Quaternion Atomic Function Wavelet for Applications in Image Processing

E. Ulises Moya-Sánchez and Eduardo Bayro-Corrochano

CINVESTAV, Unidad Guadalajara
Electrical and Computer Sciences Department
{emoya,edb}@gdl.cinvestav.mx

Abstract. Atomic Functions are widely used in different applications in image processing, pattern recognition, computational physics and also in the digital interpretation of signal measurements. The main contribution of this work is to develop a Quaternionic Atomic Function Wavelet as a new quaternionic image wavelet transform. This filter have a real part and three imaginary parts (i, j, k) of the Quaternion Atomic Function, as a result we can extract more information from the image by the three phases (ϕ, θ, φ) of the quaternion representation. The experimental part shows clearly that the phase information of the image is not afected by illumination changes.

Keywords: Quaternion Algebra, Atomic functions, Image Processing, 2D Phase Information.

1 Introduction

One of the main fields of Atomic functions AFs application is pattern recognition and image processing [1]. This work presents the theory and some results of the Quaternion Atomic Function QAF, as a new quaternionic wavelet. We use the AF because, it is novel and versatile, easy to derivate (only a shift), it is compact in space domain, and it has the possibility of representing any polynomial by means of its translations. We develop the AF in a hypercomplex algebra (quaternion algebra H), this framework permits to extract the phase information of the image. The combination of this function AF with this framework H makes a new useful image filter.

We apply the QAF or *qup* on a test image (squares) in 3 ways as follows: firstly, convolution of each part (real, i, j, k) of QAF is applied on the test image, secondly, we calculate the three phases of the filtered image, thirdly and we uses a wavelet for multiscale image processing. We structure this work as follows, the first section is devoted to present the AF and the main characteristics, the subject of the second section is the quaternion algebra, the next section introduces the QAF, in the section four we present the Quaternion Atomic Wavelet Function, in the section five we present the results and finally the conclusions.

I. Bloch and R.M. Cesar, Jr. (Eds.): CIARP 2010, LNCS 6419, pp. 346–353, 2010.

2 Atomic Functions

By definition, AF are compactly supported infinitely differentiable solutions of differential equations with a shifted argument [1] i.e.

$$Lf(x) = \lambda \sum_{k=1}^{M} c(k) f(ax - b(k)), |a| > 1, \tag{1}$$

where $L = \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + ... + a_n$ is a linear differential operator with constant coefficients. In the AF class, the function $up(x)$ is the simplest and at the same time, the most useful primitive function to generate other kinds of atomic functions [1]. It satisfies the equation

$$f(x)' = 2\left(f(2x+1) - f(2x-1)\right), \tag{2}$$

Function $up(x)$ is infinitely differentiable but non-analytical; $up(0) = 1$, $up(-x) = up(x)$. Other types of AF satisfying equation (1): $fup_n(x)$, $\Xi_n(x)$, $h_a(x)$ [6]. In this work we only use $up(x)$. In general the Atomic Function $up(x)$ is generated by infinite convolutions of rectangular impulses. The function $up(x)$ has the following representation in terms of the Fourier transform:

$$up(x) = \frac{1}{2\pi} \int_{\infty}^{\infty} e^{iux} \prod_{k=1}^{\infty} \frac{\sin(u2^{-k})}{u2^{-k}} du. \tag{3}$$

Figure 1 shows the $up(x)$ and the Fourier Transform of $F(up)$. Atomic windows were compared with classic ones [1,6] by means of the system of parameters such as: the equivalent noise bandwidth, the 50% overlapping region correlation, the parasitic modulation amplitude, the maximum conversion losses (in decibels), the maximum side lobe level (in decibels), the asymptotic decay rate of the side lobes (in decibels per octave), the window width at the six-decibel level, the coherent gain. All atomic windows exceed classic ones in terms of the asymptotic decay rate [1,6].

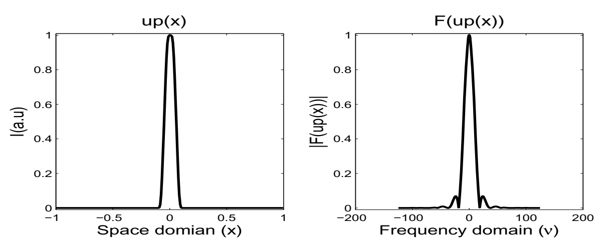

Fig. 1. Atomic function $up(x)$ and the Fourier Transform of $up(x)$

Figure 2 illustrate the first derivate, dup see equation (2) in convolution with the image, this function can be used as a oriented line detector with a simple rotation. We show three orientations 0^o, 45^o, 135^o.

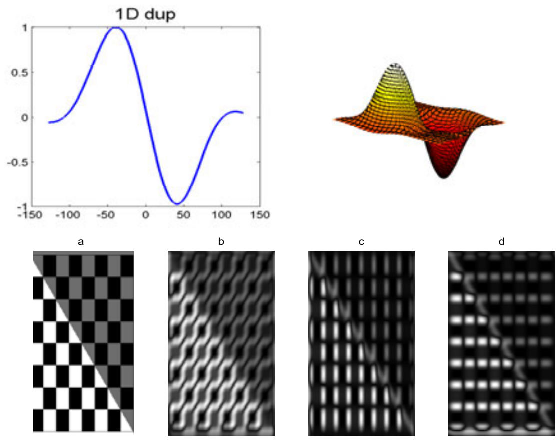

Fig. 2. Convolution of $dup(x, y)$ with the test image. a) Test Image, b)Result of the convolution of the image with $dup(x, y, 0^o)$, c)Result of the convolution of the image with $dup(x, y, 45^o)$, d)Result of the convolution of the image with $dup(x, y, 135^o)$.

3 Quaternion Algebra

The quaternion algebra is a framework we need to define the QAF. Quaternion algebra was invented by Hamilton in 1843. It is an associative non-commutative four-dimensional algebra [2,3].

$$q = a + bi + cj + dk \;\; a, b, c, d \;\epsilon\; \Re \tag{4}$$

The units i, j obey the relations

$$i^2 = j^2 = -1, ij = -k, \tag{5}$$

the norm of a quaternion is defined $|q| = \sqrt{q\bar{q}}$ where $\bar{q}$ is a conjujate of q. Similarly to the complex numbers. In 2D the phase component carries the main part of image information [4,5].

Since the quaternions constitute a 4D algebra we can represent q in a polar representation of the form equation (4) i.e. $(|q|, \phi, \theta, \psi)$, where $|q|$ is the magnitude and the angles (ϕ, θ, ψ) represent a novel kind of phase vector. By definition [3]:

$$q = |q|e^{i\phi}e^{k\psi}e^{j\theta} \tag{6}$$

the phase range are delimited (ϕ,θ,ψ),$[\ -\pi, \pi\ ]\times[\ -\pi/2, \pi/2\ ]\times[\ -\pi/4, \pi/4\ ]$.

4 Quaternion Atomic Function

The $up(x)$ function is easily extendable to two dimensions. Since a 2D signal can be split into an even (e) and odd (o) parts [3]

$$f(x, y) = f_{ee}(x, y) + f_{oe}(x, y) + f_{eo}(x, y) + f_{oo}(x, y), \tag{7}$$

one can then separate the four components of equation (3) and represent it as a quaternion as follows:

$$QAF(x, y) = up(x, y)[\cos(w_x)\cos(w_y) + i(\sin(w_x)\cos(w_y)) + \tag{8}$$

$$+ j(\cos(w_x)\sin(w_y)) + k(\sin(w_x)\sin(w_y))]$$

$$= QAF_{ee} + iQAF_{oe} + jQAF_{eo} + kQAF_{oo} \tag{9}$$

Figure 3 shows a Quaternion Atomic Function QAF or qup in the space domain with its four components: real part QAF_{ee} is observed in a , and the imaginary parts $QAF_{eo}, QAF_{oe}, QAF_{oo}$ are illustrated in b, c, d respectively. We can see clearly the differences in each part of our filter.

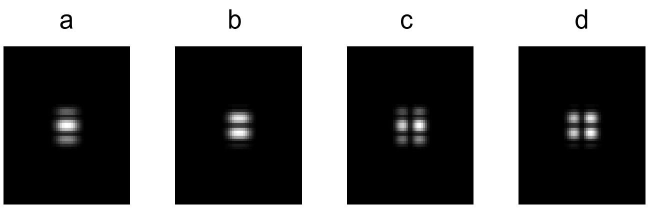

Fig. 3. Quaternion Atomic function up(x). a) QAF_{ee}, b) QAF_{oe}, c) QAF_{eo} and d) QAF_{oo}.

5 Quaternion Atomic Wavelet Function

In the Fourier Transform of a 2D signal, the phase component carries the main part of image information. We use this phase information in the quaternionic wavelet multiresolution analysis. This technique can be easily formulated in terms of the quaternion AF mother wavelet, for a more detail explanation see [2]. For the 2D image function $f(x, y)$, a quaternionic wavelet can be written as

$$f(x, y) = A_n^q f + \sum_{j=1}^{n} [D_{j,1}^q f + D_{j,2}^q f + D_{j,3}^q f]. \tag{10}$$

The upper index q indicates a *quaternion* 2D signal. We can characterize each approximation function $A_j^q f(x, y)$ and the difference components $D_{j,p}^q f(x, y)$ for $p = 1, 2, 3$ via a 2D scaling function $\mathbf{\Phi}^q(x, y)$ and its associated wavelet functions $\mathbf{\Psi}_p^q(x, y)$ as follows:

$$A_j^q f(x, y) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} a_{j,k,l} \mathbf{\Phi}_{j,k,l}^q(x, y), \tag{11}$$

$$D_{j,p}^q f(x, y) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} d_{j,p,k,l} \mathbf{\Psi}_{j,p,k,l}^q(x, y),$$

where

$$\mathbf{\Phi}^q_{j,k,l}(x,y) = \frac{1}{2^j}\mathbf{\Phi}^q\left(\frac{x-k}{2^j}, \frac{y-l}{2^j}\right), \quad (j,k,l) \in Z^3,$$
$$\mathbf{\Psi}^q_{j,p,k,l}(x,y) = \frac{1}{2^j}\mathbf{\Psi}^q_p\left(\frac{x-k}{2^j}, \frac{y-l}{2^j}\right) \qquad (12)$$

and

$$a_{j,k,l}(x,y) =< f(x,y), \mathbf{\Phi}^q_{j,k,l}(x,y) >, \qquad (13)$$
$$d_{j,p,k,l} =< f(x,y), \mathbf{\Psi}^q_{j,p,k,l}(x,y) > .$$

In order to carry out a separable quaternionic multiresolution analysis, we decompose the scaling function $\mathbf{\Phi}^q(x,y)_j$ and the wavelet functions $\mathbf{\Psi}^q_p(x,y)_j$ for each level j as follows:

$$\mathbf{\Phi}^q(x,y)_j = \phi^{\boldsymbol{i}}(x)_j\phi^{\boldsymbol{j}}(y)_j,$$
$$\mathbf{\Psi}^q_1(x,y)_j = \phi^{\boldsymbol{i}}(x)_j\psi^{\boldsymbol{j}}(y)_j,$$
$$\mathbf{\Psi}^q_2(x,y)_j = \psi^{\boldsymbol{i}}(x)_j\phi^{\boldsymbol{j}}(y)_j,$$
$$\mathbf{\Psi}^q_3(x,y)_j = \psi^{\boldsymbol{i}}(x)_j\psi^{\boldsymbol{j}}(y)_j, \qquad (14)$$

where $\phi^{\boldsymbol{i}}(x)_j$ and $\psi(x)^{\boldsymbol{i}}_j$ are 1D complex filters applied along the rows and columns respectively. Note that in ϕ and ψ, we use the imaginary number $\boldsymbol{i}$, $\boldsymbol{j}$ of quaternions that satisfies $\boldsymbol{j}\boldsymbol{i} = \boldsymbol{k}$.

By using these formulas, we can build quaternionic wavelet pyramids. Figure 4 shows the two primary levels of the pyramid (fine to coarse). According to equation (14), the approximation after the first level $A^q_1 f(x,y)$ is the output of $\mathbf{\Phi}^q(x,y)_1$, and the differences $D^q_{1,1}f$, $D^q_{1,2}f$, $D^q_{1,3}f$ are the outputs of $\mathbf{\Psi}^q_{1,1}(x,y)$, $\mathbf{\Psi}^q_{1,2}(x,y)$ and $\mathbf{\Psi}^q_{1,3}(x,y)$. The procedure continues through the j levels decimating the image at the outputs of the levels (indicated in figure 4 within the circle).

The quaternionic wavelet analysis from level $j-1$ to level j corresponds to the transformation of one quaternionic approximation to a new quaternionic approximation and three quaternionic differences, i.e.,

$$\{A^q_{j-1}\} \rightarrow \{A^q_j,\, D^q_{j,p},\, p = 1,2,3\}. \qquad (15)$$

Note that we do not use the idea of a mirror tree [2]. As a result, the quaternionic wavelet tree is a compact and economic processing structure which can be used for the case of n-dimensional multi-resolution analysis.

6 Results

Figure 5 shows the original image and four resulting images after convolution with components of the filter. The real part is observed in a , and i, j, k, imaginary

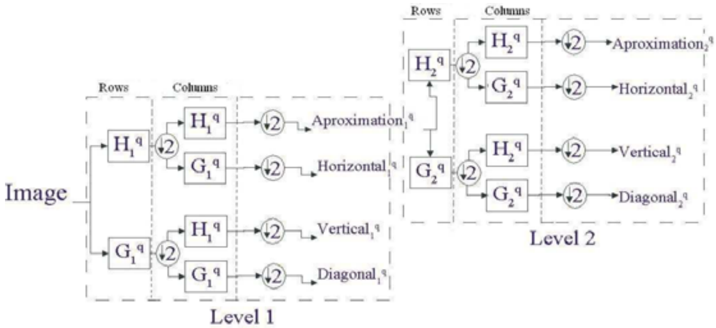

Fig. 4. Abstraction of two levels of the quaternionic wavelet pyramid

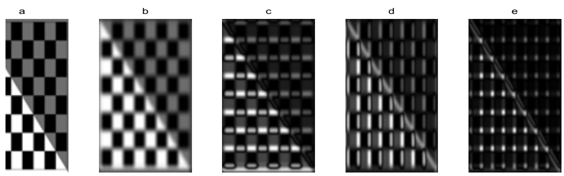

Fig. 5. Convolution of the test image with the *qup*. a) Original image. b) real-part c) i-part, d) j-part and e) k-part.

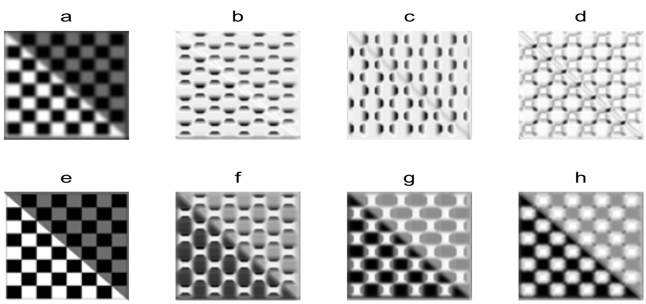

Fig. 6. The amplitude of the filtered Image a) and the three phases (ϕ, θ, φ), $b), c), d)$ respectively. The second row shows the original image f) and the subtraction of the original image and the three phases (ϕ, θ, φ) of the filtered image.

parts can be appreciate in b, c, d respectively. This figure illustrates how the QAF or qup filter works in different directions such as vertical, horizontal and combination of both. The direct convolution with the image is sensitive to the contrast of the image. Figure 6 shows the amplitude (a) (real part) and the three phases (ϕ, θ, φ) of quaternionic phase of the filtered image. The phase information is immune to changes of the contrast. In the second row of the Figure 6 we can see a subtraction of the original image and the three phases, it shows how the phase information can be used to localize and extract more information independently of the contrast of the image.

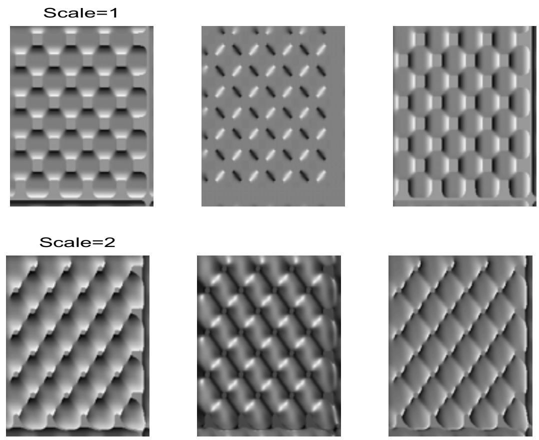

Fig. 7. (upper row) Thresholded quaternionic phases (ϕ, θ, φ) at first scale. (second row) Thresholded quaternionic phases (ϕ, θ, φ) at a second scale.

The *QAF qup* kernel was used as the mother wavelet in the multi-resolution pyramid. Figure 7 presents the three quaternionic phases at two scale levels of the pyramid. The lower row shows the phases after thresholding to enhance the phase structure. You can see how vertical lines and crossing points are highlighted.

7 Conclusion

This work introduces the theory and some applications of the quaternion Atomic Function Wavelet in image processing. This work indicates that the *QAF* (*qup*) can be more useful than simply *up* because it exploits the quaternion phase concept. The information of the three phases is independent of illumination changes. We present the use of this *AF* quaternionic mother wavelet for multi-resolution analysis which can be applied for optical flow, texture segmentation and image matching. The *QAF* wavelet filter disentangles structure symmetries of the image through the levels of an multi-resolution pyramid. A future work will include multi-resolution analysis for optical flow using real images.

References

1. Kravchenko, V.F., Perez-Meana, H.M., Ponomaryov, V.I.: Adaptive digital processing of multidimensional signals with applications. MOSCOW FIZMATLIT, MOSCOW (2009)
2. Bayro-Corrochano: Geometric Computing for Wavelet Transforms, Robot Vision, Learning, Control and Action. Springer, London (2010)
3. Bülow, T.: Hypercomplex Spectral Signal Representations for the Processing and Analysis of Images. Christian- Albert, Kiel University, Ph.D Thesis (1999)

4. Jähne, B.: Digital Image Processing. Springer, Germany (2002)
5. Kovesi, P.: Invariant measures of images features from phase information. University of Western Australia, PhD Thesis (1996)
6. Guyaev Yu, V., Kravchenko, V.F.: A New class of WA-Systems of Kravchenko-Rvachev Functions. MOSCOW Doklady Mathematics 75(2), 325–332 (2007)