Proving or Disproving Planar Straight-Line Embeddability onto Given Rectangles

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Encoding as SAT Problem

Given a plane graph $G = (V, E)$ and a rectangle we ask whether there exists a planar straight-line embedding of G onto the grid-points of the rectangle. For this NP-hard problem [5] some powerful heuristics have been developed to minimise the area of an embedding of a given graph [5,4]. Moreover, for particular families of graphs upper and lower bounds on the area have been proven [2]. However, in the general case it is not possible to ensure whether there is an embedding that preserves a particular area restriction $A = h \cdot w$. We present an implementation[1] based on a translation into SAT to tackle this kind of problems for small graphs. We only describe the direct encoding into CNF[2] that turned out to be most suitable.

Matching Vertices to Grid Positions. We introduce boolean variables $x_{i,j}$ representing whether or not vertex $0 < i \leq N$ ($N = |V|$) is located at grid position $0 < j \leq A$. This causes $N \cdot A$ variables. Moreover, in case $N < A$ we further introduce one variable $x_{\cdot,j}$ per grid position to represent the fact that position j is not used by any vertex. Another possibility would be to introduce $A - N =: d$ disconnected dummy vertices, but this causes $d \cdot A$ additional variables.

The following clauses ensure that each vertex is placed at (at least) one position: $(x_{i,1} \vee x_{i,2} \vee \ldots \vee x_{i,A}) \quad \forall\, 0 < i \leq N$. Analogously we ensure that each grid position is either used by a vertex or, in case of $N < A$ may be free:
$(x_{\cdot,j} \vee x_{1,j} \vee x_{2,j} \vee \ldots \vee x_{N,j}) \quad \forall\, 0 < j \leq A \qquad$ *[$x_{\cdot,j}$ is omitted if $N = A$]*
In case $N = A$ the conjunction of the above *position clauses* would be sufficient to guarantee that each vertex is placed at exactly one postion and vice versa. When $N < A$ a valid mapping of vertices to positions has to be enforced by additional constraints. One possibility is to introduce binary clauses $(\overline{x_{i,j}} \vee \overline{x_{k,l}})$ for each pair of literals within a clause. Note, that this is necessary for each of the above clauses. In practice it is important to have these – possibly redundant – constraints to guide the SAT-solver by enabling early recognition of conflicting assignments. We modified or SAT-solver *SApperloT*[3] to treat the position clauses as special constraints where **exactly one** literal has to be true. This simulates all $O(N \cdot A^2 + A \cdot N^2)$ binary clauses by simultaneously using a linear amount of memory.

[1] `www-pr.informatik.uni-tuebingen.de/?site=forschung/sat/algo_engineering`

[2] A formula in CNF (conjunctive normal form) is a conjunction of clauses; clauses are disjunctions of literals, whereas a literal is a boolean variable or its negation.

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Planar Embedding and Symmetry Breaking. To achieve a planar embedding of the graph crossings have to be prohibited explicitely. In order to avoid symmetric constraints this is done by introducing further A^2 variables: For each possible straight-line connection between any two grid positions we hold a variable $y_{k,l}$ $(0 < k, l \leq A)$ indicating whether or not the edge between grid position k and position l is present in an embedding of G.

With this, any placement of any two adjacent vertices onto grid positions causes a particular edge embedding to be drawn. If, for instance, two adjacent vertices i and j are placed at the positions k and l then the edge between these two positions is actually drawn. Hence, $x_{a,k} \wedge x_{j,l}$ implies variable $y_{k,l}$ to be true. This can be expressed by the clause $(\overline{x_{a,k}} \vee \overline{x_{j,l}} \vee y_{k,l})$. Note that there will be another clause $(\overline{x_{a,l}} \vee \overline{x_{j,k}} \vee y_{k,l})$ for the symmetric case. Given that the number of adjacent vertex pairs in a planar graph is bounded by $O(N)$ the number of introduced ternary clauses by this kind of constraints is bounded by $O(N \cdot A^2)$. It remains to prohibit the crossing of any two embedded edges. For this reason we disallow all combinations of crossing edge embeddings by introducing binary clauses of the form $(\overline{y_{k,l}} \vee \overline{y_{q,t}})$. At the same time we forbid any edge embedding that crosses a grid position k unless k is chosen to contain no vertex $(x_{\cdot,k} = true)$. The number of binary clauses introduced by these constraints is bounded by A^4.

Experimental Results and Conclusion

The purpose of our approach is to realise both: either proving or disproving the existence of a graph embedding onto a specified rectangle for small graphs. For our experimental setup we chose seven different kinds of planar graphs (using [1]): biconnected, triconnected, not biconnected, nested triangles, nested triangles completely triangulated, trees and grids. We generated the graphs with $16 \leq N \leq 100$ vertices. For the first three types of graphs each N was combined with $|E| \in \{N, \frac{3}{2}N, 2N, \frac{5}{2}N, 3N-6\}$ edges. For 393 graphs from a total of 576 test graphs our software was able to prove (269) or disprove (124) the existence of a planar straight-line embedding on a specified tight rectangle. We see strong potential to extend our tool to confirm conjectures for special kinds of graphs (as e.g. in [2]). For such cases additional constraints could be added to prune the search space of the solver.

References

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