Universal Spaces for $(k, \overline{k})$−Surfaces*

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Introduction

In the graph–theoretical approach to Digital Topology, the search for a definition of digital surfaces as subsets of voxels is still a work in progress since it was started in the early 1980's. Despite the interest of the applications in which it is involved (ranging from visualization to image segmentation and graphics), there is not yet a well established general notion of digital surface that naturally extends to higher dimensions (see [5] for a proposal). The fact is that, after the first definition of surface, proposed by Morgenthaler [13] for $\mathbb{Z}^3$ with the usual adjacency pairs $(26, 6)$ and $(6, 26)$, each new contribution [10,4,9], either increasing the number of surfaces or extendeding the definition to other adjacencies, has still left out some objects considered as surfaces for practical purposes [12].

In this paper we find, for each adjacency pair $(k, \overline{k})$, $k, \overline{k} \in \{6, 18, 26\}$ and $(k, \overline{k}) \neq (6, 6)$, a homogeneous $(k, \overline{k})$-connected digital space whose set of digital surfaces is larger than any of those quoted above; moreover, it is the largest set of surfaces within that class of digital spaces as defined in [3]. This is an extension of a previous result for the $(26, 6)$-adjacency in [7].

1 A Framework for Digital Topology

This section summarizes the framework for Digital Topology we introduced in [3]. In this approach a *digital space* is a pair (K, f), where K is a polyhedral complex, representing the spatial layout of voxels, and f is a *lighting function* from which we associate to each digital image an Euclidean polyhedron, called its continuous analogue, that intends to be a continuous interpretation of the image.

In this paper we will only consider spaces of the form (R^3, f), where the complex R^3 is determined by the unit cubes in the Euclidean space $\mathbb{R}^3$ centered at points of integer coordinates. Each 3-cell in R^3 represents a voxel, so that a digital object displayed in an image is a subset of the set $\mathrm{cell}_3(R^3)$ of 3-cells in R^3; while the lower dimensional cells in R^3 (actually, d-cubes, $0 \leq d < 3$) are used to describe how the voxels could be linked to each other. Notice that each d-cell $\sigma \in R^3$ can be associated to its center $c(\sigma)$. In particular, if $\dim \sigma = 3$

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then $c(\sigma) \in \mathbb{Z}^3$, so that every digital object O in R^3 can be naturally identified with a subset of the discrete space $\mathbb{Z}^3$. Henceforth we shall use this identification without further comment.

Lighting functions are maps of the form $f : \mathcal{P}(\mathrm{cell}_3(R^3)) \times R^3 \to \{0,1\}$, where $\mathcal{P}(\mathrm{cell}_3(R^3))$ stands for the family of all subsets of $\mathrm{cell}_3(R^3)$; i.e, all digital objects. Each of these maps may be regarded as a "face membership rule", in the sense of Kovalevsky [11], that assigns to each digital object O the set of cells $f_O = \{\alpha \in R^3 \,;\, f(O,\alpha) = 1\}$. This set yields a continuous analogue as the counterpart of O in ordinary topology. Namely, the *continuous analogue* of O is the polyhedron $|\mathcal{A}^f_O| \subseteq \mathbb{R}^3$ triangulated by the subcomplex of the first derived subdivision of R^3, $\mathcal{A}^f_O$, consisting of all simplexes whose vertices are centers $c(\sigma)$ of cells $\sigma \in f_O$.[1] However, to avoid continuous analogues which are contrary to our usual topological intuition, lighting functions must satisfy the five properties below. We need some more notation to introduce them.

As usual, given two cells $\gamma, \sigma \in R^3$ we write $\gamma \leq \sigma$ if γ is a face of σ, and $\gamma < \sigma$ if in addition $\gamma \neq \sigma$. The interior of a cell σ is the set $\mathring{\sigma} = \sigma - \partial\sigma$, where $\partial\sigma = \cup\{\gamma \,;\, \gamma < \sigma\}$ stands for the boundary of σ. We refer to [14] for further notions on polyhedral topology.

Next, we introduce two types of neighbourhoods of a cell $\alpha \in R^3$ in a given digital object $O \subseteq \mathrm{cell}_3(R^3)$: the *star of α in O* which is the set $\mathrm{st}_3(\alpha; O) = \{\sigma \in O \,;\, \alpha \leq \sigma\}$ of voxels in O having α as a face, and the set $\mathrm{st}^*_3(\alpha; O) = \{\sigma \in O \,;\, \alpha \cap \sigma \neq \emptyset\}$ called the *extended star of α in O*. Finally, the *support* of O is the set $\mathrm{supp}(O)$ of cells of R^3 (not necessarily voxels) that are the intersection of 3-cells in O; that is, $\alpha \in \mathrm{supp}(O)$ if and only if $\alpha = \cap\{\sigma \,;\, \sigma \in \mathrm{st}_3(\alpha; O)\}$. To ease the writing, we use the following notation: $\mathrm{st}_3(\alpha; R^3) = \mathrm{st}_3(\alpha; \mathrm{cell}_3(R^3))$ and $\mathrm{st}^*_3(\alpha; R^3) = \mathrm{st}^*_3(\alpha; \mathrm{cell}_3(R^3))$.

A *lighting function* on R^3 is a map $f : \mathcal{P}(\mathrm{cell}_3(R^3)) \times R^3 \to \{0,1\}$ for which the following five axioms hold for all $O \in \mathcal{P}(\mathrm{cell}_3(R^3))$ and $\alpha \in R^3$; see [2,3].

(1) *object axiom*: if $\alpha \in O$ then $f(O,\alpha) = 1$;
(2) *support axiom*: if $\alpha \notin \mathrm{supp}(O)$ then $f(O,\alpha) = 0$;
(3) *weak monotone axiom*: $f(O,\alpha) \leq f(\mathrm{cell}_3(R^3),\alpha)$;
(4) *weak local axiom*: $f(O,\alpha) = f(\mathrm{st}^*_3(\alpha; O),\alpha)$; and,
(5) *complement axiom*: if $O' \subseteq O \subseteq \mathrm{cell}_3(R^3)$ and $\alpha \in R^3$ are such that $\mathrm{st}_3(\alpha; O) = \mathrm{st}_3(\alpha; O')$, $f(O',\alpha) = 0$ and $f(O,\alpha) = 1$, then the set $\alpha(O',O) = \cup\{\mathring{\omega} \,;\, \omega < \alpha, f(O',\omega) = 0,\ f(O,\omega) = 1\} \subseteq \partial\alpha$ is non-empty and connected.

If $f(O,\alpha) = 1$ we say that f *lights* the cell α for the object O, otherwise f *vanishes* on α for O.

Example 1. The following are lighting functions on R^3: (a) $f_{\max}(O,\alpha) = 1$ if and only if $\alpha \in \mathrm{supp}(O)$; (b) $g(O,\alpha) = 1$ if and only if $\mathrm{st}_3(\alpha; R^3) \subseteq O$.

A digital space (R^3, f) is *Euclidean* if its continuous analogue is $\mathbb{R}^3$ (that is, if $f(\mathrm{cell}_3(R^3),\alpha) = 1$ for each cell $\alpha \in R^3$) and, in addition, it is *homogeneous* in

[1] We often drop the "f" from the notation and also write $\mathcal{A}_{R^3}$ instead $\mathcal{A}^f_{\mathrm{cell}_3(R^3)}$.

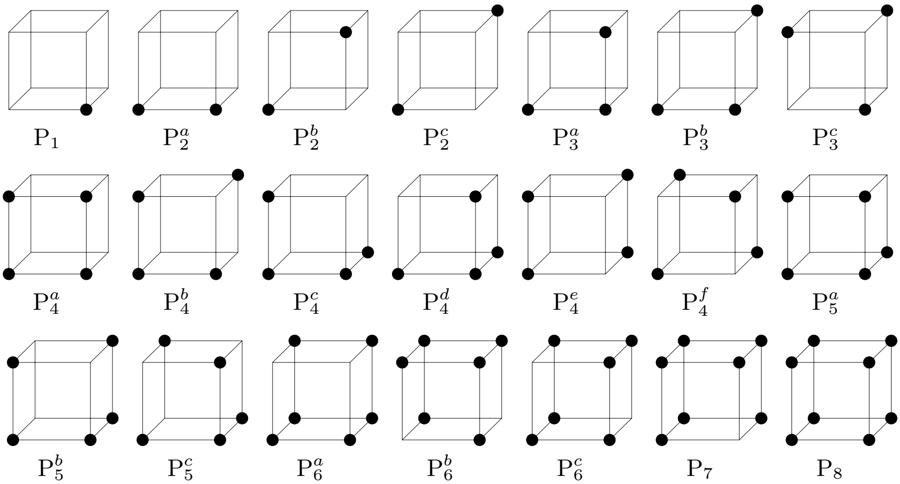

Fig. 1. Non-empty canonical 0-patterns around a vertex. For $d = 1, 2$, the list of d-patterns is longer because $\mathrm{st}_3^*(\alpha; O)$ may contain up to 12 (respectively, 18) voxels.

the sense that the continuous analogue of any object O does not change under isometries $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ preserving $\mathbb{Z}^3$ (i.e, $\varphi(|\mathcal{A}_O^f|) = |\mathcal{A}_{\varphi(O)}^f|$). It is obvious that Axiom 3 above is redundant for Euclidean spaces. Moreover, homogeneity allows us to rewrite Axiom 4 in terms of a minimal family of objects, called *(canonical) d-patterns*, consisting of all subsets of $\mathrm{st}_3^*(\alpha_d; R^3)$ which are distinct up to rotations or symmetries, where α_d is a fixed d-cell for $0 \leq d < 3$; see Fig. 1. More precisely, Axiom 4 becomes $f(O, \alpha) = f(\mathrm{P}(O, \alpha), \alpha_d)$, where $\mathrm{P}(O, \alpha) \subseteq \mathrm{st}_3^*(\alpha_d; R^3)$, called the *pattern of O in α*, is the unique canonical d-pattern for which there exists a isometry such that $\varphi(\mathrm{st}_3^*(\alpha; O)) = \mathrm{P}(O, \alpha)$.

2 $(k, \overline{k})$-Connected Digital Spaces

In this section we assume that (R^3, f) is Euclidean. Our aim is to find some necessary conditions on the lighting function f so that the space (R^3, f) provides us with the $(k, \overline{k})$-connectedness, as usually defined on $\mathbb{Z}^3$ by mean of adjacency pairs, $k, \overline{k} \in \{6, 18, 26\}$. For this, we first recall that a digital object O in (R^3, f) is said to be connected if its continuous analogue $|\mathcal{A}_O|$ is connected. On the other hand, the set of voxels $\mathrm{cell}_3(R^3) - O$, regarded as the complement of O, is declared to be O-connected if $|\mathcal{A}_{R^3}| - |\mathcal{A}_O|$ is connected. Moreover, we call $C \subseteq \mathrm{cell}_3(R^3)$ a component of O ($\mathrm{cell}_3(R^3) - O$) if it consists of all voxels σ whose centers $c(\sigma)$ belong to a component of $|\mathcal{A}_O|$ ($|\mathcal{A}_{R^3}| - |\mathcal{A}_O|$, respectively). These notions of connectedness are characterized by the following notions of adjacency

Definition 1. *Two cells $\sigma, \tau \in O$ are* $\emptyset$-adjacent *iff $f(O, \alpha) = 1$ for some common face $\alpha \leq \sigma \cap \tau$. Moreover, $\sigma, \tau \in \mathrm{cell}_3(R^3) - O$ are* O-adjacent *iff $f(O, \alpha) = 0$ for some $\alpha \leq \sigma \cap \tau$.*

More precisely, for $X \in \{\emptyset, O\}$, the notion of X-adjacency directly leads to the notions of X-path, X-connectedness and X-component. Then, it can be proved that $C \subseteq O$ is a component of O if and only it is an $\emptyset$-component, while $C \subseteq \mathrm{cell}_3(R^3) - O$ is a component of the complement $\mathrm{cell}_3(R^3) - O$ if and only if it is a O-component. See Section 4 in [2] for a detailed proof of this fact in a much more general context.

By using this characterization, one may get intuitively convinced that the lighting functions $f_{\max}$ and g in Example 1 describe the $(26, 6)$- and $(6, 26)$-adjacencies usually defined on $\mathbb{Z}^3$. Actually the digital spaces $(R^3, f_{\max})$ and (R^3, g) are $(26, 6)$- and $(6, 26)$-spaces, respectively, in the sense of the following

Definition 2. *Given an adjacency pair $(k, \overline{k})$ in $\mathbb{Z}^3$ we say that the digital space (R^3, f) is a $(k, \overline{k})$-*space *if the two following properties hold for any digital object $O \subseteq \mathrm{cell}_3(R^3)$:*

1. *C is a $\emptyset$-component of O if and only if it is a k-component of O; and,*
2. *C is an O-component of the complement $\mathrm{cell}_3(R^3) - O$ if and only if it is a $\overline{k}$-component.*

From now on, we assume that (R^3, f) is a Euclidean $(k, \overline{k})$-space, $k, \overline{k} \in \{6, 18, 26\}$ and $(k, \overline{k}) \neq (6, 6)$. Examples of these spaces can be found in [1,3,6].

Proposition 1. *Let $\mathrm{P}(O, \alpha)$ the pattern of a digital object $O \subseteq \mathrm{cell}_3(R^3)$ in a vertex $\alpha \in R^3$. The following properties hold:*

1. *If $\mathrm{P}(O, \alpha) = \mathrm{P}_8$ then $f(O, \alpha) = 1$.*
2. *$f(O, \alpha) = 0$ whenever $\mathrm{P}(O, \alpha) \in \{\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2^a, \mathrm{P}_2^b, \mathrm{P}_3^a, \mathrm{P}_4^a\}$.*
3. *If $\mathrm{P}(O, \alpha) = \mathrm{P}_2^c$, then $f(O, \alpha) = 1$ iff $k = 26$.*
4. *For $\overline{k} = 6$, $f(O, \alpha) = 1$ whenever $\mathrm{P}(O, \alpha) \in X = \{\mathrm{P}_3^c, \mathrm{P}_4^b, \mathrm{P}_4^e, \mathrm{P}_4^f, \mathrm{P}_5^b, \mathrm{P}_5^c, \mathrm{P}_6^b\}$.*
5. *If $\mathrm{P}(O, \alpha) = \mathrm{P}_6^c$, then $f(O, \alpha) = 1$ iff $\overline{k} \in \{6, 18\}$.*

Proof. Property (1) is an immediate consequence of the definition of Euclidean digital spaces and Axiom 4. Similarly, (2) follows from Axiom 2. To show (3) notice that $f(\mathrm{st}_3(\alpha; O), \alpha) = 1$ iff $\mathrm{st}_3(\alpha; O)$ is $\emptyset$-connected, and hence k-connected; however, it is a 26-connected set but is not 18-connected. For the proof of (4) and (5) let us consider the object $O_1 = (\mathrm{cell}_3(R^3) - \mathrm{st}_3^*(\alpha; R^3)) \cup \mathrm{st}_3^*(\alpha; O)$. If $f(O, \alpha) = 0$, Axiom 4 implies that its complement $\mathrm{cell}_3(R^3) - O_1$ would be O_1-connected, but it is not 6-connected if $\mathrm{P}(O, \alpha) \in X$ and is not 18-connected if $\mathrm{P}(O, \alpha) = \mathrm{P}_6^c$. □

Proposition 2. *Let $O \subseteq \mathrm{cell}_3(R^3)$ be a digital object and $\beta = \langle \alpha_1, \alpha_2 \rangle \in R^3$ be a 1-cell such that $f(O, \alpha_1) = f(O, \alpha_2)$. If $\mathrm{st}_3(\beta; O) = \mathrm{st}_3(\beta; R^3)$ then $f(O, \beta) = 1$. Moreover, if (R^3, f) is a $(k, 6)$-space and $\mathrm{st}_3(\beta; O) = \{\sigma, \tau\}$, with $\beta = \sigma \cap \tau$, then $f(O, \beta) = 1$ as well.*

Proof. It suffices to find an object $O' \supseteq O$ such that $\mathrm{st}_3(\beta; O') = \mathrm{st}_3(\beta; O)$ and $f(O', \delta) = 1$ for each cell $\delta \in \{\beta, \alpha_1, \alpha_2\}$. In these conditions, it is readily checked that $\beta(O, O')$ is either empty or equal to the non-connected set $\{\alpha_1, \alpha_2\}$. Hence, $f(O, \beta) = 1$ by Axiom 5.

If $\mathrm{st}_3(\beta;O)=\mathrm{st}_3(\beta;R^3)$ we use the object $O_1=\mathrm{cell}_3(R^3)$, while we take $O_2=\mathrm{cell}_3(R^3)-\{\sigma',\tau'\}$, where $\{\sigma',\tau'\}=\mathrm{st}_3(\beta;R^3)-\mathrm{st}_3(\beta;O)$, if $\mathrm{st}_3(\beta;O)=\{\sigma,\tau\}$. In the first case, $f(O_1,\delta)=1$, $\delta\in\{\beta,\alpha_1,\alpha_2\}$, since (R^3,f) is Euclidean. In the second, the equalities $f(O_2,\delta)=1$ also hold since, in addition, the complement $\mathrm{cell}_3(R^3)-O_2$ is not 6-connected. □

Proposition 3. *Let $O\subseteq\mathrm{cell}_3(R^3)$ be a digital object and $\gamma\in\mathrm{supp}(O)$ a 2-cell. Then γ or some of its faces are lighted for O.*

Proof. Assume $f(O,\delta)=0$ for each proper face $\delta<\gamma$, and let us consider the object $O_1=(\mathrm{cell}_3(R^3)-\mathrm{st}_3(\beta_0;R^3)-\mathrm{st}_3(\beta_2;R^3))\cup\mathrm{st}_3(\beta_0;O)\cup\mathrm{st}_3(\beta_2;O)$, where β_i, $0\le i\le 3$, are the four 1-faces of γ, and $\alpha_i=\beta_i\cap\beta_{i+1(\mathrm{mod}\ 4)}$ its vertices. Since (R^3,f) is homogeneous we know that $f(O_1,\alpha_0)=f(O_1,\alpha_1)$ and $f(O_1,\alpha_2)=f(O_1,\alpha_3)$. Thus, for $i=0,2$, the sets $\beta_i(O,O_1)$ are empty or non-connected and $f(O_1,\beta_i)=0$ by Axiom 5. On the other hand, by Proposition 2 $f(O_1,\beta_1)=1$ if f vanishes on both vertices α_1,α_2 for O_1. Hence, some cell in the set $\{\alpha_1,\beta_1,\alpha_2\}$, and similarly for $\{\alpha_3,\beta_3,\alpha_0\}$, is lighted for O_1. This way the sets $\gamma(O,O_1)$ and $\gamma(O_1,\mathrm{cell}_3(R^3))$ are non-connected, and $f(O_1,\gamma)=f(O,\gamma)=1$ also by Axiom 5. □

3 $(k,\overline{k})$-Surfaces

Similarly to our previous definition of connectedness on a digital space (R^3,f), we use continuous analogues to introduce the notion of digital surface. Namely, a digital object S is a *digital surface* in the digital space (R^3,f), an *f-surface* for short, if $|\mathcal{A}_S|$ is a (combinatorial) surface without boundary; that is, if for each vertex $v\in\mathcal{A}_S$ its link $\mathrm{lk}(v;\mathcal{A}_S)=\{A\in\mathcal{A}_S\,;\,v,A<B\in\mathcal{A}_S \text{ and } v\notin A\}$ is a 1-sphere.

Along this section S will stand for an arbitrary f-surface S in a given Euclidean $(k,\overline{k})$-space (R^3,f), where $k,\overline{k}\in\{6,18,26\}$ and $(k,\overline{k})\neq(6,6)$. Our goal is to compute its continuous analogue; actually, we will determine the value $f(S,\delta)$ for almost each cell $\delta\in R^3$. A simple although crucial tool for this task is the following

Remark 1. By the definition of continuous analogues, each vertex in $\mathcal{A}_S$ is the center $c(\gamma)$ of a cell $\gamma\in R^3$ lighted for the surface S. Moreover, the cycle $\mathrm{lk}(c(\gamma);\mathcal{A}_S)$ is the complex determined by the set of cells $X_S^\gamma=\{\delta\in R^3\,;\,\delta<\gamma \text{ or } \gamma<\delta \text{ and } f(S,\delta)=1\}$, which is contained in $Y_S^\gamma=\{\delta\in R^3\,;\,\delta<\gamma \text{ or } \gamma<\delta \text{ and } \delta\in\mathrm{supp}(S)\}$ by Axiom 2.

If δ is a 3-cell, Axioms 1 and 2 in the definition of lighting functions imply that $f(S,\delta)=1$ if and only if $\delta\in S$. For 1-cells we will prove the following

Theorem 1. *Given a 1-cell $\beta\in R^3$, let $\alpha_1,\alpha_2<\beta$ be its vertices and $\gamma_j\in R^3$ be the four 2-cells such that $\beta<\gamma_j$, $1\le j\le 4$. Then $f(S,\beta)=1$ if and only if one of the two following properties holds:*

1. *$\mathrm{st}_3(\beta;S)=\{\sigma,\tau\}$, with $\beta=\sigma\cap\tau$ and, moreover, $f(S,\alpha_i)=1$, $i=1,2$, and $f(S,\gamma_j)=0$, $1\le j\le 4$.*
2. *$\mathrm{st}_3(\beta;S)=\mathrm{st}_3(\beta;R^3)$, $f(S,\alpha_i)=0$ and $f(S,\gamma_j)=1$, $i=1,2$, $1\le j\le 4$.*

The proof of Theorem 1 needs the results in Sect. 2 as well as to know the lighting of some vertices of R^3 for the surface. However, the "only if" part is a consequence of the following

Proposition 4. *Let $\beta \in R^3$ be a 1-cell. Then $f(S,\beta) = 0$ if one of the following conditions holds:*

1. $\mathrm{st}_3(\beta; S)$ *consists of exactly three elements.*
2. $\mathrm{st}_3(\beta; S) = \mathrm{st}_3(\beta; R^3)$ *and $f(S,\alpha) = 1$ for some vertex $\alpha < \beta$.*

Proof. Assume on the contrary that $f(S,\beta) = 1$. We shall prove that the link $L = \mathrm{lk}(c(\beta); \mathcal{A}_S)$ is not a 1-sphere, which is a contradiction. If condition (1) holds, it is readily checked that the 2- and 3-cells of the set Y_S^β, as defined in Remark 1, do not determine a cycle in $\mathcal{A}_S$ unless $c(\alpha)$ also belongs to L for some vertex $\alpha < \beta$. But then $c(\alpha)$ is an end of all edges $\langle c(\alpha), c(\sigma_i)\rangle \in L$, where σ_i ranges over the star of β in S, and thus L is not a 1-sphere if $\mathrm{st}_3(\beta; S)$ has three or four elements. □

Proof. of Theorem 1 ("only if" part) If $f(S,\beta) = 1$ then $\beta \in \mathrm{supp}(S)$, and either $\mathrm{st}_3(\beta; S) = \{\sigma, \tau\}$, with $\beta = \sigma \cap \tau$, or $\mathrm{st}_3(\beta; S) = \mathrm{st}_3(\beta; R^3)$ by Proposition 4. In the first case only the two vertices of β may belong to Y_S^β, which must be lighted since $L = \mathrm{lk}(c(\beta); \mathcal{A}_S)$ is a cycle. In the second case, we already know, also by Proposition 4, that $f(S,\alpha_i) = 0$, $i = 1, 2$. Then $f(S,\gamma_j) = 1$ in order that L is a 1-sphere. □

Remark 2. If a vertex $\alpha \in R^3$ is lighted for the surface S the "only if" part of Theorem 1 allows us to be more precise than in Remark 1. Indeed, the set of cells X_S^α, determining the link $\mathrm{lk}(c(\alpha); \mathcal{A}_S)$, is contained in the disjoint union $\mathrm{st}_3(\alpha; S) \cup Z_S^\alpha$, where $Z_S^\alpha = \{\delta > \alpha\,;\, \mathrm{st}_3(\delta; S) = \{\sigma, \tau\}, \delta = \sigma \cap \tau\}$.

Using this remark we are able to describe locally the continuous analogue of an f-surface S around any vertex α which is lighted for it. Before, and in addition to those found in Proposition 1, we next identify some more patterns of an f-surface S around a vertex α for which $f(S,\alpha) = 0$.

Proposition 5. *Let $\mathrm{P}(S,\alpha)$ be the pattern of an f-surface S around a vertex $\alpha \in R^3$. If $\mathrm{P}(S,\alpha) \in \{\mathrm{P}_2^c, \mathrm{P}_3^b, \mathrm{P}_4^c, \mathrm{P}_4^d, \mathrm{P}_5^a, \mathrm{P}_5^c, \mathrm{P}_6^b, \mathrm{P}_7\}$ then $f(S,\alpha) = 0$.*

Proof. Assume that $f(S,\alpha) = 1$. For each voxel $\sigma \in \mathrm{st}_3(\alpha; S)$ let us consider the set $Z_\sigma = \{\delta \in Z_S^\alpha\,;\, \delta < \sigma\}$; see Remark 2. Since $L = \mathrm{lk}(c(\alpha); \mathcal{A}_S)$ is a cycle, $c(\sigma) \in L$ is in two edges of L; in other words, exactly two elements of Z_σ belong to X_S^α. However, if $\mathrm{P}(S,\alpha) \in \{\mathrm{P}_2^c, \mathrm{P}_3^b, \mathrm{P}_4^c, \mathrm{P}_4^d, \mathrm{P}_5^a\}$ it is easily found a voxel $\sigma \in \mathrm{st}_3(\alpha; S)$ for which the set Z_σ is a sigleton or the emptyset. On the other hand, for each of the patterns $\mathrm{P}_5^c, \mathrm{P}_6^b$ and P_7 there is a proper subset of voxels $A \subset \mathrm{st}_3(\alpha; S)$ such that, for each $\sigma \in A$, $Z_\sigma \subset X_S^\alpha$ since it consists of exactly two elements. But then one observes that the cells in $A \cup (\cup_{\sigma \in A} Z_\sigma)$ determines a cycle in L which leaves out the centers of the voxels in $\mathrm{st}_3(\alpha; S) - A$. □

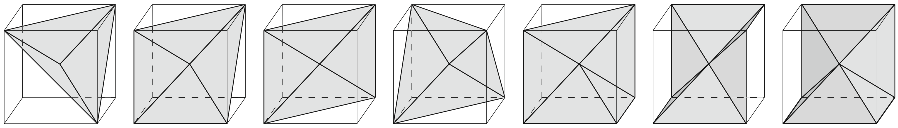

Fig. 2. Continuous analogue of a surface S around a vertex $\alpha \in R^3$

Remark 3. Given an f-surface S and a vertex $\alpha \in R^3$ such that $f(S, \alpha) = 1$ we know that $\mathrm{P}(S, \alpha) \in \{\mathrm{P}_3^c, \mathrm{P}_4^b, \mathrm{P}_4^e, \mathrm{P}_4^f, \mathrm{P}_5^b, \mathrm{P}_6^a, \mathrm{P}_6^c, \mathrm{P}_8\}$ from Propositions 1 and 5. For each of these patterns, except for P_8, we can determine the lighting of every cell in Z_S^α by using the same technique as in the proof above. That is, we completely know the continuous analogue of the surface inside the unit cube $C_\alpha \subset \mathbb{R}^3$ whose vertices are the centers of the eight voxels containing α; see Fig. 2.

Indeed, if $\mathrm{P}(S, \alpha) \in \{\mathrm{P}_3^c, \mathrm{P}_4^b, \mathrm{P}_4^e, \mathrm{P}_5^b, \mathrm{P}_6^c\}$ each set Z_σ, $\sigma \in \mathrm{st}_3(\alpha; S)$, consists of two cells. Therefore, all of them are lighted for the surface (i.e., $Z_S^\alpha \subset X_S^\alpha$).

For P_6^a, let σ, τ be the two voxels in $\mathrm{st}_3(\alpha; S)$ which are 6-adjacent to three other voxels in the star of α in S. One readily checks that for each voxel $\rho \in A = \mathrm{st}_3(\alpha; S) - \{\sigma, \tau\}$ the set Z_ρ has two elements and then it is contained in X_S^α; moreover, $X_S^\alpha = \mathrm{st}_3(\alpha; S) \cup (\cup_{\rho \in A} Z_\rho)$ since this set determines a cycle in $\mathrm{lk}(c(\alpha); \mathcal{A}_S)$. In other words, $\gamma = \sigma \cap \tau$ is the only 2-cell in $\mathrm{supp}(\mathrm{st}_3(\alpha; S))$ such that $f(S, \gamma) = 0$; moreover, also $f(S, \beta_i) = 0$ for the two edges $\alpha < \beta_1, \beta_2 < \gamma$.

Finally, at this point we can only determine the continuous analogue for P_4^f up to symmetries. We know that only two of the three 1-cells in each of the sets Z_{σ_i}, where $\mathrm{st}_3(\alpha; S) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, can be lighted for the surface. However the selection of two cells in Z_{σ_1} determines what cells in $Z_{\sigma_2}, Z_{\sigma_3}, Z_{\sigma_4}$ must be lighted in order that $\mathrm{lk}(c(\alpha); \mathcal{A}_S)$ is a cycle.

To complete our analysis of the lighting on vertices and edges of R^3 for S we use some separation properties of f-surfaces. Firstly, notice that, from our definition of $(k, \overline{k})$-space and the characterization of connectedness on a Euclidean space (R^3, f) in Sect. 2, the $\overline{k}$-components of $\mathrm{cell}_3(R^3) - S$ are characterizing the connected components of $\mathbb{R}^3 - |\mathcal{A}_S| = |\mathcal{A}_{R^3}| - |\mathcal{A}_S|$. Therefore, we obtain the following separation result as a corollary of the well-known Jordan-Brouwer Theorem; see Sect. 5 in [3].

Theorem 2. *Let S be a k-connected f-surface in a Euclidean $(k, \overline{k})$-space (R^3, f). Then S separates its complement* $\mathrm{cell}_3(R^3) - S$ *into two $\overline{k}$-components.*

Next proposition shows that the $\overline{k}$-componentes of $\mathrm{cell}_3(R^3) - S$ can be locally determined by using the notion of relative ball from polyhedral topology. Recall that given a closed topological surface $M \subset \mathbb{R}^3$ in the Euclidean space, a *relative ball* (B^3, B^2) in $(\mathbb{R}^3, M)$ is a pair of topological balls $B^2 \subset B^3$ such that $B^2 \cap \partial B^3 = \partial B^2$ and $B^2 = B^3 \cap M$. The key property of relative balls, which is also a consequence of the Jordan-Brouwer Theorem, states that the difference $B^3 - B^2$ has exactly two connected components, each contained in a distinct component of $\mathbb{R}^3 - M$; see [2] for details.

Proposition 6. *Two 26-adjacent voxels $\sigma, \tau \notin S$ belong to the same $\overline{k}$-component of* $\mathrm{cell}_3(R^3) - S$ *if and only if* $f(S, \sigma \cap \tau) = 0$.

Proof. If $f(S, \delta) = 0$, where $\delta = \sigma \cap \tau$, then σ and τ are S-adjacent. Hence, they belong to the same $\overline{k}$-component of $\mathrm{cell}_3(R^3) - S$ by Definition 2.

Conversely, assume $f(S, \delta) = 1$. It is not difficult to show that the pair of polyhedra $(|\mathrm{st}(c(\delta), \mathcal{A}_{R^3})|, |\mathrm{st}(c(\delta), \mathcal{A}_S)|)$ is a relative ball in $(\mathbb{R}^3, |\mathcal{A}_S|)$, where $\mathrm{st}(c(\delta), \mathcal{A}_X) = \{A \in \mathcal{A}_X \,;\, c(\alpha), A < B \in \mathcal{A}_X\}$ is the *star* of $c(\delta)$ in $\mathcal{A}_X$. Therefore, it will suffice to show that each $c(\sigma)$ and $c(\tau)$ belong to a distinct component of the difference $D = |\mathrm{st}(c(\delta), \mathcal{A}_{R^3})| - |\mathrm{st}(c(\delta), \mathcal{A}_S)|$.

Notice that $\dim \delta \leq 1$; otherwise, if δ is a 2-cell then $\delta \notin \mathrm{supp}(S)$, and $f(S, \delta) = 0$ by Axiom 2, since $\sigma, \tau \notin S$. If $\dim \delta = 1$ the "only if" part of Theorem 1 yields that $\mathrm{st}_3(\delta; S) = \{\sigma', \tau'\} = \mathrm{st}_3(\delta; R^3) - \{\sigma, \tau\}$. Then $|\mathrm{st}(c(\delta), \mathcal{A}_{R^3})|$ is the double cone from the vertices $v_1, v_2 < \delta$ over the unit square whose vertices are the centers of the voxels in $\mathrm{st}_3(\delta; R^3)$, while $|\mathrm{st}(c(\delta), \mathcal{A}_S)|$ is the double cone from v_1 and v_2 over the union of edges $\langle c(\delta), c(\sigma')\rangle \cup \langle c(\delta), c(\tau')\rangle$. After this description it is easily checked that $c(\sigma)$ and $c(\tau)$ belong to distinct components of the difference D.

If $\dim \delta = 0$, $|\mathrm{st}(c(\delta), \mathcal{A}_{R^3})|$ is the cube $C_\delta \subset \mathbb{R}^3$ with vertices at the centers of the eight voxels containing δ. Moreover, $\mathrm{P}(S, \delta) \in \{\mathrm{P}_3^c, \mathrm{P}_4^b, \mathrm{P}_4^e, \mathrm{P}_5^b, \mathrm{P}_6^c\}$ by Propositions 1 and 5. Then, $|\mathrm{st}(c(\delta), \mathcal{A}_S)|$ is the continuous analogue locally described in Remark 3 and the result follows; see Fig. 2. $\square$

Corollary 1. *Assume $\overline{k} \in \{18, 26\}$. Then $f(S, \delta) = 0$ for each cell $\delta \in R^3$ satisfying one of the following conditions:*

1. *δ is an edge such that* $\mathrm{st}_3(\delta; S) = \{\sigma, \tau\}$ *with* $\delta = \sigma \cap \tau$.
2. *δ is a vertex and* $\mathrm{P}(S, \delta) \in \{\mathrm{P}_3^c, \mathrm{P}_4^b, \mathrm{P}_4^e, \mathrm{P}_4^f, \mathrm{P}_5^b\}$.

Proof. If δ is a vertex and $\mathrm{P}(S, \delta) = \mathrm{P}_4^f$ then each edge $\beta > \delta$ is the intersection of two 18-adjacent voxels in $\mathrm{st}_3(\delta; R^3) - S \subset \mathrm{cell}_3(R^3) - S$ and thus $f(S, \beta) = 0$. Therefore $f(S, \delta) = 0$ since, otherwise, the link $\mathrm{lk}(c(\delta); \mathcal{A}_S)$ is a discrete set of points consisting just of the centers of the four voxels in $\mathrm{st}_3(\delta; S)$. For all the remaining cases it is not difficult to find a pair of voxels σ', τ' in a 18-component of $\mathrm{st}_3(\delta; R^3) - S$ such that $\delta = \sigma' \cap \tau'$.

Propositions 1 and 5 as well as the corollary above provides us with all the information we need about the lighting of vertices for an f-surface S, which is summarized as follows. We are also ready to finish the proof of Theorem 1.

Theorem 3. *Let S be an f-surface in a Euclidean $(k, \overline{k})$-space (R^3, f), and let $\alpha \in R^3$ be a vertex such that* $\mathrm{P}(S, \alpha) \notin \{\mathrm{P}_6^a, \mathrm{P}_8\}$. *Then* $\mathrm{P}(S, \alpha) \notin \mathbb{FP}_{k,\overline{k}}$ *and, moreover,* $f(S, \alpha) = 1$ *if and only if* $\mathrm{P}(S, \alpha) \in \mathbb{P}_{\overline{k}}$. *The sets* $\mathbb{FP}_{k,\overline{k}}$ *and* $\mathbb{P}_{\overline{k}}$, *whose elements are respectively called* $(k, \overline{k})$-forbidden patterns *and* $\overline{k}$-plates, *are defined as follows:* $\mathbb{FP}_{6,26}$, $\mathbb{FP}_{18,26}$, $\mathbb{FP}_{6,18}$ *and* $\mathbb{FP}_{18,18}$ *are the empty set,* $\mathbb{FP}_{26,26} = \mathbb{FP}_{26,18} = \{\mathrm{P}_2^c\}$, $\mathbb{FP}_{18,6} = \{\mathrm{P}_5^c, \mathrm{P}_6^b\}$ *and* $\mathbb{FP}_{26,6} = \{\mathrm{P}_2^c, \mathrm{P}_5^c, \mathrm{P}_6^b\}$; $\mathbb{P}_6 = \{\mathrm{P}_3^c, \mathrm{P}_4^b, \mathrm{P}_4^e, \mathrm{P}_4^f, \mathrm{P}_5^b, \mathrm{P}_6^c\}$, $\mathbb{P}_{18} = \{\mathrm{P}_6^c\}$ *and* $\mathbb{P}_{26} = \emptyset$.

Proof. of Theorem1 ("if" part) If S is a $(k,\overline{k})$-f-surface for $\overline{k} \in \{18, 26\}$ and $\mathrm{st}_3(\beta; S) = \{\sigma, \tau\}$ condition (1) does not hold and, so, there is nothing to prove. Indeed, for each vertex $\alpha < \beta$, $\mathrm{P}(S, \alpha) \in \{\mathrm{P}_2^b, \mathrm{P}_3^b, \mathrm{P}_3^c, \mathrm{P}_4^b, \mathrm{P}_4^e, \mathrm{P}_4^f, \mathrm{P}_5^b, \mathrm{P}_5^c, \mathrm{P}_6^b\}$; therefore $f(S, \alpha) = 0$ by Corollary 1 and Propositions 1 and 5. For the rest of cases the result is an immediate consequence of Proposition 2. □

Notice that in the theorem above we have excluded P_8 from our analysis. This pattern is usually considered as a small blob and it is widely rejected as part of a surface. Following this criterion, from now on we will only consider regular spaces according to the next definition.

Definition 3. *An Euclidean $(k,\overline{k})$-space is said to be* regular *if* $\mathrm{P}(S, \alpha) \neq \mathrm{P}_8$ *for any surface S and any vertex $\alpha \in R^3$.*

Remark 4. We have also eluded P_6^a in Theorem 3. Actually, a surface S containing this pattern can have two different but homeomorphic continuous analogues, depending on the value of $f(S, \alpha)$ for any vertex $\alpha \in R^3$ such that $\mathrm{P}(S, \alpha) = \mathrm{P}_6^a$ (recall that we are only considering homogeneous digital spaces). If $f(S, \alpha) = 1$, we have already described locally the continuous analogue of S around α in Remark 3. For describing the intersection $|\mathcal{A}_S| \cap C_\alpha$, where $C_\alpha \subset \mathbb{R}^3$ is the unit cube with vertices at the centers of the voxels in $\mathrm{st}_3(\alpha; R^3)$, in case $f(S, \alpha) = 0$, let us consider the two edges $\beta_i = \langle \alpha, \alpha_i \rangle$ of R^3 such that $\mathrm{st}_3(\beta_i; S) = \mathrm{st}_3(\beta_i; R^3)$, $i = 1, 2$. If (R^3, f) is a regular space Theorem 3 yields that $f(S, \alpha_i) = 0$ for $i = 1, 2$ and, then, $f(S, \beta_i) = 1$ by Theorem 1; moreover, any 2-cell containing α is lighted for S. So that, the continuous analogue of S around α is the union of the two unit squares whose vertices are the centers of the voxels in $\mathrm{st}_3(\alpha; S)$.

We finish this section showing that certain patterns around an edge are forbidden in a $(k,\overline{k})$-f-surface when $k, \overline{k} \in \{18, 26\}$.

Proposition 7. *Let S be a digital surface in a Euclidean $(k,\overline{k})$-space (R^3, f), $k, \overline{k} \in \{18, 26\}$. If $\beta \in R^3$ is an edge such that* $\mathrm{st}_3(\beta; S) = \{\sigma, \tau\}$, *with $\beta = \sigma \cap \tau$, then σ and τ belong to a 6-component of* $\mathrm{st}_3^*(\beta; S)$.

Proof. (sketch) Assume, on the contrary, that σ and τ are in distinct 6-components of $\mathrm{st}_3^*(\beta; S)$. Then one checks that $\mathrm{P}(S, \alpha_i) \in \{\mathrm{P}_2^b, \mathrm{P}_3^b, \mathrm{P}_3^c, \mathrm{P}_4^b, \mathrm{P}_4^e, \mathrm{P}_4^f, \mathrm{P}_5^c\}$ for the two vertices $\alpha_1, \alpha_2 < \beta$, and thus $f(S, \delta) = 0$, for $\delta \in \{\beta, \alpha_1, \alpha_2\}$ by Corollary 1 and Theorem 3. From these facts we build a 18-connected object O_1, possibly infinite, such that $\mathrm{P}(O_1, \beta') = \mathrm{P}(S, \beta)$ for each edge $\beta' \in \mathrm{supp}(O_1)$ which is a face of exactly two voxels in O_1. However, it can be proved that O_1 is not $\emptyset$-connected in (R^3, f), which is a contradiction. □

4 Universal $(k,\overline{k})$-Spaces

In this section we reach our goal: for each adjacency pair $(k,\overline{k}) \neq (6, 6)$, $k, \overline{k} \in \{6, 18, 26\}$, we define a lighting function on the complex R^3 which gives us a regular $(k,\overline{k})$-space $E_{k,\overline{k}} = (R^3, f_{k,\overline{k}})$ whose set of digital surfaces is the largest within that class of digital spaces. Namely, we will prove the following

Theorem 4. *Any digital surface S in a regular $(k,\overline{k})$-space (R^3, f) is also a digital surface in the space $E_{k,\overline{k}} = (R^3, f_{k,\overline{k}})$, which is called the* universal $(k,\overline{k})$-space.

In the definition of $E_{k,\overline{k}}$ we use the set of forbidden patterns introduced in Theorem 3 and the set of $\overline{k}$-plates, that are the elementary "bricks" from which f-surfaces of Euclidean $(k,\overline{k})$-spaces are built. Indeed, the lighting function $f_{k,\overline{k}}$ is defined as follows. Given a digital object $O \subseteq \mathrm{cell}_3(R^3)$ and a cell $\delta \in R^3$, $f_{k,\overline{k}}(O,\delta) = 1$ if and only one of the following conditions holds:

1. $\dim \delta \geq 2$ and $\delta \in \mathrm{supp}(O)$
2. $\dim \delta = 0$ and $\mathrm{P}(O,\delta) \in \mathbb{P}_{\overline{k}} \cup \mathbb{FP}_{k,\overline{k}} \cup \{\mathrm{P}_8\}$
3. $\dim \delta = 1$ and $\mathrm{st}_3(\delta; O) = \mathrm{st}_3(\delta; R^3)$ (*a square plate*), or
4. $\dim \delta = 1$ and $\mathrm{st}_3(\delta; O) = \{\sigma, \tau\}$, with $\delta = \sigma \cap \tau$, and one of the next further conditions also holds: (a) for $\overline{k} = 6$, and $k \neq 6$, $f_{k,6}(O,\alpha_1) = f_{k,6}(O,\alpha_2)$, where α_1, α_2 are the two vertices of the 1-cell δ; or (b) if σ, τ belong to distinct 6-components of $\mathrm{st}_3^*(\delta; O)$, for $k, \overline{k} \in \{18, 26\}$.

Notice that if $k = 6$, and $\overline{k} \in \{18, 26\}$, none of the conditions 4(a) and 4(b) hold. So, $f_{6,\overline{k}}(O,\delta) = 1$ for a 1-cell $\delta \in R^3$ if and only if $\mathrm{st}_3(\delta; O) = \mathrm{st}_3(\delta; R^3)$.

It is not difficult, but a tedious task, to check that each $f_{k,\overline{k}}$ is a lighting function, and to prove that $E_{k,\overline{k}}$ is actually a homogeneous $(k,\overline{k})$-space. To show that all of them are regular spaces, assume that O is a digital objet and $\alpha \in R^3$ is a vertex such that $\mathrm{P}(O,\alpha) = \mathrm{P}_8$; that is, $\mathrm{st}_3(\alpha; O) = \mathrm{st}_3(\alpha; R^3)$. Then, it is readily checked from the definition that $f_{k,\overline{k}}(O,\delta) = 1$ for each cell $\delta \geq \alpha$. Therefore, the unit cube $C_\alpha \subset \mathbb{R}^3$ centered at α, and whose vertices are the centers of all voxels in $\mathrm{st}_3(\alpha; R^3)$, is contained in $|\mathcal{A}_O|$; thus no surface in $E_{k,\overline{k}}$ can contain the pattern P_8. This also proves that $E_{k,\overline{k}}$ is Euclidean since $f(\mathrm{cell}_3(R^3), \delta) = 1$ for every cell $\delta \in R^3$.

In Remark 4 we suggested that the continuous analogue of a given surface in two different $(k,\overline{k})$-spaces may differ only if it contains the pattern P_6^a. Next results make more precise this in relation with the universal $(k,\overline{k})$-space.

Proposition 8. *Let S be a surface in a regular $(k,\overline{k})$-space (R^3, f). If $\delta \in R^3$ is a cell such that $\mathrm{P}(S,\alpha) \neq \mathrm{P}_6^a$ for all vertices $\alpha \leq \delta$, then $f(S,\delta) = f_{k,\overline{k}}(S,\delta)$.*

Proof. We may assume that $\delta \in \mathrm{supp}(S)$, otherwise both f and $f_{k,\overline{k}}$ vanish on δ for S by Axiom 2 of lighting functions.

Assume firstly that δ is a vertex of R^3, and so $f(S,\delta) = 1$ iff $\mathrm{P}(S,\delta) \in \mathbb{P}_{\overline{k}}$ by Theorem 3. Moreover, since (R^3, f) is regular the same theorem and the hypothesis ensures that $\mathrm{P}(S,\delta) \notin \mathbb{FP}_{k,\overline{k}} \cup \{\mathrm{P}_6^a, \mathrm{P}_8\}$ and, under these conditions, the definition of $f_{k,\overline{k}}$ also yields that $f_{k,\overline{k}}(S,\delta) = 1$ iff $\mathrm{P}(S,\delta) \in \mathbb{P}_{\overline{k}}$.

If δ is a 2-cell, $f_{k,\overline{k}}(S,\delta) = 1$ by definition. We reach a contradiction if $f(S,\delta) = 0$. Indeed, in that case Proposition 3 gives us a face $\alpha < \delta$ which is lighted for S. Since $\mathrm{st}_3(\delta; S) \subseteq \mathrm{st}_3(\alpha; S)$, condition (2) in Theorem 1 must hold if $\dim \alpha = 1$. Otherwise, if α is a vertex, then $\mathrm{st}_3(\alpha; S)$ is a $\overline{k}$-plate by Theorem 3 and we showed that δ is then lighted in Remark 3.

Finally, if $\dim \delta = 1$ and $\delta \in \mathrm{supp}(S)$ three cases are possible: (a) $\mathrm{st}_3(\delta; S) = \{\sigma, \tau\}$, with $\delta = \sigma \cap \tau$; (b) $\mathrm{st}_3(\delta; S) = \{\sigma, \tau, \rho\}$; and (c) $\mathrm{st}_3(\delta; S) = \mathrm{st}_3(\delta; R^3)$. In case (b) both $f_{k,\overline{k}}$ and f vanish on δ by definition and Theorem 1, respectively.

We claim that $f(S,\delta) = 1$ in case (c). Indeed, let α_1, α_2 be the vertices of the edge δ. Since $\mathrm{st}_3(\delta; S) \subseteq \mathrm{st}_3(\alpha_i; S)$ and (R^3, f) is regular we know that $\mathrm{P}(S,\alpha_i) \in \{\mathrm{P}_4^a, \mathrm{P}_5^a, \mathrm{P}_6^b, \mathrm{P}_7\}$; moreover, $\mathrm{P}(S,\alpha_i) \neq \mathrm{P}_6^b$ if $\overline{k} = 6$. In any case these patterns are not $\overline{k}$-plates. Then $f(S,\alpha_i) = 0$, $i = 1,2$, by Theorem 3, and Proposition 2 yields our claim. The case (a) requires two different arguments for $\overline{k} = 6$ and $\overline{k} \in \{18, 26\}$. If $\overline{k} = 6$, and so $k \neq 6$, we have already proved that $f(S,\alpha_i) = f_{k,\overline{k}}(S,\alpha_i)$ for the two vertices $\alpha_1, \alpha_2 < \delta$. Then also $f(S,\delta) = f_{k,\overline{k}}(S,\delta)$ by definition of $f_{k,\overline{k}}$ and Theorem 1. Finally, if $\overline{k} \in \{18, 26\}$ we know that $f(S,\delta) = 0$ by Corollary 1 and, moreover, $\mathrm{st}_3^*(\delta; S)$ is 6-connected if $k \neq 6$ by Proposition 7. Under these conditions we get $f_{k,\overline{k}}(S,\delta) = 0$ by definition. □

Proposition 9. *Let S be a surface in a regular $(k,\overline{k})$-space (R^3, f). Given a vertex $\alpha \in R^3$, let $C_\alpha \subset \mathbb{R}^3$ be the unit cube centered at α. If* $\mathrm{P}(S,\alpha) = \mathrm{P}_6^a$ *the two following properties hold:*

1. *$D_f^\alpha = |\mathcal{A}_S^f| \cap C_\alpha$ and $D_U^\alpha = |\mathcal{A}_S^{f_{k,\overline{k}}}| \cap C_\alpha$ are both 2-balls with common border; that is, $\partial D_f^\alpha = \partial D_U^\alpha$.*
2. *There exists a pl-homeomorphism $\varphi_\alpha : D_f^\alpha \to D_U^\alpha$ which extends the identity in the border; moreover, if $f(S,\alpha) = 0$ then $\varphi_\alpha = \mathrm{id}$.*

Proof. From the definition of the lighting function $f_{k,\overline{k}}$ it is readily checked that all the 2- and 3-cells $\delta > \alpha$, $\delta \in \mathrm{supp}(S)$, and also the two 1-cells $\beta_1, \beta_2 > \alpha$ such that $\mathrm{st}_3(\beta_i; S) = \mathrm{st}_3(\beta_i; R^3)$ are lighted for S, while $f_{k,\overline{k}}(S,\delta) = 0$ for any other cell $\delta \geq \alpha$. This way, the disk D_U^α is the union of the two unit squares defined by the centers of all voxels in $\mathrm{st}_3(\alpha; S)$. In particular, $D_U^\alpha = D_f^\alpha$ if $f(S,\alpha) = 0$ by Remark 4.

If $f(S,\alpha) = 1$ we know by Remark 3 that D_f^α is also a disk and from the above description of D_U^α it becomes clear that $\partial D_f^\alpha = \partial D_U^\alpha$. Moreover, φ_α can be defined as the conic extension of the identity that assigns the center $c(\alpha)$ to $c(\sigma \cap \tau)$, where $\sigma, \tau \in \mathrm{st}_3(\alpha; S)$ are the two only 3-cells which are 6-adjacent to three other 3-cells in that set.

Proof. (of Theorem 4) We claim that the polyhedra $|\mathcal{A}_S^f|$ and $|\mathcal{A}_S^{f_{k,\overline{k}}}|$ are *pl*-homeomorphic. Thus, the continuous analogues of S in both digital spaces (R^3, f) and $E_{k,\overline{k}}$ are combinatorial surfaces.

To ease the reading we will write $|\mathcal{A}_S^U|$ instead $|\mathcal{A}_S^{f_{k,\overline{k}}}|$, while keep $|\mathcal{A}_S^f|$ for the continuous analogue of S in the space (R^3, f). In order to define a *pl*-homeomorphism $\varphi : |\mathcal{A}_S^f| \to |\mathcal{A}_S^U|$ let us consider the sets of disks $\{D_f^\alpha = |\mathcal{A}_S^f| \cap C_\alpha\}_{\alpha \in A}$ and $\{D_U^\alpha = |\mathcal{A}_S^U| \cap C_\alpha\}_{\alpha \in A}$, where A stands for the set of vertices $\alpha \in R^3$ such that $\mathrm{P}(S,\alpha) = \mathrm{P}_6^a$ and $C_\alpha \subset \mathbb{R}^3$ is the unit cube centered at α. For each vertex $\alpha \in A$ we set $\varphi = \varphi_\alpha$, where $\varphi_\alpha : D_f^\alpha \to D_U^\alpha$ are the *pl*-homeomorphisms provided by Proposition 9, while from Proposition 8 we can define $\varphi : \overline{|\mathcal{A}_S^f| - \cup_{\alpha \in A} D_f^\alpha} \to \overline{|\mathcal{A}_S^U| - \cup_{\alpha \in A} D_U^\alpha}$ as the identity. Notice that $\varphi = \mathrm{id}$ if $f(S,\alpha) = 0$ for some vertex $\alpha \in A$, so we may assume that $f(S,\alpha) = 1$ for all of them. In order to check that φ and φ^{-1} are well defined it suffices to prove that $D_g^\alpha \cap D_g^{\alpha'} \subseteq$

$\partial D_g^\alpha \cap \partial D_g^{\alpha'}$ for each pair of distinct vertices $\alpha, \alpha' \in A$, where $g \in \{f, U\}$, and also $D_U^\alpha \cap (\overline{|\mathcal{A}_S^U| - \cup_{\lambda \in A} D_U^\lambda}) \subseteq \partial D_U^\alpha$ for each $\alpha \in A$.

Given $\alpha \in A$, let $\beta_1, \beta_2 > \alpha$ the two edges such that $\mathrm{st}_3(\beta_i; S) = \mathrm{st}_3(\beta_i; R^3)$ and $\gamma > \alpha$ the 2-cell having β_1 and β_2 as faces. From Remark 3 we know that $f(S, \delta) = 0$ for $\delta \in \{\gamma, \beta_1, \beta_2\}$ and then no other face of γ, but α, is lighted by f for S by Axiom 5. In particular, $\mathrm{P}(S, \alpha') \notin \mathbb{P}_{\overline{k}} \cup \{\mathrm{P}_6^a\}$ for each vertex $\alpha \neq \alpha' < \gamma$ and the result follows. $\square$

Theorem 4 suggests that, for each adjacency pair $(k, \overline{k}) \neq (6, 6)$, $k, \overline{k} \in \{6, 18, 26\}$, the $f_{k,\overline{k}}$-surfaces of the universal $(k, \overline{k})$-space $E_{k,\overline{k}}$ could be identified with $(k, \overline{k})$-surfaces of the discrete space $\mathbb{Z}^3$. This leads to the problem of characterizing the $f_{k,\overline{k}}$-surfaces just in terms of the adjacency pairs $(k, \overline{k})$, similarly to Morgenthaler's definition of $(26, 6)$-surfaces in [13].

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