On the Characterization of Level Planar Trees by Minimal Patterns*

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Abstract. We consider characterizations of level planar trees. Healy *et al.* [8] characterized the set of trees that are level planar in terms of two minimal level non-planar (MLNP) patterns. Fowler and Kobourov [7] later proved that the set of patterns was incomplete and added two additional patterns. In this paper, we show that the characterization is still incomplete by providing new MLNP patterns not included in the previous characterizations. Moreover, we introduce an iterative method to create an arbitrary number of MLNP patterns, thus proving that the set of minimal patterns that characterizes level planar trees is infinite.

1 Introduction

An important application of automatic graph drawing can be found in the layout of graphs that represent hierarchical relationships. When drawing graphs in the *xy*-plane, this translates to a restricted form of planarity where the *y*-coordinate of a vertex is given and the drawing algorithm only has the freedom to choose the *x*-coordinate. This restricted form of planarity is called *level planarity*, and each given *y*-coordinate corresponds to a *level*.

Jünger, Leipert, and Mutzel [13] provide a linear-time recognition algorithm for level planar graphs. This algorithm is based on the level planarity test given by Heath and Pemmaraju [9,10]. The algorithm by Heath and Pemmaraju is based on the more restricted PQ-tree level planarity testing algorithm of *hierarchies* (level graphs of directed acyclic graphs in which all edges are between adjacent levels and all the source vertices are on the uppermost level) given by Di Battista and Nardelli in [3]. In the paper, the authors also characterize such hierarchies in terms of level non-planar (LNP) patterns. Jünger and Leipert [12] provide a linear-time level planar embedding algorithm that outputs a set of linear orderings in the *x*-direction for the vertices on each level. However, to obtain a straight-line planar drawing one needs to subsequently run an $O(|V|)$ algorithm given by Eades *et al.* [4] who demonstrate that every level planar embedding has a straight-line drawing, though it may require exponential area.

Healy *et al.* [8] use LNP patterns to provide a set of *minimal level non-planar* (MLNP) subgraph patterns that characterize level planar graphs. This is the counterpart for level graphs to the characterization of planar graphs by Kuratowski [14] in terms of forbidden subdivisions of K_5 and $K_{3,3}$. Two new MLNP tree patterns were added in [7]

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by Fowler and Kobourov to the previous set of patterns given by Healy *et al*. In this paper, we show that the characterization remains incomplete by providing new MLNP patterns not included in the previous characterizations. Moreover, we introduce an iterative method to create an arbitrary number of MLNP patterns, thus proving that the set of minimal patterns that characterizes level planar trees is infinite.

The study of MLNP patterns is motivated in part by the problem of visualizing hierarchical structures. Sugiyama *et al.* [15] described what has become the standard framework for drawing directed acyclic graphs. In this framework vertices are assigned to levels and then on each level vertices are ordered, with the overall goal of minimizing the number of edge crossings. There exists good heuristics and some exact methods based upon integer linear programs (ILPs) to find good orders within levels [11]. However, typically the assignment of vertices to levels is done with the help of greedy local optimizations [2]. Understanding the underlying obstructions to level planarity (such as MLNP patterns) could lead to better solutions to the level assignment step.

Level planarity is also related to simultaneous embedding [1]. In general, a set of restrictions on the layout of one graph may help in the layout of a second graph on the same vertex set. Specifically, when embedding a path with a planar graph, if the graph can be drawn on horizontal levels, then the path can be drawn in a y-monotone fashion without crossings. Estrella-Balderrama *et al.* [6] characterized the set of unlabeled level planar (ULP) trees on n vertices that are level planar over all possible labelings of the vertices in terms of two forbidden trees: T_8 and T_9. A level non-planar labeling of T_9 was used to obtain MLNP patterns P_3 and P_4 in [7]; see Fig. 3.

2 Preliminaries

A *k-level graph* $G(V, E, \phi)$ on n vertices is a directed graph $G(V, E)$ with a level assignment $\phi : V \rightarrow \{1, \ldots, k\}$ such that the induced partial order is strict: $\phi(u) < \phi(v)$ for every $(u, v) \in E$. A k-level graph is a k-partite graph in which ϕ partitions V into k independent sets $V_1, V_2, \ldots, V_k$, which form the k *levels* of G. A *level-j* vertex v is on the j^{th} *level* V_j if $\phi(v) = j$ (*i.e.* $v \in V_j$). In a level graph, an edge (u, v) is *short* if $\phi(v) = \phi(u) + 1$ while edges spanning multiple levels are *long*. A *proper level graph* has only short edges. Any level graph can be made proper by subdividing long edges into short edges. In this paper, a level graph is proper unless stated otherwise.

A level graph G has a *level drawing* if there exists a drawing such that every vertex in V_j is placed along the horizontal line $\ell_j = \{(x, j) \mid x \in \mathbb{R}\}$ and the edges are drawn as strictly y-monotone polylines. The order that the vertices of V_j are placed along each ℓ_j in a level drawing of a proper graph induces a family of linear orders along the x-direction, which form a *linear embedding* of G. A level drawing, and consequently its level embedding, is *level planar* if it can be drawn without edge crossings. A level graph G is level planar if it admits a level planar embedding. The definition of level drawings allowing only straight-line segments for edges is equivalent, given that Eades *et al.* [4] have shown that every level planar graph has a straight-line planar drawing.

A *path* is a non-repeating ordered sequence of vertices $(v_1, v_2, \ldots, v_n)$ for $n \geq 1$. A *star* with n vertices is a tree with one vertex of degree $n - 1$, called the *root*, and $n - 1$ vertices of degree 1. A *spider* is an arbitrarily subdivided star, where *subdividing* an

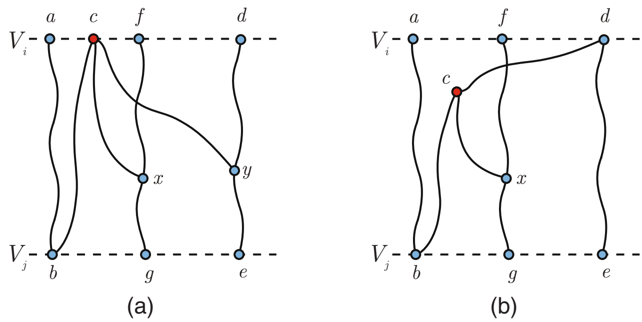

Fig. 1. Original MLNP patterns P_1 in (a) and P_2 in (b) proposed by Healy *et al*

edge (u, v) replaces the edge with a new vertex w and new edges (u, w) and (w, v). In a *degree-k spider*, the root has degree k.

A *chain-link*, denoted $u \rightsquigarrow v$, is a path from vertex u to vertex v with $u \neq v$ such that each internal vertex w that lies along the path has degree 2. Let $\phi(u \rightsquigarrow v)$ denote the set of levels of the internal vertices where $i \leq \phi(u \rightsquigarrow v) \leq j$ is a short-hand for saying that $i \leq \phi(w) \leq j$ for each internal vertex w of the chain-link $u \rightsquigarrow v$. Unless stated otherwise we assume that $\phi(u) \leq \phi(u \rightsquigarrow v) \leq \phi(v)$ for each chain-link $u \rightsquigarrow v$. A *linking chain*, or simply a *chain*, is a sequence of one or more chain-links. Notice that a vertex in the intersection of two chains is not considered a crossing between the chains. In all figures, a curve connecting two vertices, represents a chain.

In a level non-planar graph, a *pattern* is an obstructing subgraph with a level assignment that forces a crossing. Since here we define particular patterns in terms of chains, they represent a set of graphs with similar properties in terms of leveling. A level non-planar pattern is *minimal* if the removal of an arbitrary edge makes the pattern level planar. All the patterns described here (with the exception of a few that are symmetrical) have a corresponding horizontally flipped version.

3 Previous Work

3.1 Characterization of Level Planar Trees by Healy *et al.*

Healy *et al.* [8] defined MLNP patterns as follows: Let i and j be the minimum and maximum level, respectively, of any vertex in the pattern. Let x be a vertex of degree 3 with three subtrees with the following properties: (i) each subtree has at least one vertex on both extreme levels; (ii) a subtree is either a chain or it has two subtrees that are chains; (iii) all leaves are located on extreme levels (and each leaf is the only vertex in its subtree on the extreme level); and (iv) the subtrees that are chains and have non-leaf vertices on one extreme level, also have at least one leaf vertex on the opposite extreme level.

Then they distinguish two patterns; P_1 with x on an extreme level and P_2 with x on a non-extreme level (Healy *et al.* denote them T1 and T2). Figure 1 shows P_1 and P_2. Notice that these patterns are defined in terms of subtrees. This implies, for example,

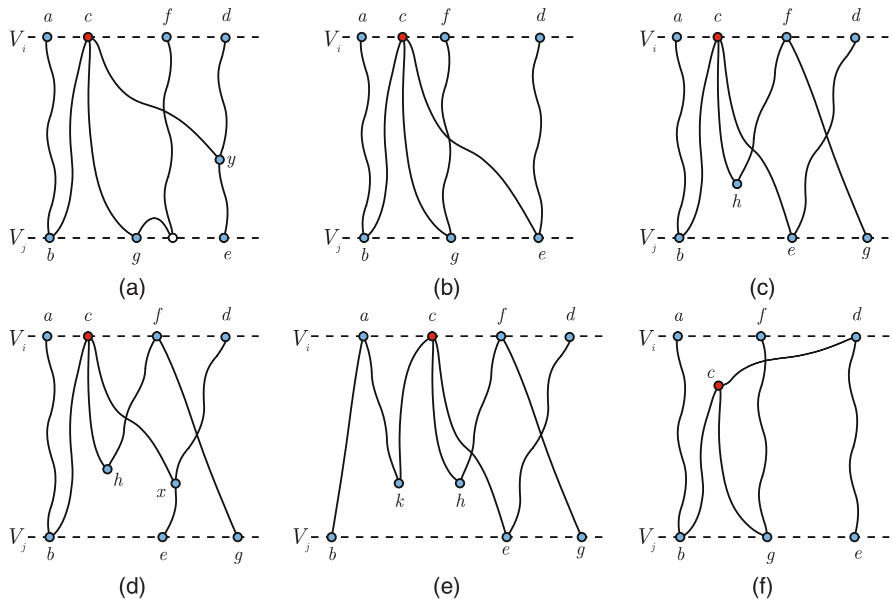

Fig. 2. (a-e) Five variations of pattern P_1 in addition to the one in Fig. 1(a); (f) One variation of pattern P_2 in addition to the one in Fig. 1(b)

that a subtree with a vertex of degree 3 may be replaced by a path. Fowler and Kobourov, on the other hand, defined the patterns in terms of paths. Hence, to properly compare the set of patterns we need to consider the different cases, or *variations*, of the subtrees in P_1 and P_2. Hence, P_1 leads to variations $P_1^A, \ldots, P_1^F$ and P_2 leads to variations P_2^A and P_2^B; see Fig. 2. Notice that when a chain reaches an extreme level with a degree-2 vertex, more degree-2 vertices of the chain can also be on the extreme level. This is illustrated in Fig. 2(a) for the chain $c \rightsquigarrow g \rightsquigarrow f$ with a second degree-2 vertex. Healy *et al.* [8] showed that both of these patterns are minimal level non-planar.

3.2 Characterization of Level Planar Trees by Fowler and Kobourov

The two trees T_8 and T_9 were shown to be the only obstructions in the context of unlabeled level planarity for trees [6]. However, as the tree T_9 does not match any of the MLNP patterns by Healy *et al.* [8], a new pattern P_3 was proposed [7]; see Fig. 3(b). Note that matching T_9 with either of the earlier patterns P_1 or P_2 would be impossible as both P_1 and P_2 are based on a central vertex of degree 3 (vertex x in Fig. 1), while T_9 and its matching pattern P_3 have a central vertex of degree 4 (vertex x in Fig. 3(b)).

Yet another pattern P_4 can be obtained from P_3 by "splitting" vertex x of degree 4 such that $i < l \leq \phi(x) \leq m < j$ into two vertices of degree 3 connected by a path. In Fig. 3(b) vertex x is replaced by a chain $x \rightsquigarrow y$ such that $l \leq \phi(x \rightsquigarrow y) \leq m$ in Fig. 3(c). Patterns P_3 and P_4 were added to the previous set of two patterns (eight variations) to obtain a new characterization consisting of four patterns (ten variations). A sketch of a

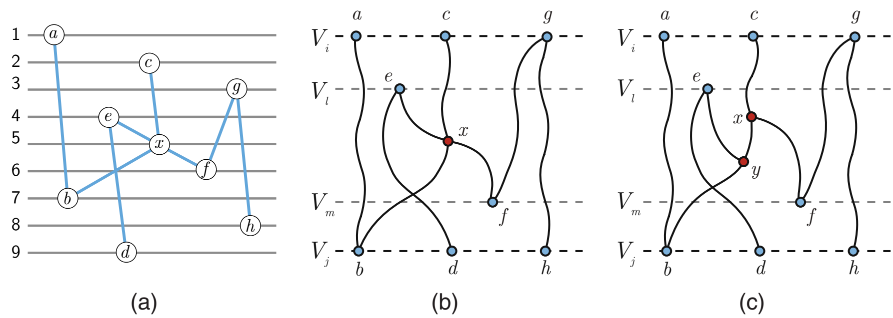

Fig. 3. Fowler and Kobourov generalized the forbidden ULP tree T_9 in (a) to produce the MLNP patterns P_3 in (b) and P_4 in (c)

proof for the claim that this new characterization is complete was made in [7], but in the next section we show that the characterization remains incomplete.

4 New Minimal Level Non-planar Patterns

In this section, we show that the characterization of level planar trees by minimal patterns is still incomplete. In Sect. 4.1, we show that there are variations of P_3 and P_4 that were not considered. Then in Sect. 4.2, we describe a new pattern previously not considered as it has a vertex of degree 5, whereas, all of the previously known MLNP patterns have maximum degree 4.

4.1 Variations of Patterns P_3 and P_4

The previous characterization introduces the new patterns P_3 and P_4. Just as with the variations of P_1 and P_2, different variations of P_3 and P_4 can be produced by replacing some chains with degree-3 spiders. We describe these variations next.

- *Pattern* P_3^A. This is the original pattern P_3; see Fig. 3(b).
- *Pattern* P_3^B. This pattern is similar to P_3^A but replaces the chain $x \rightsquigarrow f \rightsquigarrow g$ such that $l \leq \phi(x) \leq m$, $\phi(f) = m$, $\phi(g) = i$, and $i \leq \phi(f \rightsquigarrow g) \leq m$, with a degree-3 spider rooted at f' and leaves f, g, and x such that $l \leq \phi(f') \leq m$, $\phi(x) = \phi(f) = m$, $\phi(g) = i$, and $l \leq \phi(f \rightsquigarrow f') \leq m$; see Fig. 4(a).
- *Pattern* P_3^C. This pattern is similar to P_3^A but replaces the chain $x \rightsquigarrow e \rightsquigarrow d$, such that $l \leq \phi(x) \leq m$, $\phi(e) = l$, $\phi(d) = j$, and $l \leq \phi(x \rightsquigarrow e) \leq m$ with a degree-3 spider rooted at e' and leaves e, d, and x such that $l \leq \phi(e') \leq m$, $\phi(x) = \phi(e) = l$, $\phi(d) = j$, and $l \leq \phi(e \rightsquigarrow e') \leq m$; see Fig. 4(b).
- *Pattern* P_3^D. This pattern makes both replacements made by patterns P_3^B and P_3^C on P_3^A such that $\phi(e) = \phi(f') = l$, $\phi(e') = \phi(f) = m$, $l \leq \phi(x) \leq m$, $i \leq \phi(x \rightsquigarrow g) \leq m$, and $l \leq \phi(x \rightsquigarrow h) \leq j$; see Fig. 4(c).
- *Pattern* P_4^A. This is the original pattern P_4; see Fig. 3(c).
- *Patterns* P_4^B, P_4^C, *and* P_4^D. These patterns make analogous replacements on P_4^A as those made by P_3^B, P_3^C, and P_3^D on P_3^A; see Fig. 4(d-f).

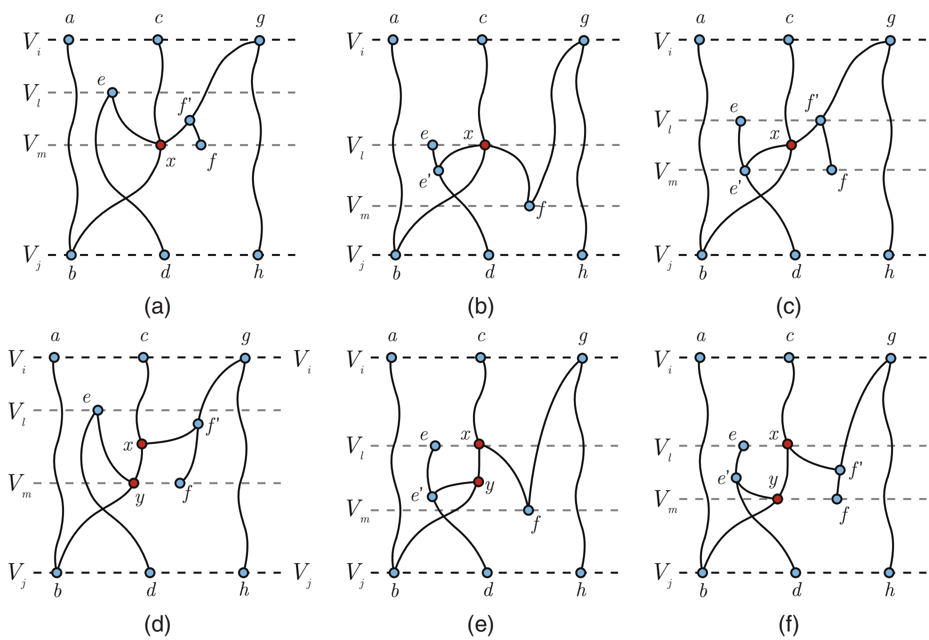

Fig. 4. (a-c) Variations of pattern P_3 (P_3^B, P_3^C, and P_3^D); (d-f) Variations of pattern P_4 (P_4^B, P_4^C, and P_4^D)

The importance of the new variations of P_3 and P_4 is that they break the fundamental assumption made in the early attempts at characterizations, namely that in any minimal level non-planar pattern, leaves must lie on extreme levels i or j. All of the new patterns have leaves on non-extreme levels. We omit the proofs for the variations of P_3 and P_4 as in the next section we formally show that a new pattern, P_5 with non-extreme leaves is MLNP. Moreover, in Sect. 5, we show that the set of MLNP patterns for trees is not just missing a few more patterns but is actually infinite.

4.2 New Pattern P_5

In this section, we describe a new pattern P_5 and its variations. The main characteristic of this pattern is the presence of a vertex x with degree 5.

- *Pattern* P_5^A. This pattern is a degree-5 spider, rooted at x, with two levels l and m between the extreme levels i and j such that $i < l < \phi(x) \leq m < j$. There is a chain $x \rightsquigarrow c$ such that $\phi(c) = i$, a chain $x \rightsquigarrow d$ such that $\phi(d) = j$; a chain $x \rightsquigarrow p \rightsquigarrow q$ such that $\phi(p) = m$ and $\phi(q) = l$; a chain $x \rightsquigarrow e \rightsquigarrow f \rightsquigarrow g \rightsquigarrow h$ such that $\phi(e) = l$, $\phi(f) = m$, $\phi(g) = i$, and $\phi(h) = j$; and a chain $x \rightsquigarrow k \rightsquigarrow b \rightsquigarrow a$ such that $l < \phi(k) < \phi(x)$, $\phi(b) = j$, $\phi(a) = i$ and $l < \phi(x \rightsquigarrow k \rightsquigarrow b) \leq j$; see Fig. 5(a).
- *Pattern* P_5^B. Similar to P_5^A but replaces the chain $x \rightsquigarrow e \rightsquigarrow f \rightsquigarrow g \rightsquigarrow h$ with a degree-3 spider rooted at f' such that $l < \phi(f') < m$, with x, g, and f such that $l < \phi(x) \leq m$, $\phi(e) = l$, $\phi(g) = i$, $\phi(f) = m$, and there is a chain $x \rightsquigarrow e \rightsquigarrow f'$

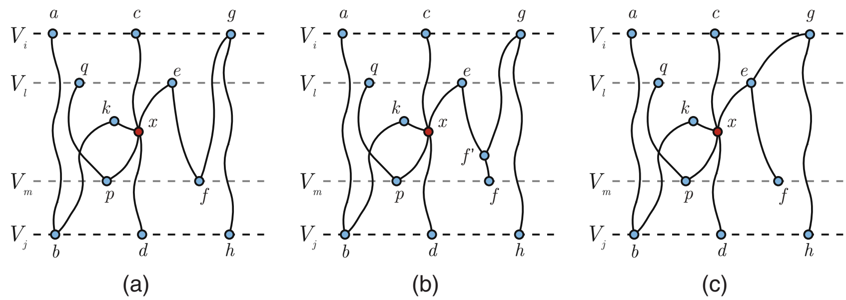

Fig. 5. Patterns P_5^A, P_5^B, and P_5^C

such that $\phi(e) = l$ where $l \leq \phi(x \rightsquigarrow e \rightsquigarrow f') \leq m$, $l \leq \phi(f \rightsquigarrow f') \leq m$, and $i \leq \phi(f' \rightsquigarrow g) \leq m$; see Fig. 5(b).

– *Pattern* P_5^C. Similar to P_5^A but replaces the chain $x \rightsquigarrow e \rightsquigarrow f \rightsquigarrow g \rightsquigarrow h$ with a degree-3 spider rooted at e such that $\phi(e) = l$ with leaves g, x, and f; see Fig. 5(c).

In the following two lemmas we show that this new pattern is MLNP.

Lemma 1. *Pattern P_5 is level non-planar.*

Proof. We show that P_5^A is level non-planar (the cases for P_5^B and P_5^C are similar). First notice that to avoid a crossing with chain $c \rightsquigarrow x \rightsquigarrow d$, all the vertices of the chain $x \rightsquigarrow e \rightsquigarrow f \rightsquigarrow g \rightsquigarrow h$ must lie to the right of the chain $c \rightsquigarrow x \rightsquigarrow d$ while all the vertices of the chain $x \rightsquigarrow k \rightsquigarrow b \rightsquigarrow a$ must lie to the left, or vice versa; see Fig. 5(a). Assume w.l.o.g. that $x \rightsquigarrow k \rightsquigarrow b \rightsquigarrow a$ lies to the left and $x \rightsquigarrow e \rightsquigarrow f \rightsquigarrow g \rightsquigarrow h$ lies to right of chain $c \rightsquigarrow x \rightsquigarrow d$ (as in Fig. 5(a)). Now observe that in order to avoid a crossing of chain $x \rightsquigarrow p \rightsquigarrow q$ with chains $a \rightsquigarrow b$, $c \rightsquigarrow x \rightsquigarrow d$ or $g \rightsquigarrow h$, the chain $x \rightsquigarrow p \rightsquigarrow q$ must lie between chains $a \rightsquigarrow b$ and $c \rightsquigarrow x \rightsquigarrow d$ or lie between chains $c \rightsquigarrow x \rightsquigarrow d$ and $g \rightsquigarrow h$. However, in the first case a crossing will occur with chain $x \rightsquigarrow k \rightsquigarrow b$ (since $\phi(k) < \phi(x)$ and $\phi(x) \leq \phi(x \rightsquigarrow p) \leq m$) and in the later case a crossing will occur with chain $x \rightsquigarrow e \rightsquigarrow f \rightsquigarrow g$. □

Lemma 2. *The removal of any edge in pattern P_5 makes it level planar.*

Proof. We consider the different cases of edge removal from the chains in P_5^A (P_5^B and P_5^C are similar):

case 1) If any edge is removed from chain $x \rightsquigarrow p \rightsquigarrow q$, then the crossing with chain $x \rightsquigarrow e \rightsquigarrow f$ is avoided when $x \rightsquigarrow p \rightsquigarrow q$ is to the right of $c \rightsquigarrow x \rightsquigarrow d$ as in Fig. 6(a).

case 2) If any edge is removed from chains $x \rightsquigarrow k \rightsquigarrow b \rightsquigarrow a$ or $x \rightsquigarrow e \rightsquigarrow f \rightsquigarrow g \rightsquigarrow h$, then all the vertices in the chain (except x) can be to the left or to the right of chain

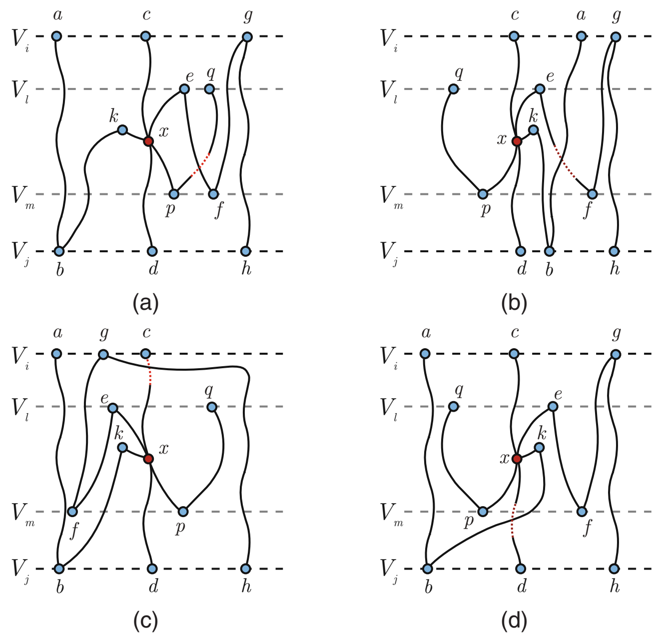

Fig. 6. Different cases of removing an edge (dotted) from pattern P_5^A

$c \rightsquigarrow x \rightsquigarrow d$ where chain $x \rightsquigarrow p \rightsquigarrow q$ can be on the other side avoiding the crossing as in Fig. 6(b).

case 3) If any edge is removed from chain $c \rightsquigarrow x$, then chains $x \rightsquigarrow k \rightsquigarrow b \rightsquigarrow a$ and $x \rightsquigarrow e \rightsquigarrow f \rightsquigarrow g$ can be on the same side with respect to $c \rightsquigarrow x \rightsquigarrow d$. Thus avoiding the crossing with chain $x \rightsquigarrow p \rightsquigarrow q$; see Fig. 6(c).

case 4) If any edge is removed from chain $x \rightsquigarrow d$, then chain $x \rightsquigarrow k \rightsquigarrow b$ can lie to the right of chain $x \rightsquigarrow p \rightsquigarrow q$ as in Fig. 6(d). □

We now use Lemmas 1 and 2 to show that P_5 is indeed MLNP.

Theorem 1. *P_5 is a minimal level non-planar pattern for trees.*

Proof. By Lemma 1, P_5 is level non-planar and by Lemma 2, P_5 is minimal. Minimality also implies that P_5 does not contain any MLNP pattern as a subgraph. Moreover, pattern P_5 does not match any of the previous patterns given that vertex x has degree 5, while all of the previously known patterns have maximum degree 4. □

In this section, we have shown that a new pattern P_5 is MLNP. However, P_5 is not the only pattern missing from earlier characterizations. New patterns $P_6, \ldots, P_{11}$ are shown along with their variations in [5]. The proofs of level non-planarity and minimality of these patterns are similar to the one given for P_5. Thus, instead of proving that each of these patterns is MLNP, we describe a constructive method for generating an infinite number of distinct MLNP patterns in the next section.

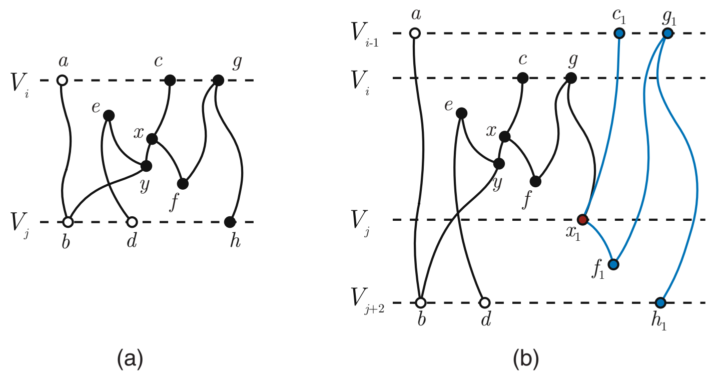

Fig. 7. Construction of a new pattern $(P_4^A)_1$ in (b) from pattern P_4^A in (a)

5 Infinite Minimal Level Non-planar Patterns

Our approach for creating new MLNP patterns is to take a known pattern as a base and then repeat a subgraph of the pattern making modifications on the leveling such that the new pattern does not strictly contain the previous one. Here we use P_4^A but the method applies to other patterns as well.

The first step is to make a copy of the path $p_0 = c \rightsquigarrow x \rightsquigarrow f \rightsquigarrow g \rightsquigarrow h$ such that $\phi(c) = \phi(g) = i < \phi(x) < \phi(f) < \phi(h) = j$ as in Fig. 7(a) in order to get a new path $p_1 = c_1 \rightsquigarrow x_1 \rightsquigarrow f_1 \rightsquigarrow g_1 \rightsquigarrow h_1$ such that $\phi(c_1) = \phi(g_1) = i - 1$, $\phi(x_1) = j$, $\phi(f_1) = j + 1$, and $\phi(h_1) = j + 2$ as in Fig. 7(b). The second step is to add p_1 to P_4^A by merging vertices x_1 and h creating a new vertex of degree 3 that takes the place of h. This new level assignment creates two new extreme levels $i - 1$ and $j + 2$. We complete the construction of the new pattern by moving vertices a, b, and d to the new extreme levels, specifically, we set $\phi(a) = i - 1$ and $\phi(b) = \phi(d) = j + 2$.

We now generalize the previous construction to an arbitrary number of iterations. We denote the pattern created at iteration t from pattern P as $(P)_t$. Thus, the original P_4^A is $(P_4^A)_0$ and the pattern created in Fig. 7(b) is $(P_4^A)_1$. The vertices in the pattern are labeled in the same way, for example $x_0 = x$. Therefore, in order to create a new pattern $(P_4^A)_{t+1}$ from pattern $(P_4^A)_t$, we first copy the path $p_t = c_t \rightsquigarrow x_t \rightsquigarrow f_t \rightsquigarrow g_t \rightsquigarrow h_t$ to get a new path $p_{t+1} = c_{t+1} \rightsquigarrow x_{t+1} \rightsquigarrow f_{t+1} \rightsquigarrow g_{t+1} \rightsquigarrow h_{t+1}$ such that $\phi(c_{t+1}) = \phi(g_{t+1}) = i - t - 1$, $\phi(x_{t+1}) = j + 2t$, $\phi(f_{t+1}) = j + 2t + 1$, and $\phi(h_{t+1}) = j + 2t + 2$. We then merge x_{t+1} with h_t to obtain the new x_{t+1}. Finally, we set the levels as $\phi(a) = i - t - 1$, and $\phi(b) = \phi(d) = j + 2t + 2$; see Fig. 8.

In the next lemma we show that a pattern, $(P_4^A)_t$, generated with the previous method is level non-planar.

Lemma 3. *Pattern $(P_4^A)_t$ for $t \geq 0$, is level non-planar.*

Proof. We use induction on t, the number of iterations in the generation method. The base case is $t = 0$; this is the original pattern P_4 which is proven to be level non-planar in the characterization by Fowler and Kobourov [7]. We now assume that $(P_4^A)_t$ is level

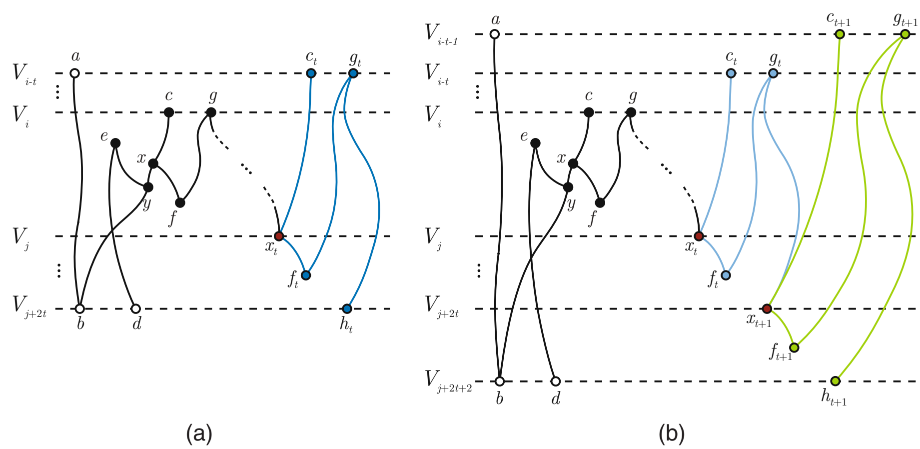

Fig. 8. Construction of a new pattern $(P_4^A)_{t+1}$ in (b) from pattern $(P_4^A)_t$ in (a)

non-planar in order to prove that $(P_4^A)_{t+1}$ is level non-planar. That is, we show that the modifications made to $(P_4^A)_t$ to obtain $(P_4^A)_{t+1}$ do not affect the level non-planarity of the new pattern.

Clearly, the addition of vertices and edges cannot affect the level non-planarity of a tree, hence the addition of the path p_{t+1} does not make the pattern level planar. Moreover, since the chains $a \rightsquigarrow b$ and $e \rightsquigarrow d$ in $(P_4^A)_t$ are contained in the chains $a \rightsquigarrow b$ and $e \rightsquigarrow d$ of $(P_4^A)_{t+1}$, the change on the levels of a and d are simply addition of vertices and edges that cannot affect the level non-planarity of the pattern. Finally, we consider the change of level of vertex b. Notice that the crossing between the chain $a \rightsquigarrow b \rightsquigarrow y \rightsquigarrow x \rightsquigarrow f \rightsquigarrow g$ and the chain $y \rightsquigarrow e \rightsquigarrow d$ in $(P_4^A)_t$ cannot be avoided in $(P_4^A)_{t+1}$ with the change of level of b. This is because as d is moved to the level of b the chain $f \rightsquigarrow g \rightsquigarrow \cdots \rightsquigarrow x_{t+1} \rightsquigarrow f_{t+1} \rightsquigarrow h_{t+1}$ plays an analogous role in the pattern $(P_4^A)_{t+1}$ that the chain $f \rightsquigarrow \cdots \rightsquigarrow h_t$ plays in the pattern $(P_4^A)_t$. That is, the addition of the chain $c_{t+1} \rightsquigarrow h_{t+1}$ to the pattern $(P_4^A)_{t+1}$ prevents the switch of side of the chain $a \rightsquigarrow b$ in order to avoid the crossing with $y \rightsquigarrow e \rightsquigarrow d$ as this will produce a crossing with the chain $c_{t+1} \rightsquigarrow x_{t+1}$ (as in Fig. 9(d)). Therefore, by induction the pattern $(P_4^A)_t$ is level non-planar for all non-negative integers $t \geq 0$. □

We next show the minimality of the patterns generated with the method above.

Lemma 4. *The removal of any edge in $(P_4^A)_t$ for any $t \geq 0$, makes it level planar.*

Proof. We consider the cases of edge removal in $(P_4^A)_t$.

case 1) If any edge is removed from the chain $a \rightsquigarrow b \rightsquigarrow y \rightsquigarrow e \rightsquigarrow d$, then the self-intersection is avoided as in Fig. 9(a).

case 2) If any edge is removed from the chain $x \rightsquigarrow y$, then the chain $e \rightsquigarrow d$ can use the gap to avoid the crossing as in Fig. 9(b).

case 3) If any edge is removed from the chain $x_\alpha \rightsquigarrow f_\alpha$ or $g_\alpha \rightsquigarrow h_\alpha$ for any $\alpha = 0, \ldots, t$, then chain $a \rightsquigarrow b \rightsquigarrow y$ can use the gap to be drawn between the chains $c_\alpha \rightsquigarrow x_\alpha$ and $f_\alpha \rightsquigarrow g_\alpha$ as in Fig. 9(c) or between g_α and h_α.

case 4) If any edge is removed from the chains $c_\alpha \rightsquigarrow x_\alpha$ or $f_\alpha \rightsquigarrow g_\alpha$ for any $\alpha = 0, \ldots, t$, then the chain $a \rightsquigarrow b$ can interchange sides with the chain $h_\alpha \rightsquigarrow g_\alpha$ if $\alpha = t$ as in Fig. 9(d). When $\alpha < t$, all the chains $c_\beta \rightsquigarrow x_\beta \rightsquigarrow f_\beta \rightsquigarrow g_\beta \rightsquigarrow h_\beta$ for $\beta = \alpha + 1, \ldots, t$ are moved along with the chain $h_\alpha \rightsquigarrow g_\alpha$. □

With the last two lemmas we now show that a pattern generated with the iterative method described in this section is MLNP.

Theorem 2. *Pattern $(P_4^A)_t$ for $t \geq 0$, is a minimal level non-planar pattern for trees.*

Proof. By Lemma 3, $(P_4^A)_t$ is level non-planar and by Lemma 4, $(P_4^A)_t$ is minimal. Minimality implies that $(P_4^A)_t$ does not contain any MLNP pattern as a subgraph. In particular, $(P_4^A)_t$ does not contain the previous pattern $(P_4^A)_{t-1}$. To see this in Fig. 8(b), observe that in the subgraph between levels i and j, the chain $a \rightsquigarrow b \rightsquigarrow y$ is separated by level j into two disjoint chains. Moreover, pattern $(P_4^A)_t$ does not match any of the previous patterns $(P_4^A)_\alpha$ for $\alpha = 0, \ldots, t-1$ since $(P_4^A)_t$ contains an additional vertex of degree 3, x_t. □

Theorem 2 implies that we can generate an arbitrary number of different MLNP patterns. This gives our main result.

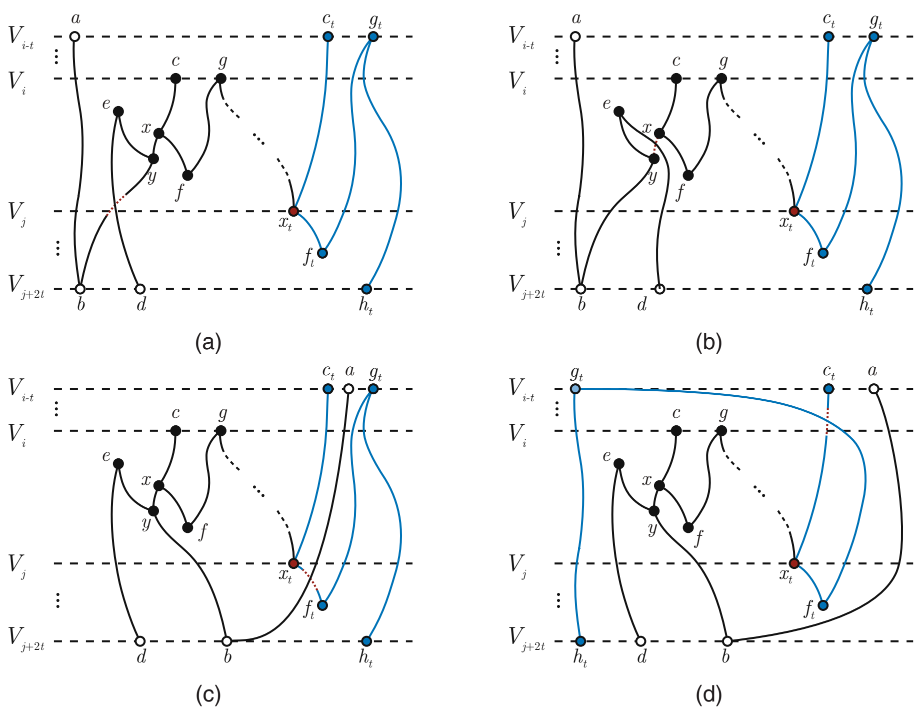

Fig. 9. Different cases of removing an edge (dotted) from pattern $(P_4^A)_t$

Theorem 3. *The set of minimal level non-planar patterns for trees is infinite.*

6 Conclusions and Future Work

In this paper, we showed why two earlier attempts to characterize the set of level non-planar trees in terms of minimal level non-planar patterns failed. In both cases, there was an implicit assumption that the set of different MLNP patterns is small and finite. However, it turns out that there are infinitely many different MLNP patterns, and an altogether different approach might be needed for a complete characterization.

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