Optimum Diffusion for Load Balancing in Mesh Networks

George S. Markomanolis⋆ and Nikolaos M. Missirlis

Department of Informatics and Telecommunications,
University of Athens,
Panepistimiopolis, 157 84, Athens, Greece

Abstract. This paper studies the Diffusion method for the load balancing problem in case of weighted mesh graphs. Closed form formulae for the optimum values of the edge weights are determined using local Fourier analysis. It is shown that an extrapolated version of Diffusion (EDF) can become twice as fast for orthogonal mesh graphs. Also, as a byproduct of our analysis it is shown that EDF on tori is four times faster than on meshes.

1 Introduction

The performance of a balancing algorithm can be measured in terms of number of iterations it requires to reach a balanced state and in terms of the amount of load moved over the edges of the underlying processor graph. In the Diffusion (DF) method [2], [4] a processor simultaneously sends workload to its neighbors with lighter workload and receives from its neighbors with heavier workload. It is assumed that the system is synchronous, homogeneous and the network connections are of unbounded capacity. Under the synchronous assumption, the DF method has been proved to converge in polynomial time for any initial workload [4]. More specifically DF is given by the following iterative scheme

$$u_i^{(n+1)} = u_i^{(n)} - \sum_{j \in N(i)} c_{ij}(u_i^{(n)} - u_j^{(n)}), \quad n = 0, 1, 2, \ldots, \tag{1}$$

where $c_{ij} > 0$ are the edge weights (or diffusion parameters), $N(i)$ is the set of the nearest neighbors of node i of the graph $G = (V, E)$ and $u_i^{(n)}, i = 0, 1, 2, \ldots, |V|$ is the load after the nth iteration on node i. The main problem here is the determination of the parameters c_{ij} such that the rate of convergence of DF is maximized. However, this is an optimization problem with multiple parameters. Indeed, one has to find a set of values for c_{ij}'s which minimize the convergence factor γ of DF. Initially, Cybenko [2] suggested choosing $c_{ij} = 1/(1 + d(i))$, where $d(i) = |N(i)|$ and proved that for binary n-cubes this is an optimal choice. Similar convergence results were also obtained in [4] and [3]. A special case of

⋆ University of Lyon, LIP Laboratory, ENS Lyon, INRIA 46, Allée d'Italie 69364 Lyon cedex 07, France.

P. D'Ambra, M. Guarracino, and D. Talia (Eds.): Euro-Par 2010, Part I, LNCS 6271, pp. 230–241, 2010.

the problem was solved by Xu and Lau [9] assuming all edge weights are equal to a single parameter (unweighted case) and determined optimal values for the cases of the N-ary n-cube and its variant, the nD-mesh using circulant matrix theory. A first attempt to find optimal values for c_{ij}'s (weighted case), using semi-definite programming, was made in [6], where optimal numerical values for edge weights of certain graphs with small cardinality were computed. A second step towards this direction using results of Cartesian product of graphs was [7]. In [5] we found optimum values for c_{ij}'s in case of a torus graph.

In the present work we follow a similar approach and determine optimum values for c_{ij}'s in case of the 2D-mesh. As a consequence we show that the convergence rate of DF may reach an improvement of approximately twice as fast for stretched 2D-mesh compared to the unweighted case. Moreover, it is shown that EDF on orthogonal tori is four times faster than on orthogonal meshes. We begin our study by considering the EDF method [5], an extrapolated version of DF, and study its convergence analysis. Next, we consider the local EDF method, a multiparametric version of EDF, which involves a set of parameters τ_i. By applying local Fourier analysis, in a similar way to [5] we are able to find a closed form formula for the set of the parameters τ_i in the sense that the rate of convergence of local EDF is maximized for mesh graphs. The optimum values of τ_i depend only upon the local edge weights hence their computation requires only local communication.

The rest of the paper is organized as follows. In section 2, we introduce the local EDF method. In section 3, we determine the optimum values of the parameters τ_i such that the convergence rate of the local EDF method is maximized. The determination of optimum values for the edge weights as well as a theoretical comparison with DF is presented in section 4. Section 5 compares the reciprocal rates of convergence of EDF for stretched torus and mesh networks. Finally, section 6 presents our numerical experiments whereas our conclusions are stated in section 7.

2 The Extrapolated Diffusion (EDF) Method

Let $G = (V, E)$ be a connected, weighted undirected graph with $|V|$ nodes and $|E|$ edges. Let $c_{ij} \in \Re^+$ be the weight of edge $e_{ij} \in E, u_i \in \Re$ be the load of node $v_i \in V$ and $u \in \Re^{|V|}$ be the vector of load values. Further, let $A \in \Re^{|V|\times|V|}$ be the weighted adjacency matrix of G. The matrix A is symmetric as the graph G is undirected. Let us consider the following iterative scheme that requires communication with adjacent nodes only

$$u_i^{(n+1)} = u_i^{(n)} - \tau \sum_{j\in N(i)} c_{ij}(u_i^{(n)} - u_j^{(n)}), \tag{2}$$

where $\tau \in \Re \setminus \{0\}$ and $c_{ij} > 0$ for $i = 1, 2, \ldots, |V|$ and $j \in N(i)$ are parameters that play an important role in the convergence of the whole system to the equilibrium state. Then, the overall workload distribution at step n, denoted by

$u^{(n)}$, is the transpose of the vector $(u_1^{(n)}, u_2^{(n)}, \ldots, u_{|V|}^{(n)})$ and $u^{(0)}$ is the initial workload distribution. In matrix form (2) becomes

$$u^{(n+1)} = Mu^{(n)}, \tag{3}$$

where M is called the *diffusion matrix*. The elements of M, m_{ij}, are equal to τc_{ij}, if $j \in N(i)$, $1 - \tau \sum_{j \in N(i)} c_{ij}$, if $i = j$ and 0 otherwise. With this formulation, the features of diffusive load balancing are fully captured by the iterative process (3) governed by the diffusion matrix M. The diffusion matrix of EDF can be written as

$$M = I - \tau L, \qquad L = D - A, \tag{4}$$

where $D = diag(L)$. Because of (4), (3) becomes $u^{(n+1)} = (I - \tau D)\, u^{(n)} + \tau A u^{(n)}$ or in component form

$$u_i^{(n+1)} = (1 - \tau \sum_{j \in N(i)} c_{ij}) u_i^{(n)} + \tau \sum_{j \in N(i)} c_{ij} u_j^{(n)},\ i = 1, 2, \ldots, |V|. \tag{5}$$

The diffusion matrix M must have the following properties [2], [4]: nonnegative, symmetric and stochastic. Before we close this section we consider the following version of EDF, which involves a set of parameters $\tau_i, i = 1, 2, \ldots, |V|$

$$u_i^{(n+1)} = (1 - \tau_i \sum_{j \in N(i)} c_{ij}) u_i^{(n)} + \tau_i \sum_{j \in N(i)} c_{ij} u_j^{(n)}. \tag{6}$$

Note that if $\tau_i = \tau$ for any $i = 1, 2, \ldots, |V|$, then (6) yields the EDF method. The iterative scheme (6) will be referred to as the local EDF method.

3 Determination of the Parameters τ_i

We define M_{ij} as the local EDF operator for the $N_1 \times N_2$ mesh. The local EDF scheme at a node (i, j) can be written as

$$u_{ij}^{(n+1)} = M_{ij} u_{ij}^{(n)}, \tag{7}$$

where $M_{ij} = 1 - \tau_{ij} L_{ij}$. Next, we impose the lexicografic ordering of the nodes and define $L_{ij} = d_{ij} - (c_{i+1,j} E_1 + c_{i-1,j} E_1^{-1} + c_{i,j+1} E_2 + c_{i,j-1} E_2^{-1})$ the local operator of the Laplacian matrix, with $d_{ij} = c_{i+1,j} + c_{i-1,j} + c_{i,j+1} + c_{i,j-1}$, where $c_{i-1,j}$, $c_{i+1,j}$, $c_{i,j-1}$ and $c_{i,j+1}$ denote the edge weights of the west, east, south and north neighbors of node (i, j). The operators E_1, E_1^{-1}, E_2, E_2^{-1} are defined as $E_1 u_{ij} = u_{i+1,j}$, $E_1^{-1} u_{ij} = u_{i-1,j}, E_2 u_{ij} = u_{i,j+1}$, $E_2^{-1} u_{ij} = u_{i,j-1}$, which are the *forward-shift* and *backward-shift* operators in the x_1-direction, (x_2-direction), respectively with $u_{ij} = u(ih_1, jh_2) = u(x_1, x_2)$, where $x_1 = ih_1$, $x_2 = jh_2$, $h_1 = \frac{1}{N_1}$ and $h_2 = \frac{1}{N_2}$. Since the Laplacian matrix L is symmetric it follows that $c_{i+1,j} = c_{i-1,j}$ and $c_{i,j+1} = c_{i,j-1}$. Next, we use the following notation for the edge weights

$$c_i^{(1)} = c_{i+1,j} \text{ and } c_j^{(2)} = c_{i,j+1},\ i = 1, 2, \ldots, N_1,\ j = 1, 2, \ldots, N_2. \tag{8}$$

Then, $L_{ij} = d_{ij} - c_i^{(1)}(E_1 + E_1^{-1}) + c_j^{(2)}(E_2 + E_2^{-1})$, where $d_{ij} = 2(c_i^{(1)} + c_j^{(2)})$. The eigenvalues μ_{ij}, λ_{ij} of the local operators M_{ij}, L_{ij}, respectively, are related as follows: $\mu_{ij} = 1 - \tau_{ij}\lambda_{ij}$.

Lemma 1. *The spectrum of the operator L_{ij} is given by*

$$\lambda_{ij}(k_1, k_2) = 2[c_i^{(1)}(1 - \cos k_1 h_1) + c_j^{(2)}(1 - \cos k_2 h_2)], \tag{9}$$

where $i=1, 2, \ldots, N_1,\ j=1, 2, \ldots, N_2$, $k_1 = \pi\ell_1$, $\ell_1 = 0, 1, 2, \ldots, N_1 - 1$, $k_2 = \pi\ell_2$ and $\ell_2 = 0, 1, 2, \ldots, N_2 - 1$.

Proof. By assuming that an eigenfunction of the local operator L_{ij} is the complex sinusoid $e^{i(k_1x_1+k_2x_2)}$ we have

$$L_{ij}e^{i(k_1x_1+k_2x_2)} = \lambda_{ij}(k_1, k_2)e^{i(k_1x_1+k_2x_2)},$$

where

$$\lambda_{ij}(k_1, k_2) = d_{ij} - [c_i^{(1)}(e^{ik_1h_1} + e^{-ik_1h_1}) + c_j^{(2)}(e^{ik_2h_2} + e^{-ik_2h_2})] \tag{10}$$

with

$$d_{ij} = 2(c_i^{(1)} + c_j^{(2)}).$$

So we may view $e^{i(k_1x_1+k_2x_2)}$ as an eigenfunction of L_{ij} with eigenvalues $\lambda_{ij}(k_1, k_2)$ given by (10). It is easily verified that (10) yields (9). □

Since

$$\gamma_{ij}(M_{ij}) = \max_{k_1,k_2} |\mu_{ij}(k_1, k_2)|, \tag{11}$$

where not both k_1, k_2 can take the value zero and

$$\mu_{ij} = 1 - \tau_{ij}\lambda_{ij}, \tag{12}$$

it follows that the minimum value of γ_{ij} with respect to τ_{ij} is attained at [10]

$$\tau_{ij}^{opt} = \frac{2}{\lambda_{i,j,2} + \lambda_{i,j,N}}, \tag{13}$$

where $\lambda_{i,j,2}, \lambda_{i,j,N}$ are the smallest and largest eigenvalues of the operator L_{ij}, respectively. Moreover, the corresponding minimum value of $\gamma_{ij}(M_{ij})$ is given by

$$\gamma_{ij}^{opt} = \frac{P_{ij} - 1}{P_{ij} + 1}, \tag{14}$$

where

$$P_{ij} = \frac{\lambda_{i,j,N}}{\lambda_{i,j,2}} \tag{15}$$

is the P-condition number of L_{ij}. The last quantity plays an important role in the behavior of γ_{ij}^{opt}. Indeed, from (14) it follows that γ_{ij}^{opt} is a decreasing function of P_{ij}. Therefore, minimization of P_{ij} has the effect of maximizing R(LEDF), the rate of convergence of the local EDF method, defined by [10]

$$R(LEDF) = -\log \gamma_{ij}^{opt}. \tag{16}$$

Theorem 1. *The convergence factor $\gamma_{ij}(M_{ij})$ of the operator M_{ij} is minimized at*

$$\tau_{ij}^{opt} = \begin{cases} (2c_i^{(1)} + c_j^{(2)}(1+\cos\frac{\pi}{N_2}))^{-1}, \ \sigma_{ij} \geq \sigma_2 \\ \\ (2c_j^{(2)} + c_i^{(1)}(1+\cos\frac{\pi}{N_1}))^{-1}, \sigma_{ij} \leq \sigma_2, \end{cases} \tag{17}$$

and its corresponding minimum is

$$\gamma_{ij}^{opt} = \begin{cases} \dfrac{2c_i^{(1)}\cos\frac{\pi}{N_1} + c_j^{(2)}(1+\cos\frac{\pi}{N_2})}{2c_i^{(1)} + c_j^{(2)}(1+\cos\frac{\pi}{N_2})}, \ \sigma_{ij} \geq \sigma_2 \\ \\ \dfrac{c_i^{(1)}(1+\cos\frac{\pi}{N_1}) + 2c_j^{(2)}\cos\frac{\pi}{N_2}}{c_i^{(1)}(1+\cos\frac{\pi}{N_1}) + 2c_j^{(2)}}, \sigma_{ij} \leq \sigma_2, \end{cases} \tag{18}$$

where σ_{ij} and σ_2 are given by (20).

Proof. The optimum value for τ_{ij} will be determined by (13), while the minimum value of γ_{ij}^{opt} by (14) and (15). It is therefore necessary to determine $\lambda_{i,j,2}$ and $\lambda_{i,j,N}$. For the determination of $\lambda_{i,j,2}$ we let ℓ_1=0 and ℓ_2=1, or ℓ_1=1 and ℓ_2=0 in (9) thus obtaining $\lambda_{i,j,2} = 2c_j^{(2)}(1-\cos\frac{\pi}{N_2})$ or $\lambda_{i,j,2} = 2c_i^{(1)}(1-\cos\frac{\pi}{N_1})$ for each of the above choices of $\ell_1, \ \ell_2$, which lead to the following

$$\lambda_{i,j,2} = \begin{cases} 2c_i^{(1)}(1-\cos\frac{\pi}{N_1}), \ \sigma_{ij} \geq \sigma_2 \\ \\ 2c_j^{(2)}(1-\cos\frac{\pi}{N_2}), \sigma_{ij} \leq \sigma_2, \end{cases} \tag{19}$$

where

$$\sigma_{ij} = \frac{c_j^{(2)}}{c_i^{(1)}} \ \ \textit{and} \ \ \sigma_2 = \frac{1-\cos\frac{\pi}{N_1}}{1-\cos\frac{\pi}{N_2}}. \tag{20}$$

The maximum eigenvalue $\lambda_{i,j,N}$ is determined by letting $\ell_1 = N_1 - 1$ and $\ell_2 = N_2 - 1$ in (9), hence

$$\lambda_{i,j,N} = 2[c_i^{(1)}(1+\cos\frac{\pi}{N_1}) + c_j^{(2)}(1+\cos\frac{\pi}{N_2})]. \tag{21}$$

Using the expressions of $\lambda_{i,j,2}$ and $\lambda_{i,j,N}$ given by (19) and (21), respectively in (13), (14) and (15), we easily verify (17) and (18). □

4 Determination of Optimum $\mathbf{c}_i^{(1)}$ and $\mathbf{c}_j^{(2)}$

Up to this point we were concerned with the determination of optimum values for the set of parameters τ_{ij} in terms of the edge weights $c_i^{(1)}, \ i = 1, 2, \ldots, N_1$ and $c_j^{(2)}, \ j = 1, 2, \ldots, N_2$. In fact, we have solved the problem of maximizing the rate of convergence of local EDF assuming the edge weights of the graph are known á priory. Next, we will try to determine the $c_i^{(1)}$s and $c_j^{(2)}$s such that P_{ij} (and hence γ_{ij}) is minimized.

Theorem 2. *The convergence factor* $\gamma_{ij}(M_{ij})$ *is minimized when*

$$c_j^{(2)} = \sigma_2 c_i^{(1)} \tag{22}$$

and

$$\tau_{ij}^{opt} = \tau^{opt}/c_i^{(1)}, \;\; c_i^{(1)} arbitrary \tag{23}$$

for any $i = 1, 2, \ldots, N_1, \; j = 1, 2, \ldots, N_2$*, where*

$$\tau^{opt} = (2 + \sigma_2(1 + \cos\frac{\pi}{N_2}))^{-1} \tag{24}$$

and its corresponding minimum is

$$\gamma^{opt} = \frac{2\cos\frac{\pi}{N_1} + \sigma_2(1 + \cos\frac{\pi}{N_2})}{2 + \sigma_2(1 + \cos\frac{\pi}{N_2})}. \tag{25}$$

Proof. The P-condition number of L_{ij} is, because of (21) and (19), given by

$$P_{ij}(L_{ij}) = \begin{cases} \dfrac{1 + \cos\frac{\pi}{N_1} + \sigma_{ij}(1 + \cos\frac{\pi}{N_2})}{1 - \cos\frac{\pi}{N_1}}, \; \sigma_{ij} \geq \sigma_2 \\ \dfrac{1 + \cos\frac{\pi}{N_1} + \sigma_{ij}(1 + \cos\frac{\pi}{N_2})}{\sigma_{ij}(1 - \cos\frac{\pi}{N_2})}, \; \sigma_{ij} \leq \sigma_2. \end{cases} \tag{26}$$

Studying the behavior of the above expression with respect to σ_{ij} we can easily verify that it is minimized at σ_2. Therefore, (13), because of (21) and (19), yields the optimum value of τ_{ij}^{opt}. The optimum value for $\gamma_{ij}(M_{ij})$ is obtained from (14). □

Similar results hold in case $c_j^{(2)}$ is arbitrary.

From (22) we observe that if the edge weights in one dimension of the 2D-mesh are all equal to a constant value, then the edge weights of the other dimension will be all equal to a constant value also. In case the edge weights in each dimension of a 2D-mesh are all equal to a constant value the assumptions for the local Fourier analysis hold [1]. This is justified as follows. The edge weights are associated with the coefficients of the Diffusion equation. As a consequence, if the edge weights in one dimension are all equal to the same constant value, then the eigenvalues of the operator L_{ij}, given by (9), coincide with the eigenvalues of the Laplacian matrix L. Moreover, local EDF degenerates into EDF as now τ_{ij}^{opt} (see(23)) denoted by τ_{EDF}^{opt}, will be constant. Therefore, we have the following.

Corollary 1. *If the edge weights in one dimension of a 2D-mesh are all equal to the same constant value and (22) holds, then* $\gamma(M)$*, the convergence factor of the diffusion matrix* M*, is minimized at* τ_{EDF}^{opt}*, given by (23) and its corresponding minimum is* γ^{opt}*, where* τ^{opt}, γ^{opt} *are given by (24) and (25), respectively.*

The above corollary shows that the optimum value of the parameter τ of DF is not 1/4 as stated in theorem 3.5.2 of [8] but is given by (23) and (25). Furthermore, the minimum value of $\gamma(M)$ is also different and depends upon both dimensions $N_1, \; N_2$, as seen in (25), rather than upon $N_{max} = \max\{N_1, \; N_2\}$ as stated in theorem 3.5.2 in [8].

Corollary 2. *Under the hypotheses of corollary 1 and if* N=N_1=N_2*, then the convergence factor* $\gamma(M)$ *is minimized at* τ_{EDF}^{opt} *given by (23) with* $\sigma_2 = 1$*,*

$$\tau^{opt} = (3 + \cos\frac{\pi}{N})^{-1} \tag{27}$$

and its corresponding minimum is given by

$$\gamma^{opt} = \frac{1 + 3\cos\frac{\pi}{N}}{3 + \cos\frac{\pi}{N}}. \tag{28}$$

Proof. If N=N$_1$=N$_2$, then σ_2=1 hence (27) and (28) are direct results of (24) and (25), respectively. □

Letting $c_i^{(1)} = 1$ for any $i = 1, 2, \ldots, N_1$ in (22) we obtain $c_j^{(2)} = \sigma_2$ for any $j = 1, 2, \ldots, N_2$, which is one of the infinite optimum values one can obtain by this relation. We will refer to this choice for the edge weights as the *normalized* one. From corollary 1 we have the following.

Corollary 3. *For the normalized edge weights the convergence factor* $\gamma(M)$ *is minimized at* $\tau_{EDF}^{opt} = \tau^{opt}$ *and its corresponding minimum is given by* γ^{opt}*, where* τ^{opt}, γ^{opt} *are given by (24) and (25), respectively.*

4.1 Comparison with DF

Corollary 4. *Under the hypotheses of corollary 1 we have*

$$\gamma_{EDF}^{opt} \leq \gamma_{DF}^{opt}, \tag{29}$$

where $\gamma_{EDF}^{opt}, \gamma_{DF}^{opt}$ *denote the convergence factors of the EDF and DF methods, respectively.*

Proof. Let us first assume that N$_1$ ≥ N$_2$ and x=$\frac{\pi}{N_1}$, y=$\frac{\pi}{N_2}$. From (25), the optimum value of γ_{EDF} is

$$\gamma_{EDF}^{opt} = \frac{2\cos x + \sigma_2(1 + \cos y)}{2 + \sigma_2(1 + \cos y)}, \tag{30}$$

where

$$\sigma_2 = \frac{1 - \cos x}{1 - \cos y} \leq 1. \tag{31}$$

On the other hand, the optimum value of γ_{DF} is given by [9]

$$\gamma_{DF}^{opt} = \frac{1 + \cos x}{2}. \tag{32}$$

A simple algebraic manipulation reveals that

$sign(\gamma_{EDF}^{opt} - \gamma_{DF}^{opt}) = sign[(\cos x - 1)(2 - \sigma_2 - \sigma_2 \cos y)] = -1$ or 0 thus proving (29). If $N_1 \leq N_2$, then

$$\gamma_{EDF}^{opt} = \frac{1 + \cos x + 2\sigma_2 \cos y}{2\sigma_2 + 1 + \cos x} \tag{33}$$

and [9]

$$\gamma_{DF}^{opt} = \frac{1+\cos y}{2} \tag{34}$$

hence $sign(\gamma_{EDF}^{opt} - \gamma_{DF}^{opt}) = sign[(1-\cos y)(1+\cos x - 2\sigma_2)] = -1$ or 0 since $\sigma_2 \geq 1$ now. Thus, (29) holds also in this case. □

4.2 The Stretched Mesh

In order to be able to have a direct comparison of the convergence behavior of the EDF and DF methods we study the case, where one dimension of the mesh is large compared to the other one (stretched mesh).

Corollary 5. *For stretched mesh and under the hypothesis of corollary 1 we have*

$$R(EDF) \simeq 2R(DF). \tag{35}$$

Proof. Let $N_1 \gg N_2$, then

$$R(EDF) = -\log \gamma_{EDF}^{opt} \simeq \frac{2\left(1-\cos\frac{\pi}{N_1}\right)}{2+\sigma_2\left(1+\cos\frac{\pi}{N_2}\right)} \tag{36}$$

since $-\log(1-x) \simeq x$ and

$$\gamma_{EDF}^{opt} = 1 - \frac{2\left(1-\cos\frac{\pi}{N_1}\right)}{2+\sigma_2\left(1+\cos\frac{\pi}{N_2}\right)}. \tag{37}$$

Similarly, for the DF method we have

$$R(DF) = -\log(\gamma_{DF}^{opt}) \simeq \frac{1}{2}\left(1-\cos\frac{\pi}{N_1}\right) \tag{38}$$

since $\gamma_{DF}^{opt} = 1 - \frac{1}{2}\left(1-\cos\frac{\pi}{N_1}\right)$. By dividing (36) and (38) we have

$$\frac{R(EDF)}{R(DF)} \simeq \frac{4}{2+\sigma_2\left(1+\cos\frac{\pi}{N_2}\right)} \tag{39}$$

and noting that $\sigma_2 \to 0$ for $N_1 \to \infty$ and N_2 fixed, (39) yields (35). If now $N_1 \ll N_2$, then

$$\gamma_{EDF}^{opt} = 1 - \frac{2\sigma_2\left(1-\cos\frac{\pi}{N_2}\right)}{2+\sigma_2\left(1+\cos\frac{\pi}{N_2}\right)} \tag{40}$$

and

$$\gamma_{DF}^{opt} = 1 - \frac{1-\cos\frac{\pi}{N_2}}{2}, \tag{41}$$

hence

$$\frac{R(EDF)}{R(DF)} \simeq \frac{4}{\frac{2}{\sigma_2}+1+\cos\frac{\pi}{N_2}}. \tag{42}$$

But now $\sigma_2 \to \infty$ for $N_2 \to \infty$ and N_1 fixed, hence (42) yields (35). □

5 Comparison of Mesh vs. Torus

In this section we compare theoretically the performance of EDF for stretched torus and mesh networks. Our comparison is based on the reciprocal rate of convergence, denoted by RR(EDF,M) and RR(EDF,T) for mesh and torus, respectively. As is known [10] this quantity is defined by the inverse of rate of convergence and is analogous to the expected number of iterations.

Theorem 3. *For streched mesh and torus and under the hypothesis of corollary 1 we have*

$$RR(EDF, M) = 4RR(EDF, T), \tag{43}$$

where $RR(EDF, \cdot) = 1/R(EDF, \cdot)$.

Proof. If N_1, N_2 are even and $N_1 \gg N_2$, then by [5]

$$\gamma_{EDF,T}^{opt} = \frac{1 + 2\sigma_2 + \cos\frac{2\pi}{N_1}}{3 + 2\sigma_2 - \cos\frac{2\pi}{N_1}} \tag{44}$$

and from (25)

$$\gamma_{EDF,M}^{opt} = \frac{2\cos\frac{\pi}{N_1} + \sigma_2(1 + \cos\frac{\pi}{N_2})}{2 + \sigma_2(1 + \cos\frac{\pi}{N_2})}. \tag{45}$$

From (44) and (45) we have

$$\frac{RR(EDF, M)}{RR(EDF, T)} = \frac{2(1 + \cos\frac{\pi}{N_1})[2 + \sigma_2(1 + \cos\frac{\pi}{N_2})]}{3 + 2\sigma_2 - \cos\frac{2\pi}{N_1}} \tag{46}$$

where $\sigma_2 \to 0$ for $N_1 \to \infty$ and N_2 fixed. Hence (46), yields (43). If N_1, N_2 odd and $N_1 \gg N_2$ we have [5]

$$\gamma_{EDF,T}^{opt} = \frac{\cos\frac{\pi}{N_1} + \cos\frac{2\pi}{N_1} + 2\sigma_2(1 + \cos\frac{\pi}{N_2})}{2 + \sigma_2(1 + \cos\frac{\pi}{N_2}) + \cos\frac{\pi}{N1} - \cos\frac{2\pi}{N1}} \tag{47}$$

and from (25)

$$\gamma_{EDF,M}^{opt} = \frac{2\cos\frac{\pi}{N_1} + \sigma_2(1 + \cos\frac{\pi}{N_2})}{2 + \sigma_2(1 + \cos\frac{\pi}{N_2})}. \tag{48}$$

From (47) and (48) we get

$$\frac{RR(EDF, M)}{RR(EDF, T)} = \frac{2(1 + \cos\frac{\pi}{N_1})\left[2 + \sigma_2(1 + \cos\frac{\pi}{N_2})\right]}{2 + \sigma_2\left(1 + \cos\frac{\pi}{N_2}\right) + \cos\frac{\pi}{N1} - \cos\frac{2\pi}{N1}} \tag{49}$$

where $\sigma_2 \to 0$ for $N_1 \to \infty$ and N_2 fixed hence (49) yields (43). If N_1 even, N_2 odd and $N_1 \gg N_2$ then [5]

$$\gamma_{EDF,T}^{opt} = \frac{1 + \cos\frac{2\pi}{N_1} + \sigma_2(1 + \cos\frac{\pi}{N_2})}{3 - \cos\frac{2\pi}{N1} + \sigma_2(1 + \cos\frac{\pi}{N_2})} \tag{50}$$

and from (25)

$$\gamma_{EDF,M}^{opt} = \frac{2\cos\frac{\pi}{N_1} + \sigma_2(1+\cos\frac{\pi}{N_2})}{2+\sigma_2(1+\cos\frac{\pi}{N_2})}. \tag{51}$$

From (50) and (51) we have

$$\frac{RR(EDF,M)}{RR(EDF,T)} = \frac{2(1+\cos\frac{\pi}{N_1})(2+\sigma_2(1+\cos\frac{\pi}{N_2}))}{3-\cos\frac{2\pi}{N1}+\sigma_2(1+\cos\frac{\pi}{N_2})} \tag{52}$$

where $\sigma_2 \to 0$ for $N_1 \to \infty$ and N_2 fixed hence (52) yields (43).

If N_1 odd, N_2 even and $N_1 \gg N_2$ then [5]

$$\gamma_{EDF,T}^{opt} = \frac{2\sigma_2 + \cos\frac{\pi}{N_1} + \cos\frac{2\pi}{N_1}}{2+2\sigma_2+\cos\frac{\pi}{N1} - \cos\frac{2\pi}{N_1}} \tag{53}$$

and

$$\gamma_{EDF,M}^{opt} = \frac{2\cos\frac{\pi}{N_1} + \sigma_2(1+\cos\frac{\pi}{N_2})}{2+\sigma_2(1+\cos\frac{\pi}{N_2})}. \tag{54}$$

From (53) and (54) we have

$$\frac{RR(EDF,M)}{RR(EDF,T)} = \frac{2(1+\cos\frac{\pi}{N_1})(2+\sigma_2(1+\cos\frac{\pi}{N_2}))}{2+2\sigma_2+\cos\frac{\pi}{N1}-\cos\frac{2\pi}{N_1}} \tag{55}$$

where $\sigma_2 \to 0$ for $N_1 \to \infty$ and N_2 fixed hence (55) yields (43). □

So EDF in a torus is faster than in a mesh network by a factor of 4 with respect to the number of iterations.

6 Numerical Experiments

In order to test our theoretical results obtained so far we applied EDF for different sizes of 2D-meshes. The initial load of the network was placed on a single node of the graph, while we normalized the balanced load $\overline{u}$=1. Hence, the total number of amount of load was equal to the total number of nodes in the graph. For purposes of comparison we considered the application of the EDF and DF methods with optimum parameters (*normalized edge weights*) and kept iterating until an almost evenly distributing flow was calculated. The iterations were terminated when the criterion

$$\|u^{(n)} - \overline{u}\|_2 / \|u^{(0)} - \overline{u}\|_2 < \epsilon$$

with $\epsilon = \frac{1}{2}10^{-6}$ was satisfied. A comparison of the number of iterations is presented in Table 1 for both of the aforementioned methods. The fourth column shows the ratio of the number of iteration of DF over EDF. These results clearly show that fixing one dimension and increasing the other dimension the rate of convergence of EDF becomes twice as fast as the DF method verifying Corollary 5. As expected (Corollary 2) the two methods produce the same results when

$N_1=N_2$. Table 2 presents a comparison of number of iterations of EDF applied to torus and mesh networks. The fourth column shows the ratio of the number of iterations of EDF for the aforementioned networks. Increasing the dimension N_2, this ratio tends to 4, verifying Theorem 3.

Table 1. Comparison of number of iterations of EDF and DF methods for N_1xN_2 meshes

$N_1 \times N_2$	DF	EDF	DF/EDF
5×5	131	131	1
5×11	596	379	1.57
5×21	2,167	1,198	1.80
5×51	11,774	6,290	1.87
5×101	47,103	24,997	1.88
6×6	181	181	1
6×10	496	361	1.37
6×20	1,634	1,098	1.49
6×50	11,677	6,379	1.83
6×100	46,621	24,470	1.90

Table 2. Comparison of number of iterations of EDF for N_1xN_2 meshes and tori

$N_1 \times N_2$	Torus	Mesh	$\frac{Mesh}{Torus}$
5×5	36	131	3.63
5×11	98	379	3.87
5×21	308	1,198	3.89
5×51	1,704	6,379	3.74
5×101	6,647	24,997	3.76
6×6	53	181	3.41
6×10	97	361	3.72
6×20	291	1,098	3.77
6×50	1,633	6,379	3.91
6×100	6,270	24,470	3.90

7 Conclusions and Future Work

The problem of determining the edge weights c_{ij} such that DF attains its maximum rate of convergence is an active research area [6], [7]. Introducing the set of parameters $\tau_i, i = 1, 2, \ldots, |V|$, the problem moves to the determination of the parameters τ_i's in terms of c_{ij}'s. By adopting the local Fourier analysis we were able to determine quasi-optimum values for τ_i's. These values become optimum in case the edge weights are constant in each dimension and satisfy the relation (22) (Corollaries 1, 2 and 3). This result improves theorem 3.5.2 of [8], where the edge weights were all equal. Apart from the fact that our approach produces a monoparametric set of quasi-optimum values for the edge weights (see (22)) it also has the advantage of determining a closed form formula for the parameter τ and the convergence factor γ. These facts have two consequences. First, we avoid the computation of the second smallest and largest eigenvalue of the Laplacian matrix for the determination of the optimum value for τ. This is a time consuming process and was an open problem up to now as far as we know. Secondly, using the expression for γ we were able to study the convergence behavior of the EDF method and predict its performance. As a result we found that for square meshes EDF has the same rate of convergence with DF, whereas for orthogonal meshes EDF may become twice as fast as DF (stretched mesh). In addition, it was shown that EDF for orthogonal torus is four times faster than orthogonal mesh networks.

Acknowledgement

This research is supported by the University of Athens, Project: “Kapodistrias”, grant no. 70/4/4917.

References

1. Brandt, A.: Multi–level adaptive solutions to boundary–value problems (1977)
2. Cybenko, G.: Dynamic load balancing for distributed memory multi-processors. J. Parallel and Distr. Comp. 7, 279–301 (1989)
3. Hong, J.W., Tau, X.N., Cheu, M.: From local to global: an analysis of nearest neighbor balancing on hypercube. In: ACM Symp. on SIGMETRICS, pp. 73–82 (1988)
4. Boillat, J.: Load balancing and poisson equation in a graph. In: Concurrency: Practice and Experience, pp. 289–313 (1990)
5. Karagiorgos, G., Missirlis, N.M.: Convergence of the diffusion method for weighted torus graphs using fourier analysis. Theoretical Computer Science 401, 1–16 (2008)
6. Diekmannn, R., Muthukrishnan, M., Nayakkankuppam, M.V.: Engineering diffusive load balancing algorithms using experiments. In: Lüling, R., Bilardi, G., Ferreira, A., Rolim, J.D.P. (eds.) IRREGULAR 1997. LNCS, vol. 1253, Springer, Heidelberg (1997)
7. Elsässer, R., Monien, B., Preis, R.: Optimal diffusion schemes and load balancing on product graphs. Parallel Proc. Letters 14, 61–73 (2004)
8. Xu, C.Z., Lau, F.C.M.: Optimal parameters for load balancing using the diffusion method in k-ary n-cube networks. Inform. Proc. Letters 47, 181–187 (1993)
9. Xu, C.Z., Lau, F.C.M.: Load balancing in parallel computers: Theory and Practice. Academic, New York (1997)
10. Young, D.M.: Iterative Solution of Large Linear Systems. Academic, New York (2003)