Characterization of Simple Closed Surfaces in $\mathbb{Z}^3$: A New Proposition with a Graph-Theoretical Approach

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Abstract. In the present paper, we propose a topological characterization of digital surfaces. We introduce simple local conditions on the neighborhood of a voxel. If each voxel of a 26-connected digital set satisfies them, we prove a Jordan theorem and ensure that this set is strong 6-separating in $\mathbb{Z}^3$. Thus, we consider it as a digital surface.

Introduction

The mathematical study of digital images is composed of digital topology and digital geometry. They aim at providing digital analogues for the usual continuous geometrical and topological notions and should fit the conception of integer only algorithms.

The main difficulty in finding appropriate digital analogues comes from the fact that we consider a discrete space ($\mathbb{Z}^3$), that is, a set of isolated points with integer only coordinates, usually called voxels, in which continuity is no more defined and Euclidean distance can not be used easily.

Intuitively, a simple closed surface may be seen as a closed surface which *does not fold upon itself.* The study of their digital analogues, namely digital simple closed surfaces, remains a key point for representation of digital objects. These shapes are indeed well fitted for representing borders of digital objects. Moreover, they are at the crossing of digital geometry and digital topology: They should be good digital approximation of continuous simple closed surfaces and also should satisfy topological properties such as separating the space or being without singularities.

In the continuous case, it is possible to detect locally the singularities of a surface, this leads to a local characterization of simple closed surfaces. In the digital space $\mathbb{Z}^3$, we look for a local, that is, in a bounded neighborhood of the considered voxel, characterization allowing the same detection.

In the present paper, we study digital surfaces with a graph theoretical approach. A surface is then defined as a thin set of voxels linked by adjacency relations. Our contribution consists in a new notion of 26-connected digital surfaces. We propose some specific properties. If each voxel of a 26-connected digital set satisfies them in its neighborhood, then this set separates the space in two 6-connected components. Moreover, it is a strong 6-separating set in $\mathbb{Z}^3$, i.e., the removal of any of its voxels leads to a non-separating set.

S. Brlek, C. Reutenauer, and X. Provençal (Eds.): DGCI 2009, LNCS 5810, pp. 288–299, 2009.

In the sequel, we first recall some basic notions on digital adjacencies, sets and paths. Then, we present the well accepted definition of closed digital curves and introduce some notions of closed digital surfaces before stating our local characterization and proving a Jordan theorem for it.

1 Basic Notions

1.1 Digital Adjacencies and Connectedness

In the digital plane and the digital space, we consider different adjacency relations:

- two pixels $\mathbf{p}$ et $\mathbf{q} \in \mathbb{Z}^2$ are 4-*adjacent* (respectively 8-*adjacent*) if $\|\mathbf{p}-\mathbf{q}\|_1 = 1$ (respectively $\|\mathbf{p}-\mathbf{q}\|_\infty = 1$),
- two voxels $\mathbf{v}$ et $\mathbf{w} \in \mathbb{Z}^3$ are 6-*adjacent* (respectively 26-*adjacent*) if $\|\mathbf{p}-\mathbf{q}\|_1 = 1$ (respectively $\|\mathbf{p}-\mathbf{q}\|_\infty = 1$).

They lead to different types of digital neighborhoods, called k-neighborhood ($k \in \{4,8\}$ in the plane and $k \in \{6,26\}$ in the space). The *k-neighborhood* of a digital set E, $\mathcal{N}_k(\mathrm{E})$ is the set of voxels k-adjacent to at least one voxel of E.

$\mathcal{N}_k^\star(\mathrm{E})$ denotes $\mathcal{N}_k(\mathrm{E}) \setminus \mathrm{E}$ and, if F is also a digital set, $\mathcal{N}_k^{\mathrm{F}}(\mathrm{E})$ denotes $\mathcal{N}_k(\mathrm{E}) \cap \mathrm{F}$.

A sequence of voxels $\mathrm{P} = (\mathbf{p_i})_{1 \leqslant i \leqslant n_\mathrm{P}}$ is called a *k-connected digital path* if, for all $i \in \{1, \ldots, n_\mathrm{P} - 1\}$, $\mathbf{p_i}$ and $\mathbf{p_{i+1}}$ are k-adjacent. In the sequel, a path X contains n_X elements denoted by $\mathbf{x_1}, \ldots, \mathbf{x_{n_X}}$. P^{-1} is the reverse path of P and we have $(\mathrm{P}^{-1})^{-1} = \mathrm{P}$. If P and Q are two k-connected digital paths such that $\mathbf{p_{n_P}}$ and $\mathbf{q_1}$ are k-adjacent, $\mathrm{P} \cdot \mathrm{Q}$ means the concatenation of the paths P and Q, and we have $(\mathrm{P} \cdot \mathrm{Q})^{-1} = \mathrm{Q}^{-1} \cdot \mathrm{P}^{-1}$.

A digital set E is said to be k-connected if, for all $\mathbf{v}, \mathbf{w} \in \mathrm{E}$, there exists a k-connected path linking them. A digital set E is said to be *k-separating* in a superset F if the complement of E in F, $\overline{\mathrm{E}}$, admits exactly two distinct k-connected components. Moreover, if, for all $\mathbf{v} \in \mathrm{E}$, $\mathrm{E} \setminus \{\mathbf{v}\}$ is no more k-separating in F, E is a *strong k-separating* set in F.

1.2 Digital Curve

The notion of digital path is not sufficient to introduce topological properties and digital curves. A. Rosenfeld has proposed different improvements [1].

Definition 1 (Infinite digital curve [1]). *A k-*connected infinite digital curve *is a k-connected digital path* P *such that:*

1. $\mathbf{p_i} = \mathbf{p_j}$ *if and only if* $i = j$*,*
2. $\mathbf{p_i}$ *and* $\mathbf{p_j}$ *are k-adjacent if and only if* $j = i \pm 1$*.*

This definition is a discrete analogue to the notion of non closed simple curves if we consider infinite arc. It does not allow to reach closed simple curves without a slightly change.

Definition 2 (Closed digital curve [1]). *A* k-connected closed digital curve *is a* k*-connected digital path* P *such that:*

1. $\mathbf{p_i} = \mathbf{p_j}$ *if and only if* $i = j$,
2. $\mathbf{p_i}$ *and* $\mathbf{p_j}$ *are* k*-adjacent if and only if* $j = i \pm 1 \mod n_{\mathrm{P}}$.

This definition has the same properties as the latter, unless it ensures a k-adjacency relationship between the first voxel, $\mathbf{p_1}$, and the last one, $\mathbf{p_{n_P}}$.

They both can be expressed as strong separating digital set in the digital plane.

Proposition 1. *Let* E *be a digital subset of* $\mathbb{Z}^2$*. Then,* E *is an infinite or closed* 4*-connected (respectively* 8*-connected) digital curve if and only if* E *is a strong* 8*-separating (respectively strong* 4*-separating) digital set in* $\mathbb{Z}^2$*.*

Note that the adjacency relationship used for the curve and the adjacency relationship used for the complement are different. In the digital plane, good pair of adjacencies are $(4, 8)$ and $(8, 4)$, in the digital space, they are in $\{(26, 6), (18, 6), (6, 18), (6, 26)\}$, but we mainly focus on $(26, 6)$.

1.3 Digital Surface

The definition of digital curve as strong separating set is very simple but does not extend easily in higher dimension: R. Malgouyres has shown that there is no local characterization of strong 6-separating digital sets in $\mathbb{Z}^3$ [2]. D. G. Morgenthaler and A. Rosenfeld have early proposed a definition of digital surfaces more restrictive than the notions of strong separating set [3], the definition of digital (m, n)-surfaces. It involves two adjacency relations, the first, m, for the digital surface itself and the second, n, for its complement.

This definition is too restrictive, at least in the case of the $(26, 6)$-surfaces: even digital planes [4] are not included in such a class of digital surfaces. More general definitions, accepting digital planes, have been proposed, in particular, by R. Malgouyres and G. Bertrand [5,6,7]. Finally, the class of digital simplicity surfaces is the largest proposed [8,9] with the graph theoretical approach. It is based on the notion of simple points [10] and includes previously mentioned definitions. Nevertheless, this definition does not accept strong 6-separating sets which are good candidates to be digital surfaces. For example, the gray set drawn in Figure 1 does not satisfy the condition to be a simplicity surface.

2 A New Proposition of Digital Surfaces

2.1 Definition

Definition 3. *Let* E *be a* 26*-connected digital set such that, for all* $\mathbf{v} \in \mathrm{E}$*:*

1. $\mathcal{N}_{26}^{\overline{\mathrm{E}}}\left(\mathcal{N}_{26}^{\mathrm{E}}(\mathbf{v})\right)$ *admits exactly two* 6*-connected components,*
2. *for all* $\mathbf{w} \in \mathcal{N}_{26}^{\mathrm{E}}(\mathbf{v})$*,* $\mathbf{w}$ *is* 6*-connected to each of these* 6*-connected components.*

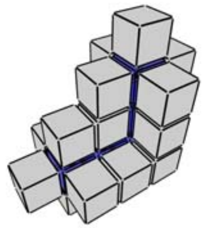

Fig. 1. Strong 6-separating digital set rejected by the definition of simplicity surfaces

Then E is called a digital surface.

The gray digital set drawn in Figure 1 is a digital surface according to definition 3.

Such a definition remains useless without a Jordan theorem. Throughout the present paper, S will denote a digital set satisfying Definition 3. In order to prove that the set S separates its complement $\overline{\mathrm{S}}$ in two distinct 6-connected components, we base our approach on the following assertion: considering two voxels not in S, the parity of the number of intersections with S remains the same for all paths between them. Then, by highlighting paths with odd and even number of intersections, we can state that S separates $\mathbb{Z}^3$. We need first to define what we mean by intersection:

Definition 4 (intersection). *Let* $\mathrm{P} = \mathrm{A} \cdot \mathrm{B} \cdot \mathrm{C}$ *be the concatenation of three* 6*-connected paths such that:*

1. $\mathbf{a_1}$, $\mathbf{a_{n_A}}$, $\mathbf{c_1}$, $\mathbf{c_{n_C}} \notin \mathrm{S}$
2. *for all* $i \in \{1, \ldots, n_\mathrm{B}\}$, $\mathbf{b_i} \in \mathrm{S}$.

Then, B *is called an* intersection *of* P *with* S*, if* $\mathbf{a_{n_A}}$, $\mathbf{c_1}$ *belong to distinct* 6*-connected components of* $\mathcal{N}_{26}(\mathrm{B}) \cap \overline{\mathrm{S}}$.

For the sake of clarity, we denote by $\mathbb{I}_S(\mathrm{P})$ the set of all intersections of P with S and by $\sharp\mathbb{I}_S(\mathrm{P})$ its cardinality, that is the number of intersections between P and S.

The property we look for can be formalized by: two 6-connected digital paths with same end points not in S, namely P and P′, should satisfy:

$$\sharp\mathbb{I}_S(\mathrm{P}) \equiv \sharp\mathbb{I}_S(\mathrm{P}') \pmod 2. \quad (1)$$

A 6-connected path P between voxels not in S can be viewed as the concatenation of n paths, $\mathrm{R_1}, \ldots, \mathrm{R_n}$ such that for all $i \in \{1, \ldots, n\}$, $\mathrm{R_i}$ is a path such that its end points are not in S. According to this decomposition:

$$\sharp\mathbb{I}_\mathrm{S}(\mathrm{P}) = \sum_{i=1}^{n} \sharp\mathbb{I}_\mathrm{S}(\mathrm{R_i}). \quad (2)$$

3 Jordan Theorem

To prove a Jordan's theorem for our digital surfaces, we reduce the problem according to different results. First, thanks to digital homotopy, we only study directly equivalent paths. Then, results on the neigborhood of an intersection between a path and a digital surface allow us to only consider two sorts of paths in our proofs: a path constituted by a unique voxel not in S and a path constituted of three elements with end points not in S and middle point in it. Endly, we highlight local deformations which preserve the parity of the number of intersections of paths with S.

3.1 Digital Homotopy

We first present some results on digital homotopy due to T. Y. Kong [11] which allow us to later restrict our study.

Definition 5 (Unit square [11]). *Let* $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ *be the canonical basis of the Euclidean three-dimensional space. Then a* unit square K *is a set of four pixels such that:*

$$\mathrm{K} = \{\mathbf{v}, \mathbf{v} + \mathbf{e_i}, \mathbf{v} + \mathbf{e_j}, \mathbf{v} + \mathbf{e_i} + \mathbf{e_j}\},$$

with $(i, j) \in \{1, 2, 3\}^2$ *and* $i \neq j$.

Definition 6 (Reduced form of digital path [11]). *Let* P *be a* 6*-connected digital path. The* reduced form R *of* P *is obtained by removing all but one voxel from every set of consecutive equal voxels. Thus,* R *is such that for all* $i \in \{1, \ldots, r_{n_\mathrm{R}} - 1\}$, $\mathbf{r_i} \neq \mathbf{r_{i+1}}$ *and we write:*

$$\mathrm{P} \equiv \mathrm{R}. \tag{3}$$

The equivalence and the direct equivalence between digital paths are then define as follows:

Definition 7 (Directly equivalent digital paths [11]). *Let* P *and* P′ *be two* 6*-connected digital paths in a digital set* E *such that* $\mathbf{p_1} = \mathbf{p'_1}$ *and* $\mathbf{p_{n_P}} = \mathbf{p'_{n_{P'}}}$. P *and* P′ *are directly equivalent if one of the following properties is satisfied:*

- P *and* P′ *have the same reduced form,*
- P *and* P′ *differ only in a unit square* K.

Definition 8 (Equivalent digital paths [11]). *Let* P *and* P′ *be two* 6*-connected digital paths with same end points in a digital set* E. P *and* P′ *are said to be equivalent if it exists a sequence of* 6*-connected digital paths* $\mathrm{P_1}, \ldots, \mathrm{P_n}$ *such that:*

1. P *is directly equivalent to* $\mathrm{P_1}$,
2. P′ *is directly equivalent to* $\mathrm{P_n}$,
3. *for all* $i \in \{1, \ldots, n-1\}$, $\mathrm{P_i}$ *is directly equivalent to* $\mathrm{P_{i+1}}$.

According to Definition 8, we limit our study to prove that two directly equivalent digital paths in $\mathbb{Z}^3$ have the same parity for their number of intersections with S. The first case of Definition 7 can be directly stated:

Proposition 2. *Let* P *and* P′ *be two* 6*-connected paths such that* $\mathrm{P} \equiv \mathrm{P}'$*. Then, we have:*

$$\sharp\mathbb{I}_{\mathrm{S}}(\mathrm{P}) = \sharp\mathbb{I}_{\mathrm{S}}(\mathrm{P}'). \tag{4}$$

Proof. The repetition of a voxel in a path cannot add or delete intersection since such a configuration involves at least three voxels. □

In the second case of Definition 7, the preservation of the parity of $\sharp\mathbb{I}_{\mathrm{S}}(\mathrm{P})$ requires more investigation and we introduce some useful technical lemmas before stating it.

3.2 Preliminaries

We present results on the number of 6-connected components adjacent to a path included in S, we will use thoughout the next part.

Lemma 1. *Let* P *be a* 6*-connected digital path with all of its elements in a surface* S *and such that* $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P})\right)$ *admits* c 6*-connected components. Let also* $\mathbf{v}$ *be a voxel belonging to* S *and* 6*-adjacent to* $\mathbf{p_{n_P}}$*. Then,* $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P}\cdot\mathbf{v})\right)$ *admits at most* c 6*-connected components.*

Proof. By definition of S, $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathbf{v})\right)$ admits only two 6-connected components and $\mathbf{p_{n_P}}$ is 6-adjacent to both of them. The voxels included in $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathbf{v})\right)$ and not in $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P})\right)$ are all 6-connected to voxels of $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P})\right)$. Consequently, $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P}\cdot\mathbf{v})\right)$ admits at most c 6-connected components. □

Proposition 3. *Let* P *be a* 6*-connected digital path with all of its elements in a surface* S*. Then* $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P})\right)$ *admits at most two distinct* 6*-connected components.*

Proof. P is at least constituted of a voxel $\mathbf{v} \in \mathrm{S}$. By definition of S, $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathbf{v})\right)$ admits exactly two 6-connected components. Then, by Lemma 1 and induction, $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P})\right)$ admits at most two 6-connected components. □

A stronger result can be enounced in the case of intersections:

Corollary 1. *Let* I *be an intersection of a* 6*-connected digital path* P *with a surface* S*. Then* $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{I})\right)$ *admits exactly two distinct* 6*-connected components.*

Proof. By definition of an intersection, this set admits at least two 6-connected components. By Proposition 3, it admits at most two 6-connected components. □

A last result on connected components in the complement of a path in a digital surface S is:

Lemma 2. *Let* P *be a* 6*-connected digital path with all of its elements in a surface* S*. Then, for all* $i \in \{1, \ldots, n_{\mathrm{P}}\}$*,* $\mathbf{p_i}$ *is* 6*-adjacent to each* 6*-connected component of* $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P})\right)$.

Proof. First, consider that $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P})\right)$ admits only one 6-connected component. By definition of S, for all $i \in \{1, \ldots, n_{\mathrm{P}}\}$, $\mathbf{p_i}$ is 6-adjacent to at least two voxels of $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathbf{p_i})\right)$ Since $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P})\right) = \bigcup_{i=1}^{n_{\mathrm{P}}}\left(\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathbf{p_i})\right)\right)$, $\mathbf{p_i}$ is 6-adjacent to the 6-connected set $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P})\right)$.

Then, consider that $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P})\right)$ admits exactly two 6-connected components:

1. By Definition 3, $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathbf{p_1})\right)$ admits two 6-connected components such that $\mathbf{p_1}$ is 6-adjacent to both of them.
2. Now suppose that $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\{\mathbf{p_1}, \ldots, \mathbf{p_k}\})\right)$ with $1 < k < n_{\mathrm{P}}$ admits two 6-connected components and for all $i \in \{1, \ldots, k\}$, $\mathbf{p_i}$ is 6-adjacent to both of them. By condition 2 of Definition 3, $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathbf{p_{k+1}})\right)$ admits two 6-connected components such that $\mathbf{p_k}$ and $\mathbf{p_{k+1}}$ are 6-adjacent to both of them. Since $\mathbf{p_k}$ is 6-adjacent to both 6-connected components of $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\{\mathbf{p_1}, \ldots, \mathbf{p_k}\})\right)$, so is $\mathbf{p_{k+1}}$.
3. By induction, for all $i \in \{1, \ldots, n_{\mathrm{P}}\}$, $\mathbf{p_i}$ is 6-adjacent to the two 6-connected components of $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\left(\mathcal{N}_{26}^{\mathrm{S}}(\mathrm{P})\right)$.

□

Thanks to those results, we deduce some simplifications on the problem we consider and formalized by Equation 1. We have:

Lemma 3. *Let* $\mathrm{P} = \mathbf{w_1} \cdot \mathrm{Q} \cdot \mathbf{w_2}$ *be a* 6*-connected digital path such that* Q *is only composed of voxels of* S *and* $\mathbf{w_1}$ *and* $\mathbf{w_2} \notin \mathrm{S}$*. Then, for all* $i \in \{1, \ldots, n_{\mathrm{Q}}\}$*, it exists voxels* $\mathbf{w_1'}$ *and* $\mathbf{w_2'}$ 6*-adjacent to* $\mathrm{q_i}$ *such that:*

$$\sharp\mathbb{I}_{\mathrm{S}}(\mathrm{P}) = \sharp\mathbb{I}_{\mathrm{S}}(\mathbf{w_1'} \cdot \mathbf{q_i} \cdot \mathbf{w_2'}). \tag{5}$$

Proof. By Proposition 3 and Lemma 2, it exists $\mathbf{w_1'}$ and $\mathbf{w_2'}$ 6-adjacent to $\mathbf{q_i}$ such that it exists 6-connected paths in $\mathcal{N}_{26}(\mathrm{Q}) \cap \overline{\mathrm{S}}$ linking, respectively, $\mathbf{w_1}$ and $\mathbf{w_1'}$, and, $\mathbf{w_2}$ and $\mathbf{w_2'}$.

In the sequel, we will consider deformations which are the substitution in a path of a unique voxel $\mathbf{v}$ by a specific path with end points on $\mathbf{v}$. If the unique voxel is not in S, we can consider without loss of generality a path only constituted by $\mathbf{v}$. When it belongs to S, we consider, again, without loss of generality, a path $\mathbf{w_1} \cdot \mathbf{v} \cdot \mathbf{w_2}$ with $\mathbf{w_1}, \mathbf{w_2} \notin \mathrm{S}$.

3.3 Deformation in a Unit Square

Studying the number of intersections, with a surface S, of digital paths differing in K, requires to take in account all possible subpaths in K. We consider here different local configurations, namely U-turn, minimal loop and minimal path, to define equivalence classes and reduce the problem.

Adding or deleting a U-turn in a digital path. A U-turn is the following configuration:

Definition 9 (U-turn). *A* U-turn *is a* 6*-connected digital path* $\mathrm{U} = (\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3})$ *such that* $\mathbf{u_1} = \mathbf{u_3}$. *A* pointed U-turn $\mathrm{U}(\mathbf{v})$ *is a U-turn such that* $\mathbf{u_1} = \mathbf{v}$.

The deletion or the addition of a U-turn in a 6-connected digital path does not change the parity of its number of intersections with a digital surface:

Lemma 4. *Let* $\mathbf{v} \in \mathbb{Z}^3$. *Let also* Q *and* R *be two* 6*-connected digital paths such that* $\mathrm{P} = \mathrm{Q}\cdot\mathbf{v}\cdot\mathrm{R}$ *and* $\mathrm{P}' = \mathrm{Q}\cdot\mathrm{U}(\mathbf{v})\cdot\mathrm{R}$ *are* 6*-connected digital paths with extreme points not in* S. *Then, we have:*

$$\sharp\mathbb{I}_S(\mathrm{P}) \equiv \sharp\mathbb{I}_S(\mathrm{P}') \pmod 2. \tag{6}$$

Proof.

1. $\mathbf{v} \notin \mathrm{S}$: without loss of generality, we consider $\mathrm{P} = \mathbf{v}$. Thus, we have $\sharp\mathbb{I}_\mathrm{S}(\mathrm{P}) = 0$.
   (a) $\mathbf{u_2} \notin \mathrm{S}$: $\mathrm{P}' = \mathrm{U}(\mathbf{v})$ contains no voxel of S, thus we have $\sharp\mathbb{I}_S(\mathrm{P}') = 0$.
   (b) $\mathbf{u_2} \in \mathrm{S}$: $\mathbf{u_2}$ is not an intersection of P′ with S since $\mathbf{u_1} = \mathbf{u_3}$. Thus, we have $\sharp\mathbb{I}_S(\mathrm{P}') = 0$.
2. $\mathbf{v} \in \mathrm{S}$: without loss of generality, we consider $\mathrm{P} = \mathbf{w_1}\cdot\mathbf{v}\cdot\mathbf{w_2}$ with $\mathbf{w_1}$ and $\mathbf{w_2} \notin \mathrm{S}$.
   (a) $\mathbf{u_2} \notin \mathrm{S}$:
      i. $\mathbf{v} \in \mathbb{I}_\mathrm{S}(\mathrm{P})$: it means that $\sharp\mathbb{I}_S(\mathrm{P}) = 1$ and that $\mathbf{w_1}$ and $\mathbf{w_2}$ are in distinct 6-connected components of $\mathcal{N}_{26}(\mathbf{v}) \cap \overline{\mathrm{S}}$. Either $\mathbf{u_2}$ is in the component of $\mathbf{w_1}$ or of $\mathbf{w_2}$. Without loss of generality, we focus on the first case (for the second one, consider P^{-1}). $\mathbf{u_1}$ is not an intersection of P′ with S while $\mathbf{u_3}$ is such an intersection. Then, we have $\sharp\mathbb{I}_S(\mathrm{P}') = \sharp\mathbb{I}_S(\mathrm{P})$.
      ii. $\mathbf{v} \notin \mathbb{I}_\mathrm{S}(\mathrm{P})$: it means that $\sharp\mathbb{I}_S(\mathrm{P}) = 0$ and that $\mathbf{w_1}$ and $\mathbf{w_2}$ are in the same 6-connected components of $\mathcal{N}_{26}(\mathbf{v}) \cap \overline{\mathrm{S}}$. Either $\mathbf{u_2}$ is in this component and both $\mathbf{u_1}$ and $\mathbf{u_3}$ are not intersections of P′ with S, or $\mathbf{u_2}$ is in the other component and both $\mathbf{u_1}$ and $\mathbf{u_3}$ are intersections of P′ with S. To sum up, we have $\sharp\mathbb{I}_S(\mathrm{P}) = \sharp\mathbb{I}_S(\mathrm{P}')$.
   (b) $\mathbf{u_2} \in \mathrm{S}$: since $\mathbf{v} \in \mathrm{S}$ and $\mathrm{U}(\mathbf{v}) \subseteq \mathrm{S}$, $\sharp\mathbb{I}_S(\mathrm{P}) = \sharp\mathbb{I}_S(\mathrm{P}')$.

□

A direct consequence concerns path containing the concatenation of a subpath and its reverse path:

Corollary 2. *Let* A*,* B *and* C *be three* 6*-connected paths such that* $P = A \cdot B \cdot B^{-1} \cdot C$ *and* $\mathbf{a_1}$*,* $\mathbf{c_{n_C}} \notin S$*. Then, we have:*

$$\sharp\mathbb{I}_S\,(\mathrm{P}) \equiv \sharp\mathbb{I}_S\,(\mathrm{A}\cdot\mathbf{b_1}\cdot\mathrm{C}) \pmod 2. \tag{7}$$

Proof. $B \cdot B^{-1}$ is such that $(\mathbf{b_1}, \dots, \mathbf{b_{n_B}}, \mathbf{b_{n_B}}, \dots, \mathbf{b_1})$. For all voxel $\mathbf{v}$ and $\mathbf{w_1}$, $\mathbf{w_2} \notin S$, the path $\mathbf{w_1}\cdot\mathbf{v}\cdot\mathbf{v}\cdot\mathbf{w_2}$ is directly equivalent to the path $\mathbf{w_1}\cdot\mathbf{v}\cdot\mathbf{w_2}$ and such a deformation trivially does not modify the number of intersections with S. Then, we have:

$$\begin{aligned}
\sharp\mathbb{I}_S\,(\mathrm{P}) &\equiv \sharp\mathbb{I}_S\,(\mathrm{A}\cdot(\mathbf{b_1},\dots,\mathbf{b_{n_B-1}},\mathbf{b_{n_B}},\mathbf{b_{n_B-1}},\dots,\mathbf{b_1})\cdot\mathrm{C}) \pmod 2,\\
&\equiv \sharp\mathbb{I}_S\,(\mathrm{A}\cdot(\mathbf{b_1},\dots,\mathbf{b_{n_B-2}})\cdot\mathrm{U}(\mathbf{b_{n_B-1}})\cdot(\mathbf{b_{n_B-2}},\dots,\mathbf{b_1})\cdot\mathrm{C}) \pmod 2,\\
&\equiv \sharp\mathbb{I}_S\,(\mathrm{A}\cdot(\mathbf{b_1},\dots,\mathbf{b_{n_B-2}},\mathbf{b_{n_B-1}},\mathbf{b_{n_B-2}},\dots,\mathbf{b_1})\cdot\mathrm{C}) \pmod 2,\\
&\equiv \dots,\\
&\equiv \sharp\mathbb{I}_S\,(\mathrm{A}\cdot(\mathbf{b_1},\mathbf{b_2},\mathbf{b_1})\cdot\mathrm{C}) \pmod 2,\\
&\equiv \sharp\mathbb{I}_S\,(\mathrm{A}\cdot\mathrm{U}(\mathbf{b_1})\cdot\mathrm{C}) \pmod 2,\\
&\equiv \sharp\mathbb{I}_S\,(\mathrm{A}\cdot\mathbf{b_1}\cdot\mathrm{C}) \pmod 2,
\end{aligned}$$

□

Adding or deleting a minimal loop in a digital path. The second local configuration we study is the minimal loop:

Definition 10 (Minimal loop). *A* minimal loop *is a* 6*-connected path* $L = (\mathbf{l_i})_{1 \leqslant i \leqslant 5}$ *such that:*

1. L *does not contains any U-turn,*
2. $\mathbf{l_1} = \mathbf{l_5}$.

A pointed minimal loop $L(\mathbf{v})$ *is a minimal loop such that* $\mathbf{l_1} = \mathbf{v}$*.*

Note that a loop is included in a unit square and is a parametrisation of it.

As in the case of a U-turn, the addition or deletion of a minimal loop in a path preserves the parity of its number of intersections with a digital surface.

Lemma 5. *Let* $\mathbf{v} \in \mathbb{Z}^3$ *be a voxel. Let also* Q *and* R *be* 6*-connected digital paths such that* $P = Q \cdot (\mathbf{v}) \cdot R$ *and* $P' = Q \cdot L(\mathbf{v}) \cdot R$ *are* 6*-connected digital paths with extreme points not in* S*. Then, we have:*

$$\sharp\mathbb{I}_S\,(\mathrm{P}) \equiv \sharp\mathbb{I}_S\,(\mathrm{P}') \pmod 2. \tag{8}$$

Proof. This proof is similar to the one of Lemma 4. □

Changing the irreductible path of a unit square in a digital path The last specific configuration presented is the minimal path in a unit square.

Definition 11 (minimal path in a unit square). *Let* K *be a unit square. A path is said to be* minimal in K *if it is a strict subpath of a minimal loop included in* K*. Such a path between voxels* $\mathbf{v}$ *and* $\mathbf{w}$ *in* K *is denoted as* $M_K(\mathbf{v}, \mathbf{w})$*.*

Considering two voxels in a unit square, there exists only two minimal path linking them.

Definition 12. *Let* K *be a unit square and* $\mathrm{M_K}(\mathbf{v}, \mathbf{w})$ *be a minimal path between voxels* $\mathbf{v}$ *and* $\mathbf{w}$ *in* K*. Then, the only other minimal path in* K *with same extreme points is denoted by* $\overline{\mathrm{M_K}}(\mathbf{v}, \mathbf{w})$*. Let* $\mathrm{L}(\mathbf{v}) = \mathrm{A} \cdot \mathbf{w} \cdot \mathrm{B}$ *such that* $\mathrm{M_K}(\mathbf{v}, \mathbf{w}) = \mathrm{A} \cdot \mathbf{w}$*. Then,* $\overline{\mathrm{M_K}}(\mathbf{v}, \mathbf{w})$ *is such that:*

$$\mathrm{M_K}(\mathbf{v}, \mathbf{w}) \cdot \left(\overline{\mathrm{M_K}}(\mathbf{v}, \mathbf{w})\right)^{-1} = \mathrm{A} \cdot \mathbf{w} \cdot \mathbf{w} \cdot \mathrm{B} \equiv \mathrm{L}(\mathbf{v}). \quad (9)$$

Substitute a minimal path by its complementary minimal path does not change the parity of the number of intersections with a digital surface.

Lemma 6. *Let* K *be a unit square and* $(\mathbf{v}, \mathbf{w}) \in \mathrm{K}^2$*. Let also* Q *and* R *be two* 6*-connected digital paths such that* $\mathrm{P} = \mathrm{Q} \cdot \mathrm{M_K}(\mathbf{x}, \mathbf{y}) \cdot \mathrm{R}$ *is a* 6*-connected digital path with extreme points not in* S*. Then, we have:*

$$\sharp\mathbb{I}_S\left(\mathrm{P}\right) = \sharp\mathbb{I}_S\left(\mathrm{Q} \cdot \overline{\mathrm{M_K}}(\mathbf{v}, \mathbf{w}) \cdot \mathrm{R}\right) \pmod 2. \quad (10)$$

Proof.

$$\begin{aligned}
\sharp\mathbb{I}_S\left(\mathrm{P}\right) &= \sharp\mathbb{I}_S\left(\mathrm{Q} \cdot \mathrm{M_K}(\mathbf{v}, \mathbf{w}) \cdot \mathrm{R}\right) \\
&\equiv \sharp\mathbb{I}_S\left(\mathrm{Q} \cdot \mathrm{L}(\mathbf{v})^{-1} \cdot \mathrm{M_K}(\mathbf{v}, \mathbf{w}) \cdot \mathrm{R}\right) \pmod 2 \\
&\equiv \sharp\mathbb{I}_S\left(\mathrm{Q} \cdot \overline{\mathrm{M_K}}(\mathbf{v}, \mathbf{w}) \cdot \mathrm{M_K}(\mathbf{v}, \mathbf{w})^{-1} \cdot \mathrm{M_K}(\mathbf{v}, \mathbf{w}) \cdot \mathrm{R}\right) \pmod 2 \\
&\equiv \sharp\mathbb{I}_S\left(\mathrm{Q} \cdot \overline{\mathrm{M_K}}(\mathbf{v}, \mathbf{w}) \cdot \mathbf{w} \cdot \mathrm{R}\right) \pmod 2 \\
&\equiv \sharp\mathbb{I}_S\left(\mathrm{Q} \cdot \overline{\mathrm{M_K}}(\mathbf{v}, \mathbf{w}) \cdot \mathrm{R}\right) \pmod 2
\end{aligned}$$

□

According to results on U-turns, minimal loops and paths, we can conclude on the preservation of the parity of the number of intersections between a path and a digital surface in the case of the second condition of Definition 7.

Proposition 4. *Consider two* 6*-connected digital paths* P *and* P′ *with end points not in* S *such that they differ only in a unit square* K*. Then, we have:*

$$\sharp\mathbb{I}_S\left(\mathrm{P}\right) \equiv \sharp\mathbb{I}_S\left(\mathrm{P}'\right) \pmod 2. \quad (11)$$

Proof. Deleting or adding U-turns and consecutive duplicates preserves the parity of the number of intersections with S. Without loss of generality, the study is reduced to subpaths in K constituted by the concatenation of several (0 to infinity) minimal loops and a minimal path in K. The same occurs with minimal loops. Again, without loss of generality, the study reduced and we only have to consider two subpaths between the input voxel $\mathbf{v}$ and the output voxel K of the paths P and P′, namely $\mathrm{M_K}(\mathbf{v}, \mathbf{w})$ and $\overline{\mathrm{M_K}}(\mathbf{v}, \mathbf{w})$. Lemma 6 allows to conclude and prove the proposition. □

3.4 Main Theorem

The following theorem is a direct consequence of Proposition 2 and Proposition 4:

Theorem 1. *Consider two directly equivalent* 6*-connected digital paths* P *and* P′ *with end points not in* S*. Then, we have:*

$$\sharp\mathbb{I}_S\,(\mathrm{P}) \equiv \sharp\mathbb{I}_S\,(\mathrm{P}') \pmod 2. \tag{12}$$

And, if we consider now Definition 8 about equivalent digital paths, we have:

Corollary 3. *Consider two* 6*-connected digital paths* P *and* P′ *linking two voxels* **a** *and* **b***. Then, we have:*

$$\sharp\mathbb{I}_S\,(\mathrm{P}) \equiv \sharp\mathbb{I}_S\,(\mathrm{P}') \pmod 2. \tag{13}$$

Since all path in $\mathbb{Z}^3$ are equivalent, we can easily define path with no intersection or path with only one intersection with S. We can define a partition of $\overline{\mathrm{S}}$ into two sets according to a given voxel $\mathbf{v} \notin \mathrm{S}$: the voxels linked to **v** by a path having an even number of intersections with S and the ones linked to **v** by a path having an odd number of intersections with S. Hence, we prove that S separates the space in at least two 6-connected components. Moreover, we prove a stronger result:

Proposition 5. S *is a* 6*-separating set (in* $\mathbb{Z}^3$*).*

Proof. Suppose that $\overline{\mathrm{S}}$ admits more than two 6-connected components. Since S is 26-connected, either it exists a voxel $\mathbf{v} \in \mathrm{S}$ adjacent to more than two 6-connected components, or it exists two 26-adjacent voxels $\mathbf{v}, \mathbf{w} \in \mathrm{S}$ such that $\mathcal{N}_{26}^{\overline{\mathrm{S}}}\,(\{\mathbf{v},\mathbf{w}\})$ admits three 6-connected components. By definition of S, neither the first case nor the second one can appear and $\overline{\mathrm{S}}$ admits exaclty two 6-connected components. □

We can now state the main result of the present paper:

Theorem 2. S *is a strong* 6*-separating set (in* $\mathbb{Z}^3$*).*

Proof. Proposition 5 ensures that $\overline{\mathrm{S}}$ admits exactly two distinct 6-connected components. Moreover, by definition of S, for all voxel $\mathbf{v} \in \mathrm{S}$, it exists two voxels $\mathbf{w_1}$ and $\mathbf{w_2} \notin \mathrm{S}$ 6-adjacent to **v** such that $\sharp\mathbb{I}_\mathrm{S}(\mathbf{w_1}\cdot\mathbf{v}\cdot\mathbf{w_2}) = 1$. It means that each voxel of S is 6-adjacent to both components of $\mathbb{Z}^3 \cap \overline{\mathrm{S}}$. □

Conclusion

In this paper, we have introduced a new notion of 26-connected digital surfaces which extends the one of digital curves. The local characterization stays very simple since it is only based on usual digital adjacency relations. A Jordan theorem is proved thanks to the fact that the neighborhood in $\overline{\mathrm{S}}$ of a crossing between a digital surface S and a 6-connected digital path admits at most two 6-connected components. Finally, we present digital surfaces consisting in 26-connected and strong 6-separating sets. On the same principle, it is possible to define digital surfaces consisting in 6-connected and strong 26-separating sets.

At this point, the main drawback of our characterization is the fact that we consider, for each voxel, a neighborhood larger than the 26-neighborhood. From a practical point of view, deeper investigations are required to limit the study to the 26-neighborhood and, thus, reduce the computation.

The main task to achieved is now to compare our notion of simple closed digital surfaces to previous ones. The class of simplicity surfaces is currently the largest one. The comparison with this class required some technical works. Indeed characterizations are not defined with the same tools (simplicity graph, geodesic neigborhood,...). The easiest way to proceed seems to be the study and the comparison of the sets of admissible local configurations for each definition.

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