Small Drawings of Series-Parallel Graphs and Other Subclasses of Planar Graphs*

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Abstract. In this paper, we study small planar drawings of planar graphs. For arbitrary planar graphs, $\Theta(n^2)$ is the established upper and lower bound on the worst-case area. It is a long-standing open problem for what graphs smaller area can be achieved, with results known only for trees and outer-planar graphs. We show here that series-parallel can be drawn in $O(n^{3/2})$ area, but 2-outer-planar graphs and planar graphs of proper pathwidth 3 require $\Omega(n^2)$ area.

1 Introduction

A planar graph is a graph that can be drawn without crossing. It was established 20 years ago [15,20] that it has a straight-line drawing in area $O(n^2)$ with vertices placed at grid points. This is asymptotically optimal, since there are planar graphs that need $\Omega(n^2)$ area [14].

A number of other graph drawing models (e.g., poly-line drawings, orthogonal drawings, visibility representations) exist for planar graphs. In all these models, $O(n^2)$ area can be achieved for planar graphs, see for example [17, 23]. On the other hand, $\Omega(n^2)$ area is needed, in all models, for the graph in [14]. This raises the natural question [5] whether $o(n^2)$ area is possible for subclasses of planar graphs.

Known results. Every **tree** has a straight-line drawing in $O(n \log n)$ area and in $O(n)$ area if the maximum degree is asymptotically smaller than n. See [7] for references and many other upper and lower bounds regarding drawings of trees.

It is quite easy (and appears to be folklore) to create straight-line drawings of **outer-planar graphs** that have area $O(nd)$, where d is the diameter of the dual tree of the graph. In an earlier paper [3], we showed that any outer-planar graph has a visibility representation (and hence a poly-line drawing) of area $O(n \log n)$. Since then, some work has been done on improving the bounds for straight-line drawings, with the best bounds now being $O(n^{1.48})$ [8] and $O(\Delta n \log n)$ [12].

Many drawing results are known for **series-parallel graphs**, see e.g. [1,6,16, 22]. However, the emphasis here was on displaying the series-parallel structure of the graph, and/or to use the structure to allow for additional constraints. All

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known algorithms bound the area by $O(n^2)$ area or worse. Quite recently, Frati proved a lower bound (for straight-line or poly-line drawings) of $\Omega(n \log n)$ for a series-parallel graph [13].

No graph drawing results specifically tailored to **k-outer-planar graphs** (for $k \geq 2$), or planar graphs with **small treewidth/pathwidth** appear to be known for 2-dimensional drawings. Planar graphs with small pathwidth play a critical role in drawings where the height is bounded by a constant [9], but not all graphs with small pathwidth have such a drawing.

While higher-dimensional drawings are not the focus of our paper, we would like to mention briefly that all graph classes considered in this paper can be drawn with linear area in 3D, because they are partial k-trees for constant k; see [10], and also [11] for some earlier 3D results for outer-planar graphs.

We would also like to note that all these graphs have small separators, hence all of them allow a non-planar two-dimensional orthogonal drawing in $O(n)$ area if the maximum degree is at most 4 [18].

Our Results. In this paper, we provide the following results:

- Every series-parallel graph has a visibility representation with $O(n^{3/2})$ area.
- A series-parallel graph for which at most f graphs are combined in parallel has a visibility representation with $O(fn \log n)$ area. We know $f \leq \Delta$.
- There are series-parallel graphs that require $\Omega(n^2)$ area in any poly-line drawing that respects the planar embedding.
- There are 2-outer-planar graphs that require $\Omega(n^2)$ area in any poly-line drawing. Moreover, these graphs have pathwidth 3.
- There are graphs of proper pathwidth 3 and maximum degree 4 that require $\Omega(n^2)$ area.

For algorithms, we restrict our attention to visibility representations, because any such drawing can be converted to a poly-line drawing with asymptotically the same area. Hence all our upper bounds also hold for poly-line drawings.

2 Background

Let $G = (V, E)$ be a graph with $n = n(G) = |V|$ vertices and $m = m(G) = |E|$ edges. Throughout this paper, we will assume that G is *simple* (has no loops or multiple edges) and *planar*, i.e., can be drawn without crossing. A planar drawing splits the plane into connected pieces; the unbounded piece is called the *outer-face*, all other pieces are called *interior faces*. An *outer-planar graph* is a planar graph that can be drawn such that all vertices are on the outer-face.

A *2-terminal series-parallel graph with terminals* s, t is a graph defined recursively with one of the following three rules: (a) An edge (s, t) is a 2-terminal series-parallel graph. (b) If G_i, $i = 1, 2$ are 2-terminal series-parallel graphs with terminals s_i and t_i, then in a *series composition* we identify t_1 with s_2 to obtain a 2-terminal series-parallel graph with terminals s_1 and t_2. (c) If G_i, $i = 1, \ldots, k$, are 2-terminal series-parallel graphs with terminals s_i and t_i, then in a *parallel* *composition* we identify $s_1, s_2, \ldots, s_k$ into one terminal s and $t_1, t_2, \ldots, t_k$ into one terminal t to obtain a 2-terminal series-parallel graph with terminals s and t. Here k is as large as possible, i.e., none of the graphs G_i is itself obtained via a parallel composition. The *fan-out* of a series-parallel graph is the maximum number of subgraphs k used in a parallel composition.

Given a 2-terminal series-parallel graph G, a *subgraph from the composition* is any of the subgraphs $G_1, \ldots, G_k$ that was used to create G, or recursively any subgraph from the composition of $G_1, \ldots, G_k$. Since we never consider any other subgraphs, we will say "subgraphs" instead of "subgraphs from the composition".

A *series-parallel graph*, or *SP-graph* for short, is a graph for which every biconnected component is a 2-terminal series-parallel graph. It is *maximal* if no edge can be added while maintaining a simple SP-graph. Any maximal series-parallel graph is a 2-terminal series-parallel graph where in any parallel composition there exists an edge between the terminals, and in any series composition each subgraph is either an edge or obtained from a parallel composition. We will only considering drawings of maximal series-parallel graph, since this makes no difference for asymptotic upper bounds on the area of graph drawings.

A *polyline-drawing* is an assignment of vertices to points and edges to a path of finitely many line segments connecting their endpoints. A *visibiliy representation* is an assignment of vertices to boxes[1] and edges to horizontal or vertical line segment connecting boxes of their endpoints. For a planar graph, such drawings should be planar, i.e., have no crossing. We also assume that all defining features have integral coordinates; in particular points of vertices and transition-points (*bends*) in the routes of edges have integral coordinates, and boxes of vertices have integral corner points. We allow boxes to be degenerate, i.e., to be line segments or points.

The *width* of a box is the number of vertical grid lines (*columns*) that are occupied by it. The *height* of a box is the number of horizontal grid lines (*rows*) that are occupied by it. A drawing whose minimum enclosing box has width w and height h is called a $w \times h$-drawing, and has *area* $w \cdot h$.

3 Visibility Representations of Series-Parallel Graphs

In this section, we study how to create a small visibility representation of a maximal SP-graph G. Our algorithm draws G and recursively all its subgraphs H. To ease putting drawings together, we put constraints on the drawing (see also Fig. 1):

- The visibility representation is what we call *flat*: every vertex is represented by a horizontal line segment.
- Vertex s is placed in the upper right corner of the bounding box.
- Vertex t is placed in the lower right corner of the bounding box.

With our construction we develop a recursive formula for the height: $h(m)$ is the maximum height of a drawing obtained with our algorithm over all maximal

[1] In this paper, the term "box" always refers to an axis-parallel box.

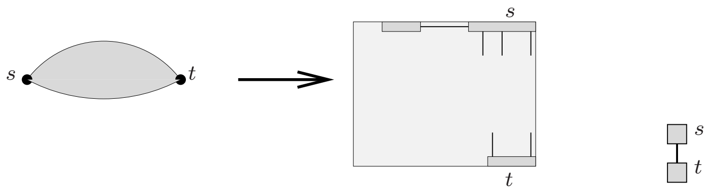

Fig. 1. Illustration of the invariant, and the base case $n = 2$

SP-graphs with m edges.(We have $m = 2n - 3$, but use m to simplify the computations.) In the base case ($m = 1$), simply place s atop t; see Fig. 1. The conditions are clearly satisfied, and we have $h(m) = 2$ for $m = 1$.

Modifying drawings. If $m \geq 2$, then we obtain the drawing by merging drawings of subgraphs together suitably. Before doing this, we sometimes modify them with an operation used earlier [3]. We say that in a drawing a vertex *spans the top (bottom) row* if its vertex box contains both the top (bottom) left point and the top (bottom) right point of the smallest enclosing box of the drawing. We can always achieve that terminal s spans the top row after adding a row; we call this *releasing terminal* s. Similarly we can also release terminal t after adding a row.

Lemma 1. *[3] Let $\Gamma(H)$ be a flat visibility representation of H of height $h \geq 2$ that satisfies the invariant. Then there exists a flat visibility representation $\Gamma'(H)$ of H of height $h+1$ that satisfies the invariant, and vertex s spans the top row.*

Subgraphs from parallel compositions. Assume H is a subgraph of G which is obtained in a parallel composition from subgraphs $H_1, \ldots, H_k$, $k \geq 2$. After possible renaming, assume that $m_i = m(H_i)$ satisfies $m_1 \geq m_2 \geq \ldots \geq m_k$. Recursively obtain drawings of $H_1, \ldots, H_k$; the drawing of H_i has height at most $h(m_i)$. Combine them after releasing both terminals in all of $H_2, \ldots, H_k$ and adding rows so that all drawings have the same height. Place H_1 leftmost, and all other H_i to the right of it; this gives a drawing of H that satisfies the invariant, see Fig. 2. Since $m_2 \geq m_3 \geq \ldots \geq m_k$, the height of this drawing is

$$h(m) \leq \max\{h(m_1), h(m_2)+2, \ldots, h(m_k)+2\} = \max\{h(m_1), h(m_2)+2\} \quad (1)$$

Subgraphs from series compositions. Now let H (with terminals s, t) be obtained from a series composition of graphs H_a and H_b with terminals s, x and x, t, respectively. Since we consider maximal SP-graphs, each of H_a and H_b is either an edge or obtained from a parallel composition. We distinguish cases.

Case (S1): One subgraph, say H_b, is an edge. Then we draw H_a recursively, extend the drawing of terminal s to the right, place t in the bottom row, and connect edge (x, t) horizontally. See Fig. 3. The case that H_a is an edge is symmetric. We have $h(m) = h(m-1)$ in this case.

Case (S2): Both subgraphs have at least two edges. Assume that $m(H_b) \leq m(H_a)$; the other case is symmetric. Graph H_b was obtained from a parallel

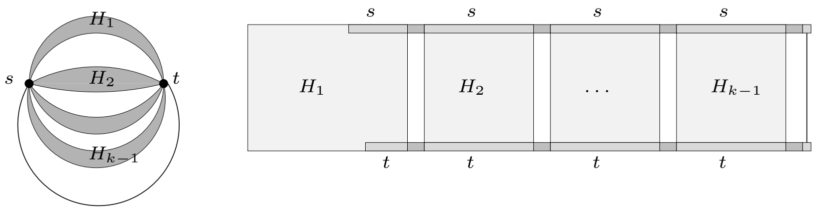

Fig. 2. Combining subgraphs in parallel

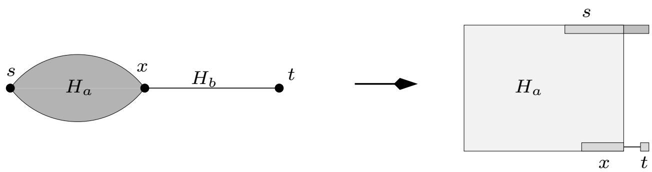

Fig. 3. A series composition when one subgraph is an edge

composition of subgraphs, say $H_1, \ldots, H_k$ such that $m(H_1) \geq \ldots \geq m(H_k)$. Note that H_k is the edge (x, t), which exists since the SP-graph is maximal.

Let L be an integer; we will discuss later how to choose L. For all $i < \min\{L, k\}$, we break subgraph H_i up further. Graph H_i is not an edge (since $i < k$ and H_k is an edge), and so is obtained in a series composition of two subgraphs H_i^a and H_i^b with terminals x, y_i and y_i, t, respectively. See also Fig. 4. Set $m_\alpha^\beta = m(H_\alpha^\beta)$ for any strings α and β.

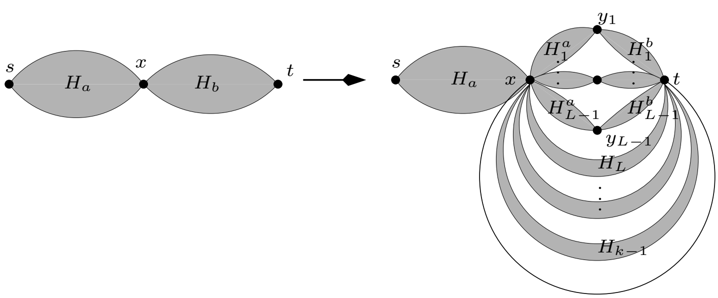

Fig. 4. Breaking down subgraph H_b

Recursively draw each of the subgraphs H_a, H_i^a, H_i^b (for $i = 1, \ldots, \min\{k, L\}-1$,) and H_i (for $i = L, \ldots, k-1$.) Before we can combine these drawings, we need to release some terminals again (recall Lemma 1). We proceed as follows:

- The drawing of H_a is unchanged and has height $h(m_a)$.
- For $i = 1, \ldots, \min\{L, k\} - 1$, release terminal x in the drawing of H_i^a, and terminal t in the drawing of H_i^b. The drawings hence have height at most $h(m_i^a) + 1$ and $h(m_i^b) + 1$.
- For $i = L, \ldots, k-1$, release both terminals in the drawing of H_i.

To explain how we put these drawings together, we distinguish two sub-cases:

Case (S2a): Assume first that $k \leq L$, and consider Fig. 5. We place H_a on the left, followed by $H_1^a, H_2^a, \ldots, H_{k-1}^a$. All these graphs share terminal x, and it spans the bottom row for $H_1^a, H_2^a, \ldots, H_{k-1}^a$, so this draws x as a horizontal segment. Now for $i = 1, \ldots, k-1$, rotate the drawing of H_i^b such that terminal t spans the bottom row and terminal y_i occupies the top left corner. We place these rotated drawings in order $H_{k-1}^b, H_{k-2}^b, \ldots, H_1^b$; then t is in the bottom row and can be connected to x with a horizontal edge.

We increase the heights of these drawings (by inserting rows, if needed) such that the two representations of y_i are in the same row, y_i is above the drawing of y_{i+1} (for $i < k-1$), and s is above all y_i's. Then all terminals can be represented as line segments and the invariant holds.

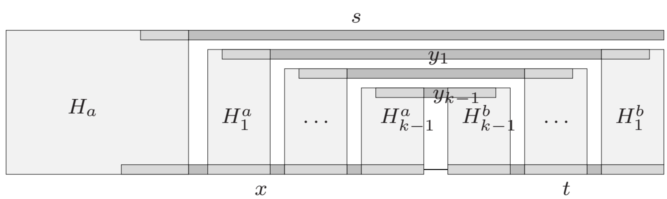

Fig. 5. Combining the subgraphs for a series composition. The case $k \leq L$.

Let h_i be the height of the drawing of H_i^a and H_i^b together in the final drawing. Then $h_{k-1} \leq \max\{h(m_{k-1}^a) + 1, h(m_{k-1}^b) + 1\} \leq h(m_{k-1}) + 1$. For $i < k - 1$, the height has been increased further to keep y_i above y_{i+1}, hence $h_i \leq \max\{h(m_i)+1, h_{i+1}+1\}$. Therefore, y_1 is at height $h_1 \leq \max\{h(m_1)+1, h(m_2)+2, \ldots, h(m_{k-1}) + k - 1\}$, s is at least one higher, and the total height is

$$h(m) \leq \max\{h(m_a), h(m_1) + 2, h(m_2) + 3, \ldots, h(m_{k-1}) + k\} \qquad (2)$$

Case (S2b): Now we study the case $k > L$, where we treat the graphs $H_L, \ldots, H_{k-1}$ differently. Place $H_a, H_1^a, \ldots, H_{L-1}^a, H_{L-1}^b, \ldots, H_1^b$ exactly as before. Add rows until $H_L, \ldots, H_{k-1}$ all have the same height, say h_d, and place them *below* the segment of x. We may have to add some columns to x if it is not wide enough for the subgraphs. To make the two occurrences of t match up, we extend the drawings of $H_{L-1}^b, \ldots, H_1^b$ downwards and draw edge (x, t) vertically. See Fig. 6.

To obtain a formula for the resulting height, we hence need to add $h_d - 1$ to the formula of (2) (after replacing k by L in it.) Since h_d is the maximum height among $H_L, \ldots, H_{k-1}$, and $m_L \geq \ldots \geq m_k$, we have $h_d \leq h(m_L) + 2$ (recall that both terminals were released for $H_L, \ldots, H_{k-1}$), and therefore

$$h(m) \leq \max\{h(m_a), h(m_1) + 2, h(m_2) + 3, \ldots, h(m_{L-1}) + L\} + h(m_L) + 1 \quad (3)$$

Analysis. Now we show that the above algorithm indeed yields a small area.

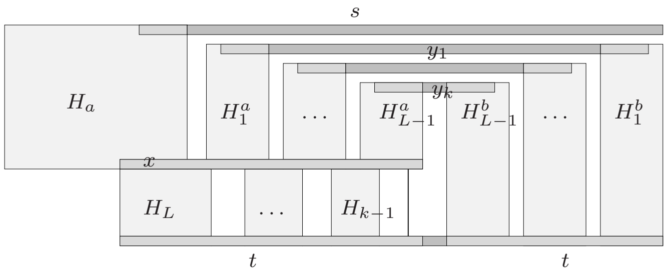

Fig. 6. Combining the subgraphs for a series composition. The case $k \geq L$.

Lemma 2. *For a suitable choice of L, we have $h(m) \leq 12\sqrt{m}$.*

Proof. This clearly holds for $m = 1$. For a parallel composition, we have $m_1 \geq m_i$ and hence $m_i \leq m/2$ for $i \geq 2$, so by (1) and $m \geq 2$

$$\begin{aligned} h(m) &\leq \max\{h(m_1), h(m_2) + 2, \ldots, h(m_k) + 2\} \\ &\leq \max\{h(m), h(m/2) + 2\} \leq \max\{12\sqrt{m}, 12\sqrt{m/2} + 2\} \leq 12\sqrt{m}. \end{aligned}$$

In case (S1), we have $h(m) = h(m_a) \leq 12\sqrt{m_a} \leq 12\sqrt{m}$. In case (S2), we assumed $m_a \geq m_b$. Also, $m_b \geq 3$ (because H_1^a and H_1^b have each an edge, and (x, t) exists), and hence $m \geq 6$. We choose $L = 3\sqrt{m_a} + 1$.[2] Now for case (S2a), we have by (2)

$$\begin{aligned} h(m) &\leq \max\{h(m_a), h(m_1) + 2, h(m_2) + 3, \ldots, h(m_{k-1}) + k\} \\ &\leq \max\{h(m_a), h(m/2) + L\} \text{ since } m_i \leq m_b \leq m/2 \text{ and } k \leq L \\ &\leq \max\{12\sqrt{m_a}, 12\sqrt{m/2} + 3\sqrt{m_a} + \frac{1}{\sqrt{6}}\sqrt{m}\} \text{ since } L = 3\sqrt{m_a} + 1 \text{ and } m \geq 6 \\ &\leq \max\{12, (\frac{12}{\sqrt{2}} + 3 + \frac{1}{\sqrt{6}})\}\sqrt{m} \leq 12\sqrt{m} \end{aligned}$$

Finally we consider case (S2b). We have $m_1 \leq m_b \leq m_a$ and $m_i \leq m_1$, hence $m_i \leq m_b/2 \leq m_a/2$ for all $i \geq 2$. Recall that the height in case (S2b) is by (3)

$$\begin{aligned} h(m) &\leq \max\{h(m_a), h(m_1) + 2, h(m_2) + 3, \ldots, h(m_{L-1}) + L\} + h(m_L) + 1 \\ &\leq \max\{h(m_a), h(m_a/2) + L - 2\} + h(m_L) + 3 \leq 12\sqrt{m_a} + 12\sqrt{m_L} + 3, \end{aligned}$$

where the last inequality holds by induction and because $h(m_a/2) + L - 2 \leq 12\sqrt{\frac{m_a}{2}} + 3\sqrt{m_a} - 1 \leq 12\sqrt{m_a}$. But

$$\begin{aligned} &(\sqrt{m_a} + \sqrt{m_L} + \frac{1}{4})^2 = m_a + m_L + \frac{1}{16} + 2\sqrt{m_a}\sqrt{m_L} + \frac{1}{2}\sqrt{m_a} + \frac{1}{2}\sqrt{m_L} \\ &\leq m_a + m_L + \frac{1}{16}\sqrt{m_a}m_L + 2\sqrt{m_a}m_L + \frac{1}{2}\sqrt{m_a}m_L + \frac{1}{3}\sqrt{m_a}m_L \end{aligned}$$

[2] Many thanks to Jason Schattman for helping with MAPLE to find small constants.

$$\text{by } \sqrt{m_a} \geq \sqrt{3} \geq \tfrac{3}{2}$$
$$\leq m_a + m_L + 3\sqrt{m_a} m_L = m_a + m_L + (L-1)m_L \text{ by } L = 3\sqrt{m_a} + 1$$
$$\leq m_a + m_L + m_1 + m_2 + \ldots + m_{L-1} \text{ by } m_i \geq m_L \text{ for } i < L$$

which is at most m. Putting it together, we get $h(m) \leq 12(\sqrt{m_a} + \sqrt{m_L} + \frac{1}{4}) \leq 12\sqrt{m}$ as desired. □

Theorem 1. *Any series-parallel graph has a visibility representation with area $O(n^{3/2})$.*

Proof. By the previous lemma, the height is $O(\sqrt{m}) = O(\sqrt{n})$ by $m = 2n-3$. To analyze the width, notice that at the most we use one column for each edge. (Each vertex obtains at least one incident vertical edge in the base case, and hence does not contribute additional width.) Hence the width is at most $m \leq 2n-3$, and the total area is $O(n^{3/2})$. □

We get better bounds if case (S2b) does not happen, i.e., if the series-parallel graph has small fan-out.

Theorem 2. *Any series-parallel graph with fan-out f has a visibility representation of area $O(fn \log n)$.*

Proof. Assume first the graph is maximal. As in Theorem 1 the width is $O(n)$, so it suffices to show that $h(m) \leq 2 + f \log m$ for a maximal SP-graph with fanout f. We proceed by induction on the number of edges. In the base case $h(1) = 2 \leq 2 + f \log m$. In case of a parallel composition, by (1) we have $m_2 \leq m/2$ and height

$$\begin{aligned} h(m) &\leq \max\{h(m_1), h(m_2) + 2\} \leq \max\{h(m_1), h(m/2) + 2\} \\ &\leq \max\{2 + f \log m_1, 2 + f \log(m/2) + 2\} \leq 2 + f \log m \end{aligned}$$

since $f \geq 2$. For case (S1), the height is $h(m) = h(m_a) \leq 2 + f \log m_a \leq 2 + f \log m$. In case (S2), we choose $L = f$, and hence always have $k \leq L$ and are in case (S2a). Here, the height is by (2)

$$\begin{aligned} h(m) &\leq \max\{h(m_a), h(m_1) + 2, h(m_2) + 3, \ldots, h(m_{k-1}) + k\} \\ &\leq \max\{h(m_a), h(m/2) + f\} \text{ since } m_i \leq m/2 \text{ and } k \leq f \\ &\leq \max\{2 + f \log m_a, 2 + f \log(m/2) + f\} \leq 2 + f \log m. \end{aligned}$$

If the graph is not maximal, then it can be made a maximal SP-graph by adding edges; this adds at most one to the fan-out f and hence the drawing of the super-graph has area $O(fn \log n)$. □

Note in particular that a series-parallel graph with maximum degree Δ has fan-out at most Δ, so any series-parallel graph has a flat visibility representation of area $O(\Delta n \log n)$. Also, any outer-planar graph is an SP-graph with fan-out $f \leq 2$, so this theorem implies our earlier result [3], and in fact yields exactly the same visibility representation.

We note here that most algorithms for visibility representations of planar graphs (e.g. [23, 21]) are *uni-directional*, i.e., all edges are drawn as vertical line segments. Our visibility representations use two directions, but since all boxes of vertices have unit height, they can be made uni-directional at the cost of at most doubling the height. Details are omitted.

4 Lower Bounds

Series-parallel graphs. Most of the previously given lower bounds for planar drawings (see e.g. [14, 2, 19]) rely on an argument that we call the "stacked cycle argument", which we briefly review here because we will modify it later. Assume we have a planar graph G with a fixed planar embedding and outer-face. A set of disjoint cycles $C_1, \dots, C_k$ is called *stacked cycles* if C_i is outside the region defined by C_{i-1} for all $i > 1$. The following is well-known:

Fact 1. *If G has k stacked cycles, then G needs at least a $2k \times 2k$-grid in any planar polyline drawing that reflects the planar embedding and outer-face.*

Therefore, to get a bound of $\Omega(n^2)$ on the area, construct graphs that consist of $n/3$ stacked triangles [14], or $\Omega(n)$ stacked cycles for some graph classes that do not allow stacked triangles [19]. The left graph in Fig. 7 is a series-parallel graph that has $n/3$ stacked cycles.

Theorem 3. *There exists a series-parallel graph that requires a $\frac{2}{3}n \times \frac{2}{3}n$-grid in any polyline drawing that respects the planar embedding and outer-face.*

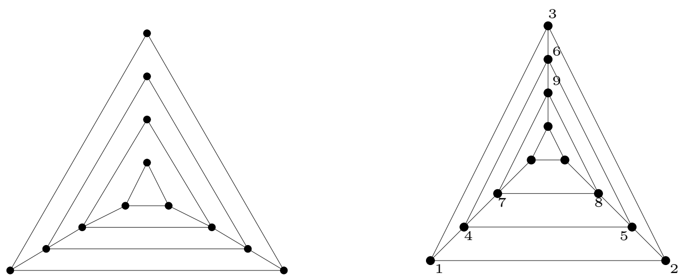

Fig. 7. Two graphs with $n/3$ stacked cycles

Note that our graph (contrary to the other lower bound graphs cited above) has many different planar embeddings, and using a different embedding one can easily construct drawings of it in area $O(n)$. Our algorithm (which changes the planar embedding) achieves area $O(n \log n)$ since the graph has fan-out 2. Frati [13] showed that another series-parallel graph (consisting of $K_{2,n}$ and a complete ternary tree) needs $\Omega(n \log n)$ in any poly-line drawing. Closing the gap between his lower bound and our upper bound of $O(n^{3/2})$ remains open.

k-outerplanar graphs. A k-outer-planar graph is defined as follows. Let G be a graph with a fixed planar embedding. G is called 1-*outer-plane* if all vertices of G are on the outer-face (i.e., if G is outer-planar in this embedding.) G is called *k-outer-plane* if the graph that results from removing all vertices from the outer-face of G is $(k-1)$-outer-plane in the induced embedding. A graph G is called *k-outer-planar* if it is k-outer-plane in some planar embedding.

Clearly, k-outer-planar graphs generalize the concept of outer-planar graphs, and hence for small (constant) k are good candidates for $o(n^2)$ area. Also, by definition we cannot use a stacked cycle argument on them (a k-outer-planar graph has at most k stacked cycles.) Nevertheless, we can show an $\Omega(n^2)$ lower bound on the area even for 2-outer-planar graphs.

To show this, we modify the stacked-cycle argument. Let G be a graph with a fixed planar embedding, and let $C_1, \ldots, C_k$ be k cycles that are edge-disjoint and any two cycles have at most one vertex in common. We say that $C_1, \ldots, C_k$ are *1-fused stacked cycles* if C_i is outside the region defined by C_{i-1} except at the one vertex that they may have in common. See Fig. 8.

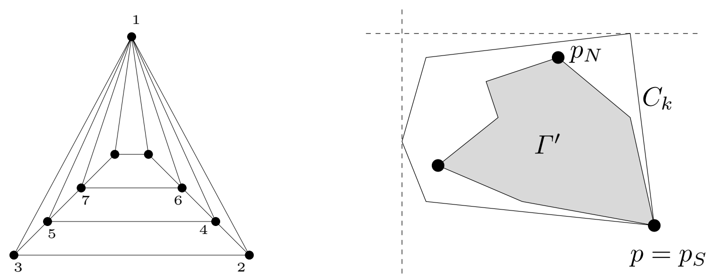

Fig. 8. A 2-outerplanar graph with $(n-1)/2$ 1-fused stacked cycles, and adding a 1-fused cycle around a drawing

Lemma 3. *Let G be a planar graph with a fixed planar embedding and outer-face, and assume G has k 1-fused stacked cycles $C_1, \ldots, C_k$. Then any poly-line drawing of G that respects the planar embedding and outer-face has width and height at least $k+1$.*

Proof. We proceed by induction on k. Clearly we need width and height 2 to draw the cycle C_1. For $k > 1$, let G' be the subgraph formed by the 1-fused stacked cycles $C_1, \ldots, C_{k-1}$.

Consider an arbitrary poly-line drawing Γ of G, and let Γ' be the induced drawing of G', which has width and height at least k by induction. Consider Fig. 8. The drawing of C_k in Γ must stay outside Γ', except at the point p where C_k and C_{k-1} have a vertex in common (if any.) Let p_N and p_S be points at a vertex or bend in the topmost and bottommost row of Γ'; by $k \geq 2$ they are distinct. So $p \neq p_N$ or $p \neq p_S$; assume the former. To go around p_N, the drawing of C_k in Γ must reach a point strictly higher than p_N, and hence uses at least one more row above Γ'. Similarly one shows that Γ has at least one more column than Γ'. □

Now we give a lower bound for 2-outerplanar graphs. The same graph also has small pathwidth (defined precisely below.)

Theorem 4. *There exists a 3-connected 2-outer-planar graph of pathwidth* 3 *that requires an* $\frac{n+1}{2} \times \frac{n+1}{2}$*-grid in any poly-line drawing that reflects the planar embedding and outer-face.*

Proof. (Sketch) Fig. 8 shows a graph that has $(n-1)/2$ 1-fused stacked cycles and hence needs an $(n+1)/2 \times (n+1)/2$-grid. Clearly it is 2-outerplanar and has pathwidth 3. □

Since this graph is 3-connected, no other planar embedding is possible. It is possible to choose a different outer-face, but at least $(n-1)/4$ 1-fused stacked cycles will remain regardless of this choice, and hence an $\Omega(n^2)$ lower bound applies to any planar drawing of this graph.

Graphs of small (proper) pathwidth. The same graph can also serve as a lower-bound example for another restriction of planar graphs, namely, graphs of bounded treewidth, pathwidth, and proper pathwidth. See for example Bodlaender's overview [4] for exact definition of treewidth and applications of these graph classes. Graphs of treewidth 2 are exactly SP-graphs. Graphs of *pathwidth* k are those that have a vertex order $v_1, \dots, v_n$ such that for any i, at most k vertices in $v_1, \dots, v_i$ have a neighbour in $v_{i+1}, \dots, v_n$. Graphs of *proper pathwidth* k are those that have a vertex order $v_1, \dots, v_n$ such that for any edge (v_i, v_j), we have $|j-i| \leq k$. Graphs of proper pathwidth k are a subset of graphs of pathwidth k, which in turn are a subset of graphs of treewidth k.

The labelling of vertices of the graph in Fig. 8 show that it has pathwidth at most 3. Many other previously given lower-bound graphs that consist of stacked cycles (see e.g. [2]) have constant pathwidth, even constant proper pathwidth, usually equal to the length of the stacked cycles. We give one more example that also has small maximum degree.

Theorem 5. *There exists a 3-connected graph of proper pathwidth* 3 *with maximum degree 4 that requires* $\Omega(n^2)$ *area in any poly-line drawing.*

Proof. The right graph in Fig. 7 shows an example with proper pathwidth at most 3 and maximum degree 4, and $n/3$ stacked cycles, hence needs a $\frac{2}{3}n \times \frac{2}{3}n$-grid in any polyline drawing. □

Since planar partial 3-trees are also partial k-trees for any $k \geq 3$, our lower bounds holds for all partial k-trees with $k \geq 3$, hence destroying the hope that the linear-area layouts in 3D [10] could be replicated in 2D.

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