphs

First, we restate, using the terminology of this paper, the well-known result of [6].

Lemma 6. *[6] Let G be a biconnected planar graph with a given planar embedding such that each split pair u, v is incident to the outer face and each maximal split component of u, v has at least one edge incident to the outer face but, possibly, for edge (u, v). Then, G admits a strictly convex drawing with the given embedding in which the outer face is drawn as an arbitrary strictly convex polygon.*

Let Γ be a monotone drawing, d any direction, and k a positive value. A *directional-scale*, denoted by $\mathcal{DS}(d, k)$, is an affine transformation defined as follows. Rotate Γ by an angle δ until d is orthogonal to the x-axis. Scale Γ by $(1, k)$ (i.e., multiply its y-coordinates by k). Rotate back the obtained drawing by an angle $-\delta$.

Lemma 7. *Let Γ be a monotone drawing and d a direction such that no edge in Γ is parallel to d. For any $\alpha > 0$ a directional-scale $\mathcal{DS}(d-\frac{\pi}{2}, k(\alpha))$ exists that transforms Γ into a monotone drawing in which the slope of any edge is between $d - \alpha$ and $d + \alpha$.*

A path monotone with respect to a direction d is (α, d)-*monotone* if, for each edge e, $d - \alpha < slope(e) < d + \alpha$. A path from a vertex u to a vertex v is an (α, d_1, d_2)-*path* if it is a composition of a (α, d_1)-monotone path from u to a vertex w and of a (α, d_2)-monotone path from w to v. Let p_N, p_S, p_W, and p_E be four points in the plane such that p_W is inside triangle $\triangle(p_N, p_S, p_E)$, $\widehat{p_W p_S p_E} = \widehat{p_W p_N p_E}$, and $\widehat{p_W p_S p_N} + 2\widehat{p_W p_S p_E} < \frac{\pi}{2}$. Quadrilateral (p_N, p_E, p_S, p_W) is a *boomerang* (see Fig. 2(a)).

Let G be a biconnected graph and $\mathcal{T}$ be the SPQR-decomposition of G rooted at an edge e. We prove that G admits a planar monotone drawing by means of an inductive algorithm which, given a component μ of $\mathcal{T}$ with poles u and v, and a boomerang $boom(\mu) = (p_N(\mu), p_E(\mu), p_S(\mu), p_W(\mu))$, constructs a drawing Γ_μ of $pert(\mu)$ satisfying the following properties. Let $d_N(\mu)$ be the half-line starting at $p_E(\mu)$ through $p_N(\mu)$, let $d_S(\mu)$ be the half-line starting at $p_E(\mu)$ through $p_S(\mu)$, let α_μ be $p_W(\mu)\widehat{p_S(\mu)}p_E(\mu) = p_W(\mu)\widehat{p_N(\mu)}p_E(\mu)$, and let $\beta_\mu = p_W(\mu)\widehat{p_S(\mu)}p_N(\mu)$. (*A*) Γ_μ is monotone; (*B*) with the possible exception of edge (u, v), Γ_μ is contained into $boom(\mu)$, with u drawn on $p_N(\mu)$ and v on $p_S(\mu)$; (*C*) each vertex $w \in pert(\mu)$ belongs to a $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path from u to v. Observe that C implies that in Γ_μ there exists a path between the poles that is monotone with respect to the line through them and that B implies the planarity of Γ_μ.

Lemma 8. *Let μ be a component of $\mathcal{T}$. Every $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path from u to v is monotone with respect to the half-line from u through v.*

Let $\mu_1, \dots, \mu_k$ be the children of μ in $\mathcal{T}$, with poles $(u_1, v_1), \dots, (u_k, v_k)$. We construct a drawing Γ_μ satisfying A–C by composing drawings $\Gamma_{\mu_1}, \dots, \Gamma_{\mu_k}$, which are constructed inductively, as follows.

If μ is a Q-node, then draw an edge between $p_N(\mu)$ and $p_S(\mu)$.

If μ is an S-node (see Fig. 2(b)), then let p be the intersection point between segment $\overline{p_W(\mu)p_E(\mu)}$ and the bisector line of $p_W(\mu)\widehat{p_N(\mu)}p_E(\mu)$. Consider k equidistant points $p_1, \dots, p_k$ on segment $\overline{p_N(\mu)p}$ such that $p_1 = p_N(\mu)$ and $p_k = p$. For each μ_i, with $i = 1, \dots, k-1$, consider a boomerang $boom(\mu_i) = (p_N(\mu_i), p_E(\mu_i), p_S(\mu_i), p_W(\mu_i))$ such that $p_N(\mu_i) = p_i$, $p_S(\mu_i) = p_{i+1}$, and $p_E(\mu_i)$ and $p_W(\mu_i)$ determine $\beta_{\mu_i} + 2\alpha_{\mu_i} < \frac{\alpha_\mu}{2}$. Apply the inductive algorithm to μ_i and $boom(\mu_i)$. Also, consider a boomerang $boom(\mu_k) = (p_N(\mu_k), p_E(\mu_k), p_S(\mu_k), p_W(\mu_k))$ such that $p_N(\mu_k) = p$, $p_S(\mu_k) = p_S(\mu)$, and $p_E(\mu_k)$ and $p_W(\mu_k)$ determine $\beta_{\mu_k} + 2\alpha_{\mu_k} < \frac{\alpha_\mu}{2}$. Apply the inductive algorithm to μ_k and $boom(\mu_k)$.

If μ is a P-node (see Fig. 2(c)), then consider $2k$ points $p_1, \dots, p_{2k}$ on segment $\overline{p_W(\mu)p_E(\mu)}$ such that $p_1 = p_W(\mu)$, $p_{2k} = p_E(\mu)$, and $p_i\widehat{p_N(\mu)}p_{i+1} = \frac{\alpha_\mu}{2k-1}$, for each

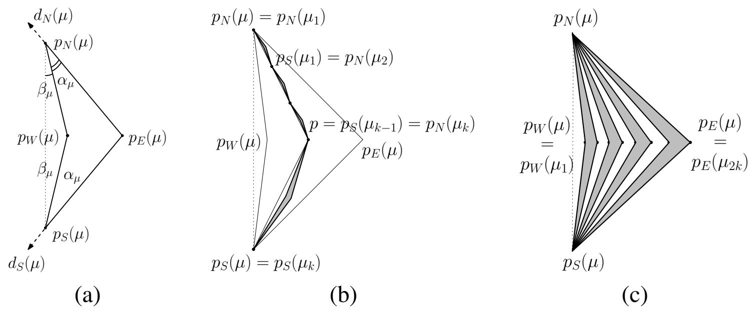


Fig. 2. (a) A boomerang. The construction rules for an S-node (b) and for a P-node (c).

$i = 1, \ldots, 2k-1$. For each μ_i, with $i = 1, \ldots, k$, consider a boomerang $boom(\mu_i) = (p_N(\mu_i),\ p_E(\mu_i),\ p_S(\mu_i),\ p_W(\mu_i))$ such that $p_N(\mu_i) = p_N(\mu)$, $p_S(\mu_i) = p_S(\mu)$, $p_W(\mu_i) = p_{2i-1}$, and $p_E(\mu_i) = p_{2i}$. Apply the inductive algorithm to μ_i and $boom(\mu_i)$.

If μ is an R-node, then consider the graph G' obtained by removing v and its incident edges from $skel(\mu)$. Since $skel(\mu)$ is triconnected, G' is a biconnected graph whose possible split pairs are on the outer face. Further, each of such split pairs separates at most three maximal split components, and in this case one of them is an edge. By Lemma 6, G' admits a convex drawing whose outer face is any strictly convex polygon. Consider a strictly convex polygon C with one vertex on p_N, one vertex on p_E, and $m-2$ vertices inside $boom(\mu)$ so that they are visible from $p_S(\mu)$ inside $boom(\mu)$ and the internal angle incident to $p_E(\mu)$ is smaller than $\frac{\pi}{2}$ (see Fig. 3(a)). Construct a convex drawing $\Gamma(G')$ of G' such that the vertices of the outer face of G' are on the vertices of C, with u on $p_N(\mu)$. By Lemma 2, $\Gamma(G')$ is monotone. Slightly perturb the position of the vertices of $\Gamma(G')$ so that no two parallel edges exist and no edge is orthogonal to $d_N(\mu)$. Apply a directional-scale $\mathcal{DS}(d_N(\mu) - \frac{\pi}{2}, k(\frac{\alpha_\mu}{2}))$ to $\Gamma(G')$. By Lemma 7, for every edge $e \in G'$, $slope(-d_N(\mu)) - \frac{\alpha_\mu}{2} < slope(e) < slope(-d_N(\mu)) + \frac{\alpha_\mu}{2})$. Further, by Lemma 4 and by the fact that the internal angle of C incident to $p_N(\mu)$ is smaller than $\frac{\alpha_\mu}{2} < \frac{\pi}{2}$, for every vertex $w \in G'$, a $(\frac{\alpha_\mu}{2}, -d_N(\mu))$-monotone path exists from u to w. Let $\Gamma(skel(\mu))$ be the drawing of $skel(\mu)$ obtained from $\Gamma(G')$ by placing v on $p_S(\mu)$ and drawing its incident edges (see Fig. 3(b)). We have the following:

Claim 1. *$\Gamma(skel(\mu))$ is monotone.*

Consider a drawing $\Gamma'(skel(\mu))$ of a subdivision of $skel(\mu)$ obtained as a subdivision of $\Gamma(skel(\mu))$. We have the following:

Claim 2. *$\Gamma'(skel(\mu))$ is monotone.*

Consider the pair of vertices x, y belonging to the subdivision of $skel(\mu)$ such that the range $range(P(x,y))$ of the monotone path $P(x,y)$ between them in $\Gamma'(skel(\mu))$ creates the largest angle $\angle(x,y)$ among all the pairs of vertices. Let $\gamma = \pi - \angle(x,y)$. Let δ be the smallest angle between two adjacent edges in $\Gamma(skel(\mu))$. For each μ_i, with $i = 1, \ldots, k$, let $p_N(\mu_i)$ and $p_S(\mu_i)$ be the points where u_i and v_i have been drawn in $\Gamma(skel(\mu))$, respectively. Consider a boomerang $boom(\mu_i) = (p_N(\mu_i),\ p_E(\mu_i),$

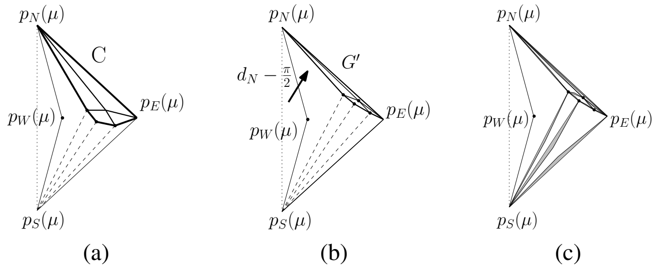


Fig. 3. Two phases of the construction for an R-node: (a) definition of the strictly convex polygon C and (b) the directional-scale applied to G'

$p_S(\mu_i), p_W(\mu_i))$ such that $p_E(\mu_i)$ and $p_W(\mu_i)$ determine $\beta_{\mu_i} + 2\alpha_{\mu_i} < \min\{\frac{\delta}{2}, \frac{\gamma}{2}\}$. For each μ_p such that either $p_N(\mu_p)$ and $p_S(\mu_p)$ lie on the vertices of C or $p_S(\mu_p) = p_S(\mu)$, choose points $p_W(\mu_p)$ and $p_E(\mu_p)$ inside $boom(\mu)$. Then, apply the inductive algorithm to μ_i, with poles u_i and v_i, and $boom(\mu_i)$ (see Fig. 3(c)).

In the following we prove that the above described algorithm constructs a planar monotone drawing of every biconnected planar graph.

Theorem 4. *Every biconnected planar graph admits a planar monotone drawing.*

Proof: Let $\mathcal{T}$ be the SPQR-tree of a biconnected graph G, rooted at any Q-node μ_e corresponding to an edge e. Consider a boomerang $boom(\mu_e) = (p_N(\mu_e), p_E(\mu_e), p_S(\mu_e), p_W(\mu_e))$ such that $x(p_N(\mu_e)) = x(p_S(\mu_e)) < x(p_W(\mu_e)) < x(p_E(\mu_e)), y(p_S(\mu_e)) < y(p_W(\mu_e)) = y(p_E(\mu_e)) < y(p_N(\mu_e))$, and $\beta_{\mu_e} + 2\alpha_{\mu_e} < \frac{\pi}{2}$. Apply the inductive algorithm described above to μ_e and $boom(\mu_e)$. We prove that the resulting drawing is monotone by showing that at each step of the induction the constructed drawing satisfies A–C. This is trivial if μ is a Q-node. Otherwise, μ is an S-node, a P-node, or an R-node and the statement is proved by the following claims:

Claim 3. *If μ is an S-node, Γ_μ satisfies A.*

Proof: Refer to Fig. 4(a). Consider any two vertices $w', w'' \in pert(\mu)$ and the components μ_a and μ_b such that $w' \in pert(\mu_a)$ and $w'' \in pert(\mu_b)$. If $a = b$, then a monotone path between w' and w'' exists by induction. Otherwise, for each μ_i, consider a vertex w_i, where $w_a = w'$ and $w_b = w''$. For each μ_i, with $i = 1, \dots, k$, consider a $(\alpha_{\mu_i}, -d_N(\mu_i), d_S(\mu_i))$-path $P(u_i, v_i)$ from u_i to v_i containing w_i. Observe that such paths exist since, for each μ_i, Γ_{μ_i} satisfies C. Consider a path $P(u_i, v_i)$ with $1 \leq i \leq k-1$. Since $\beta_{\mu_i} + 2\alpha_{\mu_i} < \frac{\alpha_\mu}{2}$, and since $p_N(\mu_i)$ and $p_S(\mu_i)$ lie on the bisector line of α_μ, for each edge $e \in P(u_i, v_i)$, it holds $slope(e) < \beta_\mu + \frac{\alpha_\mu}{2} + \beta_{\mu_i} + 2\alpha_{\mu_i} < \beta_\mu + \alpha_\mu < \beta_\mu + 2\alpha_\mu = d_N(\mu) + \alpha$, and $slope(e) > \beta_\mu + \frac{\alpha_\mu}{2} - (\beta_{\mu_i} + 2\alpha_{\mu_i}) > \beta_\mu = d_N(\mu) - \alpha_\mu$. Hence, $P(u_i, v_i)$ is $(\alpha_\mu, -d_N(\mu))$-monotone. Analogously, $P(u_k, v_k)$ is $(\alpha_\mu, d_S(\mu))$-monotone. Therefore, the path $P(u, v)$ composed of all the paths $P(u_i, v_i)$ is an $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path. By Lemma 8, $P(u, v)$ is monotone. Hence, by Prop. 2, the subpath of $P(u, v)$ between w' and w'' is monotone, as well, and Γ_μ satisfies A. □

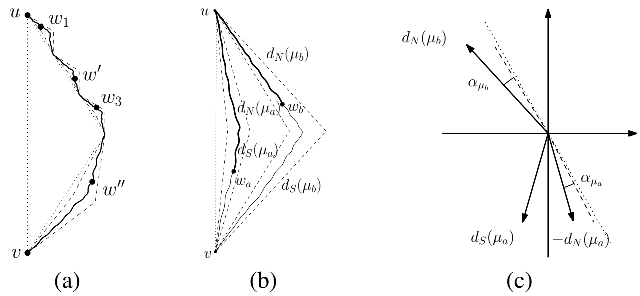


Fig. 4. Γ_μ satisfies A (a) if μ is an S-node and (b)–(c) if μ is a P-node

Claim 4. *If μ is an S-node, Γ_μ satisfies B and C.*

Claim 5. *If μ is a P-node, Γ_μ satisfies A.*

Proof: Consider any two vertices $w_a, w_b \in pert(\mu)$ and the nodes μ_a and μ_b such that $w_a \in pert(\mu_a)$ and $w_b \in pert(\mu_b)$. If $a = b$, then a monotone path from w_a to w_b exists by induction. Otherwise, consider the $(\alpha_{\mu_a}, -d_N(\mu_a), d_S(\mu_a))$-path $P_a(u, v)$ from u to v through w_a and the $(\alpha_{\mu_b}, d_N(\mu_b), -d_S(\mu_b))$-path $P_b(v, u)$ from v to u through w_b, which exist by induction (C). Suppose w_b lies on the $(\alpha_{\mu_b}, d_N(\mu_b))$-monotone path $P(w_b, u)$ from w_b to u that is a subpath of $P_b(v, u)$, the other case being analogous. Consider the $(\alpha_{\mu_a}, -d_N(\mu_a), d_S(\mu_a))$-path $P(u, w_a)$ that is a subpath of $P_a(u, v)$. We show that path $P(w_b, w_a)$ composed of $P(w_b, u)$ and $P(u, w_a)$ is monotone. When translated to the origin of the axes, $d_N(\mu_b)$, $-d_N(\mu_a)$, and $d_S(\mu_a)$ are in the 2nd, 4th, and 3rd quadrant, respectively. By construction, the wedge delimited by $d_N(\mu_b)$ and $-d_N(\mu_a)$ and containing the 3rd quadrant has an angle $\leq \pi - 2\frac{\alpha_\mu}{2k-1}$. Since, by definition, every edge of $P(w_b, u)$ creates an angle with $d_N(\mu_b)$ smaller than $\alpha_{\mu_b} = \frac{\alpha_\mu}{2k-1}$ and every edge of $P(u, w_a)$ creates an angle with $-d_N(\mu_a)$ smaller than $\alpha_{\mu_a} = \frac{\alpha_\mu}{2k-1}$, the slopes of all the edges of $P(w_b, w_a)$ lie inside a wedge having an angle smaller than π. Hence, $P(w_b, w_a)$ is monotone. □

Claim 6. *If μ is a P-node, Γ_μ satisfies B and C.*

Claim 7. *If μ is an R-node, Γ_μ satisfies Prop. A.*

Proof: Consider any two vertices $w_a, w_b \in pert(\mu)$ and the nodes μ_a and μ_b such that $w_a \in pert(\mu_a)$ and $w_b \in pert(\mu_b)$. Let e_a and e_b be the virtual edges of $skel(\mu)$ corresponding to μ_a and μ_b, respectively. If $a = b$ by induction a monotone path from w_a to w_b trivially exists. Otherwise, consider the monotone drawing $\Gamma'(skel(\mu))$ of a 1-subdivision of $skel(\mu)$ and the monotone path $P_{a,b}$ from the subdivision vertex of e_a to the subdivision vertex of e_b. By construction, $\pi - range(P_{a,b}) \geq \gamma$. As $\beta_{\mu_i} + 2\alpha_{\mu_i} < \min\{\frac{\delta}{2}, \frac{\gamma}{2}\} \leq \frac{\gamma}{2}$, for the path $P(w_a, w_b)$ obtained by replacing each edge e_i of $P_{a,b}$ with the corresponding path of $pert(\mu_{e_i})$ it holds that $range(P(w_a, w_b)) < \pi$. □

Claim 8. *If μ is an R-node, Γ_μ satisfies B and C.*

This concludes the proof of Theorem 4. □

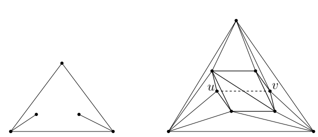
6 Conclusions and Open Problems

We initiated the study of monotone graph drawings. Concerning trees, we proved that every monotone drawing is planar, that every strictly convex drawing is monotone, and that monotone drawings exist on polynomial-size grids. We believe that simple modifications of our algorithms allow one to construct strictly convex drawings of trees on polynomial-size grids. Another possible extension of our results is to characterize monotonicity in terms of the angles between adjacent edges. Our definition of slope-disjointness goes in this direction, although it introduces some non-necessary restrictions on the slopes of the edges (like the one that all the slopes are between 0 and π).

We proved that every biconnected planar graph admits a planar monotone drawing. Extending such a result to general simply-connected graphs seems to be non-trivial. There exist planar graphs not having a monotone drawing (see Fig. 5(a)) if the embedding is given. However, we are not aware of any planar graph not admitting a planar monotone drawing for any of its embeddings.

Several area minimization problems concerning monotone drawings are, in our opinion, worth of study. First, determining tight bounds for the area requirements of grid drawings of trees appears to be an interesting challenge. Second, modifying our tree drawing algorithms so that they construct grid drawings in general position would lead to algorithms for constructing monotone drawings of non-planar graphs on a grid of polynomial size. Third, the drawing algorithm we presented for biconnected planar graphs constructs drawings in which the ratio between the lengths of the longest and of the shortest edge is exponential in n. Is it possible to construct planar monotone drawings of biconnected planar graphs in polynomial area?

Finally, we introduce a new drawing standard related to monotone drawings. A path from a vertex u to a vertex v is *strongly monotone* if it is monotone with respect to the half-line from u through v. A drawing of a graph is *strongly monotone* if a strongly monotone path connects each pair of vertices. Strong monotonicity appears to be even more desirable than general monotonicity for the readability of a drawing. However, designing algorithms for constructing strongly monotone drawings seems to be harder than for monotone drawings and only restricted graph classes appear to admit strongly monotone drawings. Note that a subpath of a strongly monotone path is, in general, not strongly monotone; also, while convexity implies monotonicity, it does not imply strong monotonicity, even for planar triangulations (see Fig. 5(b)).



(a) (b)

Fig. 5. (a) A planar embedding of a graph with no monotone drawing. (b) A drawing of a planar triangulation that is not strongly monotone.

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