Constrained Control of Weakly Coupled Nonlinear Systems Using Neural Network

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Abstract. In this paper, a new algorithm is proposed for the constrained control of weakly coupled nonlinear systems. The controller design problem is solved by solving Hamilton-Jacobi-Bellman(HJB) equation with modified cost to tackle constraints on the control input and unknown coupling. In the proposed controller design framework, coupling terms have been formulated as model uncertainties. The bounded controller requires the knowledge of the upper bound of the uncertainty. In the proposed algorithm, Neural Network (NN) is used to approximate the solution of HJB equation using least squares method. Necessary theoretical and simulation results are presented to validate proposed algorithm.

Keywords: Weak coupling, HJB equation, Bounded control, Nonlinear system, Lyapunov stability.

1 Introduction

Many physical systems are naturally weakly coupled such as power systems, flexible space structures etc. The weakly coupled linear systems were introduced to the control audience by Kokotovic [1]. Optimal control of weakly coupled system studied with parallel processing in [2-5]. The results of [5] are based on the idea of the recursive reduced-order scheme for solving the algebraic riccati equation of weakly coupled systems [2] bilinear system [6]. Recently Lim and Kim [12] proposed a similar approach for nonlinear systems using Successive Galerkin Approximation (SGA). It is to be noted that in all the above mentioned approaches, it was assumed that (coupling coefficient) ^2=0, to do parallel processing. It is well known that SGA is computationally complex [7]. It is even more difficult to handle constraints on the control input. Due to limitations of the actuators one should consider constraints on the control input. Constrained optimal controller design proposed for nonlinear system using nonquadratic performance function in [8-9]. Khalaf et. al [8] used NN based HJB solution to solve it. Compared to SGA it is less complex and can handle constraints. Also one can avoid assumptions made by earlier work to form parallel processing. However, the weak coupling theory has been studied so far only without constraints on the input. The main contribution in this paper is the constrained controller design

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of weakly coupled nonlinear system. It can be achieved in the following steps: (1) Terms related to the weak coupling has been treated as uncertainties. So, original constrained controller design for weakly coupled nonlinear system now becomes constrained robust controller design of a nonlinear uncertain system. (2) A constrained optimal control problem is formulated for the nominal dynamics of the nonlinear system. It is solved using HJB equation with modified performance functional to tackle constraints and unknown coupling. The paper is organized as follows: In section 2, controller design problem of weakly coupled nonlinear system is formulated as a robust control problem. Robust – Optimal control framework has been described in section 3. Stability issues are discussed. In section 4, NN based HJB solution is used to find constrained robust-optimal control law. Solution of NN based HJB equation found by least squares method. Numerical example is given in section 5 for the validity of the approach. Proposed work is concluded in section 6.

2 Controller Framework for Weakly Coupled Nonlinear System

Consider a weakly coupled nonlinear system

$\dot{x}_1 = f_{11}(x_1) + g_{11}(x_1)u_1 + \varepsilon\left(f_{12}(x)\right) + \varepsilon(g_{12}(x))u_2$ (1) , $\dot{x}_2 = f_{22}(x_2) + g_{22}(x_2)u_2 + \varepsilon\left(f_{21}(x)\right) + \varepsilon(g_{21}(x))u_1$ (2)

where $x_1 \in \mathbb{R}^{n1}, x_2 \in \mathbb{R}^{n2}, u_1 \in \mathbb{R}^{m1}, u_2 \in \mathbb{R}^{m2}$ and ε is a small positive coupling coefficient. Also $x = [x_1^T \; x_2^T]^T$ is state vector and $u = [u_1^T \; u_2^T]^T$ is the control input vector. Each component of u_i is bounded by a positive constant λ. i.e., $|u_i| \le \lambda \in \mathbb{R}$ (3)

We assume that f_{1i}, f_{2i} and g_{ij} are Lipschitz continuous on the set Ω. We also assume that $f_{1i}(0) = 0$ and $f_{2i}(0) = 0$. Our aim is to design a control law u such that it will stabilize coupled systems defined by (1) and (2). It can be achieved by considering controller design problem of weakly coupled nonlinear system as a controller design problem of nonlinear uncertain system having uncertainties in the form of unknown coupling terms. Rewriting equations (1) and (2) as, $\dot{x} = f(x) + g(x)u + pd(x) + ph(x)u$ (4)

where $f(x) = \left[f_{11}(x_1) \;\; f_{22}(x_2)\right]^T$, $g(x) = [g_{11}(x_1) \;\; g_{22}(x_2)]^T$, $p = \varepsilon$ and $h(x) = [g_{12}(x) \;\; g_{21}(x)]^T$.

Origin is assumed as an equilibrium point of the system (4). It is also assumed that $pd(x)$ and $ph(x)$ are bounded by a known functions, $D_{\max}(x)$ and $H_{\max}(x)$ respectively. i.e. $\|pd(x)\| \le D_{\max}(x); \|ph(x)\| \le H_{\max}(x)$ (5)

In this paper we seek a constrained optimal control that will compensate for the uncertainty related to p.

(A) Robust control problem:

For the open loop system (4), find a feedback control law $u = K(x)$ such that the closed-loop system is asymptotically stable for all admissible uncertainties p.

This problem can be formulated into an optimal control of the nominal system with appropriate cost functional.

(B) Optimal control problem:

For the nominal system

$$\dot{x} = f(x) + g(x)u \tag{6}$$

find a feedback control $u = K(x)$ that minimizes the cost functional

$$\int_0^\infty \left(\mu_1 D_{\max}^2(x) + \mu_2 H_{\max}^2(x) + x^T Q x + M(u)\right) dt \quad (7)$$

where $M(u)=2\int_0^u \lambda \tanh^{-1}(v/\lambda)R dv = 2\lambda u R \tanh^{-1}(u/\lambda)+\lambda^2 R \ln(1-u^2/\lambda^2)>0$ (8)

is non-quadratic term expressing cost related to constrained input. The matrices Q and R are positive definite matrices showing the weightage of system states and control inputs, respectively. $\mu_i > 0$, act as a design parameters.

In this paper, we addressed the following problems:

1. Solutions of the problem (A) and (B) are equivalent.
2. Solve the optimal control problem using NN.

To solve the optimal control problem, let $V(x_0)=\min_u \int_0^\infty \left(\mu_1 D_{\max}^2(x)+\mu_2 H_{\max}^2(x)+x^T Q x+M(u)\right)dt$ be the minimum cost of bringing (6) from x_0 to equilibrium point 0. The HJB equation gives $\min_u (\mu_1 D_{\max}^2(x)+\mu_2 H_{\max}^2(x)+x^T Q x+M(u)+V_x^T(f(x)+g(x)u))=0$ (9)

where $V_x = \partial V(x)/\partial x$. It is assumed that $V(x)$ is only a function of x. If $u=K(x)$ is the solution to optimal control problem then according to Bellman's optimality principle [9], it can be found by solving following HJB equation:

$$\mathrm{HJB}(V(x))=\mu_1 D_{\max}^2(x)+\mu_2 H_{\max}^2(x)+x^T Q x+M(u)+V_x^T(f(x)+g(x)u)=0 \quad (10)$$

The optimal control law is computed by solving $\partial \mathrm{HJB}(V(x))/\partial u = 0$ (11)

It gives, $u = K(x) = -\lambda \tanh\left(0.5(\lambda R)^{-1} V_x^T g(x)\right)$ (12)

With this basic introduction, following result is stated to show the equivalence of the solution of robust and the solution of optimal control problem.

Theorem 1: Consider the nominal system (6) with the performance function (7). Assume that there exit a solution of HJB equation (10). Using this solution, (12) ensures asymptotic closed loop stability of (4) if following condition is satisfied:

$$\mu_1 D_{\max}^2(x)+\mu_2 H_{\max}^2(x) \ge \left\|V_x^T(x) D_{\max}(x)\right\|^2 + \left\|V_x^T(x) H_{\max}(x)\right\|^2 + \lambda^2 \quad (13)$$

Proof: Here $u=K(x)$ is an optimal control law defined by (12) and $V(x)$ is the optimum solution of HJB equation (10). We now show that the equilibrium point system (4) is asymptotically stable for all possible uncertainties $p(x)$. To do this it is shown that $V(x)$ is a Lyapunov function. Clearly, $V(x)$ is a positive definite function i.e., $V(x)>0,\ x\neq 0$ and $V(0)=0$

Using equations (5), (10) and (13), $\dot V(x) = (\partial V/\partial x)^T (dx/dt) \le -x^T Q x \le 0$.

Thus conditions for Lyapunov local stability theory are satisfied. □

Hence by knowing the exact solution of HJB equation, one can find robust control law in the presence of uncertainties which eventually an optimal control of a weakly coupled nonlinear systems (1) and (2). Theorem 1 is valid if we know the exact solution of HJB equation, which is difficult problem. In the next section NN is used to approximate value-function V.

3 NN Based Robust-Optimal Control

It is well known that an NN can be used to approximate smooth time-invariant functions on prescribed compact sets [13]. Let $\mathbb{R}$ denote the real numbers. Given $x_k \in \mathbb{R}$, define $x=[x_0,x_1,...x_n]^T$, $y=[y_0,y_1,...y_m]^T$ and weight matrices $W_L=[w_1\ w_2\cdots w_L]^T$. Then the ideal NN output can be expressed as $y=W_L^T\sigma_L(x)$ with $\sigma_L(x)=[\sigma_1(x),\sigma_2(x),...,\sigma_L(x)]^T$ as the vector of basis function. Let NN structure be defined as $\hat{V}(x,t)=\sum_{j=1}^{L}w_j\sigma_j(x)=W_L^T\sigma_L(x)$ (14)

It gives $$\hat{V}_x(x)=\partial\hat{V}/\partial x=W_L^T\,\partial\sigma_L(x)/\partial x=W_L^T\nabla\sigma_L(x) \qquad (15)$$

$\sigma(x)$ is selected such that $\hat{V}(0)=0$ and $\hat{V}(x)>0$ for $\forall\ x\neq 0$. With this background, we propose NN based robust-optimal control framework in the next section.

3.1 NN Based Controller Formulation

Using (14) and (15) NN based HJB equation can be written as

$$\mathrm{HJB}(\hat{V}(x))=\mu_1 D_{\max}^2(x)+\mu_2 H_{\max}^2(x)+x^TQx+M(\hat{u})+\hat{V}_x^T(f(x)+g(x)\hat{u})=e \qquad (16)$$

Khalaf et. al. [8] proved existence of NN based HJB solution. The existence of it for modified performance functional can be proved in the same line of it. So, (16) can be written as $$\mathrm{HJB}(\hat{V}(x))=\mu_1 D_{\max}^2(x)+\mu_2 H_{\max}^2(x)+x^TQx+M(\hat{u})+\hat{V}_x^T(f(x)+g(x)\hat{u})\approx 0 \qquad (17)$$

The optimal control law can be found by taking derivative of (17) w.r.to $\hat{u}$. It can be found as $$\hat{u}(x)=-\lambda\tanh\left(0.5(\lambda R)^{-1}g(x)^T\hat{V}_X\right)=-\lambda\tanh\left(0.5(\lambda R)^{-1}g(x)^T W\nabla\sigma^T(x)\right) \qquad (18)$$

It is an optimal control law defined by (18) and $\hat{V}(x)$ is the solution of the HJB equation (17). We can show that with this control, the system remains asymptotically stable for all possible p. Using (14) and $\sigma(x)$, $\hat{V}(0)=0$ and $\hat{V}(x)>0$ for $\forall\ x\neq 0$. Also $\dot{\hat{V}}(x)=d\hat{V}/dt<0$ for $x\neq 0$, can be proved similarly as theorem 1 by replacing $V(x)$ by $\hat{V}(x)$ if condition (19) is satisfied. $$\mu_1 D_{\max}^2(x)+\mu_2 H_{\max}^2(x)\geq\left\|\hat{V}_x^T(x)D_{\max}(x)\right\|^2+\left\|\hat{V}_x^T(x)H_{\max}(x)\right\|^2+\lambda^2 \qquad (19)$$

From the above results, it can be proved that NN based robust control stabilizes system having uncertainty. Hence controller designed by (18) stabilizes the weakly coupled systems (1) and (2). Next section is about the utilization of the least squares method for finding a HJB solution.

4 HJB Solution by Least-Square Method

The unknown weights are determined by projecting the residual error e onto de/dW and setting it to zero using the inner product, i.e. $\langle de/dW, e\rangle=0$ for $\forall x\in\Omega\subseteq\mathbb{R}^n$ (20)

This method can be applied to solve robust-optimal control problem for the system having matched uncertainties. According to this method, by using definitions (14),(15) and (16), we can write (20) as

$$\langle\nabla\sigma(x)(f(x)+g(x)\hat{u}),\nabla\sigma(x)(f(x)+g(x)\hat{u})\rangle W+\langle\mu_1 D_{\max}^2(x)+\mu_2 H_{\max}^2(x)+x^TQx+M(\hat{u}),\nabla\sigma(x)(f(x)+g(x)\hat{u})\rangle=0 \qquad (21)$$

Hence weight updating law is

$$W = -\left\langle \nabla\sigma(x)(f(x)+g(x)\hat{u}), \nabla\sigma(x)(f(x)+g(x)\hat{u}) \right\rangle^{-1} \left\langle \mu_1 D_{\max}^2(x) + \mu_2 H_{\max}^2(x) + x^T Q x + M(\hat{u}), \nabla\sigma(x)(f(x)+g(x)\hat{u}) \right\rangle \quad (22)$$

By solving this equation, one can find control law using (18) which is the solution of controller design problem of weakly coupled systems. In the next section, simulation carried out on three uncertain systems to validate proposed algorithm.

5 Simulation Experiments

Consider the a weakly coupled nonlinear systems (1)and (2) with terms defined as, $x_1 = [x_{11} \quad x_{12}]^T$, $x_2 = [x_{21} \quad x_{22}]^T$, $u = [u_1 \quad u_2]^T$, $f_{11}(x_1) = [-1.93x_{11}^2 \quad 1.394x_{11}x_{12}]^T$, $f_{12}(x) = [0 \quad -4.2x_{21}x_{22}]^T$, $f_{21}(x) = [-1.3x_{12}^2 \quad 0.95x_{11}x_{21} - 1.03x_{12}x_{22}]^T$, $f_{22}(x_2) = [-0.63x_{21}^2 \quad 0.413x_{21} - 0.426x_{22}]^T$, $g_{11}(x_1) = [-1.274x_{11}^2 \quad 0]^T$, $g_{12}(x) = [0 \quad -6.5x_{22}]^T$, $g_{21}(x) = [0.75x_{11} \quad 0]^T$, $g_{22}(x_2) = [-0.718x_{21} \quad 0]^T$. Control input is bounded by $|u_i| \le 5,$ For simplicity let assume that $\varepsilon = 0.1$.Clearly, $|\varepsilon(f_{12}(x))| \le [0 \quad x_{21}^2 + x_{22}^2]^T$, $|\varepsilon(g_{12}(x))| \le [0 \quad x_{22}^2]^T$ $|\varepsilon(f_{21}(x))| \le [x_{12}^2 \quad x_{11}^2 + x_{12}^2 + x_{21}^2 + x_{22}^2]^T$, and $|\varepsilon(g_{21}(x))| \le [x_{11}^2 \quad 0]^T$.

It gives $D_{\max}(x) = [0 \quad x_{21}^2 + x_{22}^2 \quad x_{12}^2 \quad x_{11}^2 + x_{12}^2 + x_{21}^2 + x_{22}^2]^T$ and $H_{\max}(x) = [0 \quad x_{22}^2 \quad x_{11}^2 \quad 0]^T$.

Our aim is to find the optimal control law that will stabilize the weakly coupled nonlinear system for all possible ε. For the nominal system, $\dot{x}_1 = f_{11}(x_1) + g_{11}(x_1)u_1$ and $\dot{x}_2 = f_{22}(x_2) + g_{22}(x_2)u_2$ we have to find a feedback control law $u = K(x)$ that minimizes $\int_0^\infty (D_{\max}^T(x)\mu_1 D_{\max}(x) + H_{\max}^T(x)\mu_2 H_{\max}(x) + x^T Q x + M(u))dt$ where, $\Phi = -\tanh\left(0.5W^T \nabla\sigma(x)B(x)\right)$ Q and R have been taken as identity matrices of appropriate dimensions. $\mu_1 = 500I_{4\times4}$ and $\mu_2 = 100I_{4\times4}$ have been selected for the simulation purpose. This problem can be solved by using (18) and (22). Here we have selected $\hat{V}(x) = w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + w_4x_4^2 + w_5x_1x_2 + w_6x_1x_3 + w_7x_1x_4 + w_8x_2x_3 + w_9x_2x_4 + w_{10}x_3x_4$.

This is a power series NN of 10 activation functions containing power of the state variable of the system upto 2nd order.

It gives W =[0.54 -38.69 0.18 2857 -6.52 6.95 1.49 46.06 -31.12 60.73].

The optimal control law can be found using (18). It can be observed from figures 1(a) and 1(b) that all the system states converge to the equilibrium point. Control signal remains bounded i.e. $|u_i| \le 5,$ as shown in figure-1(c). Condition (19) is verified and shown in figure 1(d).

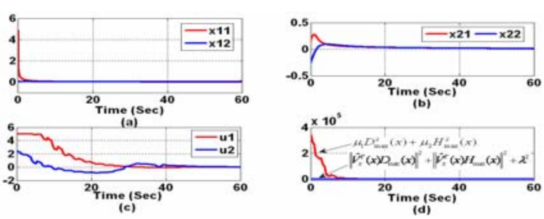

Fig. 1. (a) System (1) states Vs. time, (b) System (2) states Vs. time, (c) Variation of Control input, (d) Verification of condition (19)

6 Conclusions

The contribution of this paper is a methodology for designing constrained controllers for a weakly coupled nonlinear system. It is achieved by formulating the controller design problem of weak coupled nonlinear system into a controller design problem of a nonlinear system having uncertainty in the form of unknown coupling terms. The exact information about uncertainty is not required except some restrictive norm bound. We have adopted NN based HJB solution to design optimal control law that satisfies a prescribed bound. Modifications are done on the earlier approaches to handle constraints on the input and uncertainty related to coupling terms. Simulation results show the good agreement with that of theoretical observations.

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