Thinning Algorithms as Multivalued $\mathcal{N}$-Retractions

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Abstract. In a recent paper we have introduced a notion of continuity in digital spaces which extends the usual notion of digital continuity. Our approach, which uses multivalued maps, provides a better framework to define topological notions, like retractions, in a far more realistic way than by using just single-valued digitally continuous functions. In particular, we characterized the deletion of simple points, one of the most important processing operations in digital topology, as a particular kind of retraction.

In this work we give a simpler algorithm to define the retraction associated to the deletion of a simple point and we use this algorithm to characterize some well known parallel thinning algorithm as a particular kind of multivalued retraction, with the property that each point is retracted to its neighbors.

Keywords: Digital images, digital topology, continuous multivalued function, simple point, retraction, thinning algorithm.

1 Introduction

The notion of continuous function is the fundamental concept in the study of topological spaces, therefore it should play an important role in Digital Topology.

There have been some attempts to define a reasonable notion of continuous function in digital spaces. The first one goes back to A. Rosenfeld [14] in 1986. He defined continuous function in a similar way as it is done for continuous maps in $\mathbb{R}^n$. It turned out that continuous functions agreed with functions taking 4-adjacent points into 4-adjacent points or, equivalently, with functions taking connected sets to connected sets.

More results related with this type of continuity were proved by L. Boxer in [1,2,3,4]. In these papers, he introduces such notions as homeomorphism, retracts and homotopies for digitally continuous functions, applying these notions to define a digital fundamental group, digital homotopies and to compute the fundamental group of sphere-like digital images. However, as he recognizes in [3],

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there are some limitations with the homotopy equivalences he get. For example, while all simple closed curves are homeomorphic and hence homotopically equivalent with respect to the Euclidean topology, in the digital case two simple closed curves can be homotopically equivalent only if they have the same cardinality.

A different approach was suggested by V. Kovalevsky in [12], using multivalued maps. He calls a multivalued function continuous if the pre-image of an open set is open. He considers, however, that another important class of multivalued functions is that of connectivity preserving mappings. The multivalued approach to continuity in digital spaces has also been used by R. Tsaur and M. Smyth in [15], where a notion of continuous multifunction for discrete spaces is introduced: A multifunction is continuous if and only if it is "strong" in the sense of taking neighbors into neighbors with respect to Hausdorff metric. See the introduction of [5] for a discussion of the limitations of all these multivalued approaches.

In a recent paper [5] the authors presented a theory of continuity in digital spaces which extends the one introduced by Rosenfeld. Our approach uses multivalued maps and provides a better framework to define topological notions, like retractions, in a far more realistic way than by using just single-valued digitally continuous functions. In particular, we characterized the deletion of simple points, one of the most important processing operations in digital topology, as a particular kind of retraction.

In this work we deepen into the properties of this family of continuous maps, now concentrating on parallel deletion of simple points and thinning algorithms.

In section 2 we revise the basic notions on digital topology required throughout the paper. In section 3 we recall the notion of continuity for multivalued functions and its basic properties, in particular those related with the notion of a digital retraction. In section 4 we give a new proof of our previous result characterizing the deletion of simple points in terms of digitally continuous multivalued functions, giving a new simpler algorithm to find the multivalued function associated to the deletion of a simpler point. Sections 5 is devoted to parallel deletion of simple points. We use our new algorithm to characterize some well known parallel thinning algorithms as digital multivalued retractions with the property that each point is retracted to its neighbors.

For information on Digital Topology we recommend the survey [10] and the books by Kong and Rosenfeld [11], and by Klette and Rosenfeld [8].

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2 Single Valued Continuity in Digital Spaces

We consider $\mathbb{Z}^2$ as model for the digital plane.

Two points in the digital plane $\mathbb{Z}^2$ are 8-adjacent if they are different and their coordinates differ in at most a unit. They are said 4-adjacent if they are 8-adjacent and differ in at most a coordinate. Given $p \in \mathbb{Z}^2$ we define $\mathcal{N}(p)$ as the set of points 8-adjacent to p, i.e. $\mathcal{N}(p) = \{p_1, p_2, \ldots, p_8\}$. This is also denoted as $\mathcal{N}_8(p)$. Analogously, $\mathcal{N}_4(p)$ is the set of points 4-adjacent to p (with the above notation $\mathcal{N}_4(p) = \{p_2, p_4, p_6, p_8\}$).

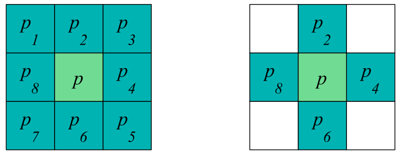

Fig. 1. $\mathcal{N}_8(p)$ and $\mathcal{N}_4(p)$ with their points labeled as used in the paper

A k-path P in $\mathbb{Z}^2$ ($k \in \{4, 8\}$) from the point q_0 to the point q_r is a sequence $P = \{q_0, q_1, q_2, \ldots, q_r\}$ of points such that q_i is k-adjacent to q_{i+1}, for every $i \in \{0, 1, 2, \ldots, r-1\}$. If $q_0 = q_r$ then it is called a closed path. A set $S \subset \mathbb{Z}^2$ is k-connected if for every pair of points of S there exists a k-path contained in S joining them. A k-connected component of S is a k-connected maximal set.

Jordan curve theorem for $\mathbb{R}^2$ states that any simple closed arc divides the plane in 2 connected components. A classical result in digital topology states that any simple closed k-arc divides the digital plane in two $\bar{k}$-connected components, where $\bar{k} = 4$ if $k = 8$, $\bar{k} = 8$ if $k = 4$ and a closed k-arc is a closed k-path $P = \{q_0, q_1, q_2, \ldots, q_r = q_0\}$ such that the only pairs of k-adjacent points of P are $(q_0, q_1), (q_1, q_2), \ldots, (q_{r-1}, q_r)$. Through the paper, given k, $\bar{k}$ will denote the complementary adjacency $\bar{k} = 12 - k$.

Let $f : X \subset \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ be a function. According to [14], f is (k, k')-continuous if and only if f sends k-adjacent points to k'-adjacent points.

Examples of (k, k)-continuous functions are the identity, any constant map, translations $f(z) = z + r$, inversions $f(z_1, z_2) = (z_2, z_1)$,... On the other hand, a expansion like $f(z) = 2z$ is not continuous if X is connected with more than one point. Moreover, any $(k, 4)$-continuous function is $(k, 8)$-continuous and any $(8, k')$-continuous function is $(4, k')$-continuous.

In [14], Rosenfeld stated and proved several results about digitally continuous functions related to operations with continuous functions, intermediate values property, almost-fixed point theorem, Lipschitz conditions, one-to-oneness, ... Boxer [1,2,3] expanded this notion to digital homeomorphisms, retractions, extensions, homotopies, digital fundamental group, induced homomorphisms, ... (see also [7] and [9] for previous related results).

3 Digitally Continuous Multivalued Functions

In the following definition we recall the concept of subdivision of $\mathbb{Z}^2$. This notion and the ones that follow can be defined for $\mathbb{Z}^n$ (the general definition can be found in [5]).

Definition 1. *The first subdivision of $\mathbb{Z}^2$ is formed by the set*

$$\mathbb{Z}_1^2 = \left\{ \left(\frac{z_1}{3}, \frac{z_2}{3} \right) \mid (z_1, z_2) \in \mathbb{Z}^2 \right\}$$

and the $3:1$ *map* $i:\mathbb{Z}^2_1 \hookrightarrow \mathbb{Z}^2$ *given by* $i\left(\frac{z_1}{3},\frac{z_2}{3}\right) = (z'_1, z'_2)$ *where* (z'_1, z'_2) *is the point in* $\mathbb{Z}^2$ *closer to* $\left(\frac{z_1}{3},\frac{z_2}{3}\right)$.

The r*-th subdivision of* $\mathbb{Z}^2$ *is the set* $\mathbb{Z}^2_r = \left\{\left(\frac{z_1}{3^r},\frac{z_2}{3^r}\right) \mid (z_1,z_2)\in\mathbb{Z}^2\right\}$ *and the* $3^r:1$ *map* $i_r:\mathbb{Z}^2_r \hookrightarrow \mathbb{Z}^2$ *given by* $i_r\left(\frac{z_1}{3^r},\frac{z_2}{3^r}\right) = (z'_1, z'_2)$ *where* (z'_1, z'_2) *is the point in* $\mathbb{Z}^2$ *closer to* $\left(\frac{z_1}{3^r},\frac{z_2}{3^r}\right)$. *Observe that* $i_1 = i$ *and* $i_r = i\circ i \circ \overset{(r)}{\ldots} \circ i$.

Moreover, if we consider in $\mathbb{Z}^2$ *a* k*-adjacency relation, we can consider in* $\mathbb{Z}^2_r$*, in an immediate way, the same adjacency relation, i.e.,* $\left(\frac{z_1}{3^r},\frac{z_2}{3^r}\right)$ *is* k*-adjacent to* $\left(\frac{z'_1}{3^r},\frac{z'_2}{3^r}\right)$ *if and only if* (z_1,z_2) *is* k*-adjacent to* (z'_1,z'_2).

Proposition 1. i_r *is* k*-continuous as a function between digital spaces.*

Definition 2. *Given* $X\subset\mathbb{Z}^2$*, the* r*-th subdivision of* X *is the set* $X_r = i_r^{-1}(X)$.

Intuitively, if we consider X made of pixels, the r-th subdivision of X consists in replacing each pixel with 9^r pixels and the map i_r is the inclusion. The reason to divide a pixel in 3×3 pixels in each subdivision (and not, for example in 2×2) is due to the fact that the 3×3 mask is at the basis of most digital topology operations (see, for example, Remark 3 or Theorem 1).

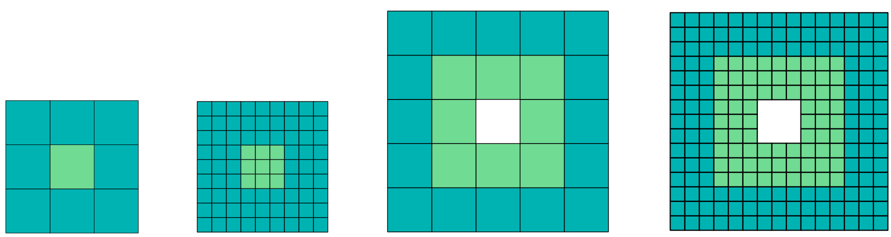

Fig. 2. Some sets and subsets and their first subdivisions (see Proposition 2)

Remark 1. *Given* $X,Y\subset\mathbb{Z}^2$*, any function* $f:X_r\longrightarrow Y$ *induces in an immediate way a multivalued function* $F:X\longrightarrow Y$ *where* $F(x)=\bigcup_{x'\in i_r^{-1}(x)} f(x')$.

Definition 3. *Consider* $X,Y\subset\mathbb{Z}^2$*. A multivalued function* $F:X\longrightarrow Y$ *is said to be a* (k,k')*-continuous multivalued function if it is induced by a* (k,k')*-continuous (single-valued) function from* X_r *to* Y *for some* $r\in\mathbb{N}$.

In the following remark we state some properties of digitally continuous multivalued functions. For more results and details the reader is referred to [5].

Remark 2. *Any single-valued digitally continuous function is continuous as a multivalued function. In particular, any constant map is continuous as a* *multivalued map. Moreover, if $F : X \longrightarrow Y$ ($X, Y \subset \mathbb{Z}^2$) is a (k, k')-continuous multivalued function, then*

i) *$F(x)$ is k'-connected, for every $x \in X$,*
ii) *if x and y are k-adjacent points of X, then $F(x)$ and $F(y)$ are k'-adjacent subsets of Y,*
iii) *F takes k-connected sets to k'-connected sets,*
iv) *if $X' \subset X$ then $F|_{X'} : X' \longrightarrow Y$ is a (k, k')-continuous multivalued function,*
v) *the composition of continuous multivalued function is a continuous multivalued function.*

Remark 3. *In [6] the morphological operations of dilation, erosion and closing have also be modeled as digitally continuous multivalued maps.*

Definition 4. *Let $X \subset \mathbb{Z}^2$ and $Y \subset X$. We say that Y is a k-retract of X if there exists a k-continuous multivalued function $F : X \longrightarrow Y$ (a multivalued k-retraction) such that $F(y) = \{y\}$ if $y \in Y$.*

If moreover $F(x) \subset \mathcal{N}(x)$ for every $x \in X \setminus Y$, we say that F is a multivalued $(\mathcal{N}, k)$-retraction.

The following results are given in [5]. Their proofs are based on the subdivisions shown in Figure 2.

Proposition 2. *i) The boundary ∂X of a square X is not a k-retract of X.*
ii) The outer boundary ∂X of an annulus X is a k-retract of X.

These results improve those for single-valued maps. For them, the boundary of a filled square is not a retract of the whole square [1] but neither is the outer boundary of a squared annulus a digital retract of it.

4 Sequential Deletion of Simple Points as Retractions

It may seem that the family of continuous multivalued functions could be too wide, therefore not having good properties. In this section we show that this is not the case. We show, in particular, that the existence of a k-continuous multivalued function from a set X to $X \setminus \{p\}$ which leaves invariant $X \setminus \{p\}$ is closely related to p being a k-simple point of X.

If $X \subset \mathbb{Z}^2$, a point $p \in X$ is k-simple ($k = 4, 8$) in X (see [10]) if its deletion does not change the topology of X in the sense that after deleting it:

- no k-connected component of X vanishes or is split in several components,
- no $\bar{k}$-connected component of $\mathbb{Z}^2 \setminus X$ is created or is merged with the background or with another such component (remind that, as defined in section 2, $\bar{k} = 4$ if $k = 8$ and $\bar{k} = 8$ if $k = 4$).

A k-simple point can be locally detected by the following characterization. A point p is k-simple if the number of k-connected components of $\mathcal{N}(p) \cap X$ which are k-adjacent to p is equal to 1 and $\mathcal{N}_{\bar{k}}(p) \cap X^c \neq \emptyset$ (this last condition is equivalent to p being a k-boundary point of X).

The following theorem is a restatement of a result given in [5] and presents a new and simpler algorithm to define the map F associated to the deletion of a simple point. The differences with the algorithm in [5] are basically two: we only require to consider the first subdivision of X (and not the second as it happened in [5]), and we obtain smaller images for the deleted simple points which are more suitable for the purposes of this paper. Although we state the result for k-connected sets, this is not a loss of generality because for a general set X it would be applied to the connected component containing the simple point we want to delete.

Theorem 1. *Let $X \subset \mathbb{Z}^2$ be k-connected and consider $p \in X$. Suppose there exists a k-continuous multivalued function $F : X \longrightarrow X \setminus \{p\}$ such that $F(x) = \{x\}$ if $x \neq p$ and $F(p) \subset \mathcal{N}(p)$. Then p is a k-simple point.*

The converse is true under the following conditions:

a) for $k = 8$ it is always true and, moreover, we can impose that $F(p) \subset \mathcal{N}_4(p)$ whenever p is not 4-isolated,
b) for $k = 4$ it is true if and only p is not 8-interior to X.

Proof. The proof is similar to that of Theorem 2 in [5]. To define our new and simpler algorithm for the converse statement, consider a simple point p in the hypothesis of the theorem. Then we have two excluding possibilities

1. $\mathcal{N}(p) \cap X \subset \{p_1, p_3, p_5, p_7\}$, then $k = 8$ and $\mathcal{N}(p) \cap X$ consists on just one of those points, say p_i, and we define $F(p) = p_i$,
2. $\mathcal{N}_4(p) \cap X \neq \emptyset$, in which case there exist 2 points in $\mathcal{N}_4(p)$ in opposite sides of p such that one of them is in X and the other one is not in X (because p is a k-simple point and for $k = 4$ we exclude the case of p being a interior point).
   In this case, we rotate the neighborhood of p in order that, if we label $\mathcal{N}(p)$ and $i^{-1}(p)$ as in the following figure, then $p_2 \notin X$ and $p_6 \in X$.

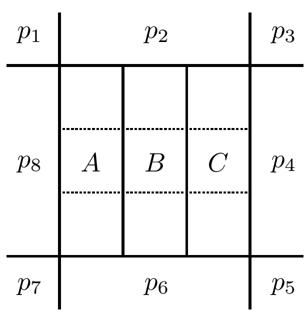

Fig. 3. Labels for $i^{-1}(p)$ and $\mathcal{N}(p)$

Then we define f according to the following cases:

a) $\underline{k=8}$. We define $f(B) = p_6$ and
   i) if $p_3 \in \mathcal{N}(p) \cap X$, then $f(C) = p_4$, otherwise $f(C) = p_6$,
   ii) if $p_1 \in \mathcal{N}(p) \cap X$, then $f(A) = p_8$, otherwise $f(A) = p_6$,

b) $\underline{k=4}$. We define $f(B) = p_6$ and
   i) if $p_4 \in \mathcal{N}(p) \cap X$, then $f(C) = p_5$, otherwise $f(C) = p_6$,
   ii) if $p_8 \in \mathcal{N}(p) \cap X$, then $f(A) = p_7$, otherwise $f(A) = p_6$.

Therefore, the multivalued map F induced by f is defined by $F(x) = \{x\}$ if $x \neq p$ and

a) for $k = 8$,

$$F(p) = \begin{cases} \{p_6\} & \text{if } \mathcal{N}(p) \cap X \subset \{p_4, p_5, p_6, p_7, p_8\} \\ \{p_4, p_6\} & \text{if } \mathcal{N}(p) \cap X = \{p_3, p_4, p_5, p_6, p_7, p_8\} \\ \{p_6, p_8\} & \text{if } \mathcal{N}(p) \cap X = \{p_4, p_5, p_6, p_7, p_8, p_1\} \\ \{p_4, p_6, p_8\} & \text{if } \mathcal{N}(p) \cap X = \{p_3, p_4, p_5, p_6, p_7, p_8, p_1\}, \end{cases}$$

b) for $k = 4$,

$$F(p) = \begin{cases} \{p_6\} & \text{if } \mathcal{N}(p) \cap X \subset \{p_1, p_3, p_5, p_6, p_7\} \\ \{p_5, p_6\} & \text{if } \{p_4, p_5, p_6, p_7\} \subset \mathcal{N}(p) \cap X \subset \{p_1, p_3, p_4, p_5, p_6, p_7\} \\ \{p_6, p_7\} & \text{if } \{p_5, p_6, p_7, p_8\} \subset \mathcal{N}(p) \cap X \subset \{p_1, p_3, p_5, p_6, p_7, p_8\} \\ \{p_5, p_6, p_7\} & \text{if } \{p_4, p_5, p_6, p_7, p_8\} \subset \mathcal{N}(p) \cap X. \end{cases}$$

Remark 4. *It is important to note that if p is a k-simple point, then $F(p)$ is contained in the component of $\mathcal{N}(p) \cap X$ which is k-adjacent to p. This is useful when defining F in a specific situation.*

Since the composition of k-continuous multivalued functions is a k-continuous multivalued function, we have the following result.

Corollary 1. *Let $X \subset \mathbb{Z}^2$ be k-connected and consider $Y \subset X$ such that Y is obtained from X by a sequential deletion of k-simple points such that none of them is 8-interior in the remainder. Then there exists a k-continuous multivalued k-retraction from X to Y.*

It is interesting to note that the ideas in Theorem 2 in [5] and in the previous theorem can be applied also to pairs of points whose simultaneous deletion does not change the k-topology. Such points, called k-simple pairs, are essential to verify the correctedness of parallel thinning algorithms and are locally characterized as follows: A pair $\{p, q\}$ is a k-simple pair if and only if at least one of them is not a k-interior point and the number of k-components of $\mathcal{N}(p, q) \cap X$ which are k-adjacent to $\{p, q\}$ is 1, where $\mathcal{N}(p, q) = ((\mathcal{N}(p) \cup \mathcal{N}(q)) \setminus \{p, q\}$.

The following result, proved in [6], characterizes the deletion of simple pairs in a similar way as Theorem 1 does it for simple points.

Theorem 2. *Consider $X \subset \mathbb{Z}^2$ k-connected and consider a pair $\{p, q\} \subset X$ of 4-adjacent points of X. Suppose that there exists a k-continuous multivalued function $F : X \longrightarrow X \setminus \{p, q\}$ such that $F(x) = \{x\}$ if $x \neq p, q$, and $F(p), F(q) \subset \mathcal{N}(p, q)$. Then $\{p, q\}$ is a k-simple pair.*

The converse is true under the following conditions:

a) for $k = 8$ it is always true and, moreover, we can impose that $F(\{p,q\}) \subset \mathcal{N}_4(p,q) = ((\mathcal{N}_4(p) \cup \mathcal{N}_4(q)) \setminus \{p,q\}$ whenever $\{p,q\}$ is not a 4-connected component of X,

b) for $k = 4$ it is true if and only p or q is not 8-interior to X.

Observe that in the above theorem we do not require that $F(p) \subset \mathcal{N}(p)$ and $F(q) \subset \mathcal{N}(q)$ as in $(\mathcal{N}, k)$-retractions. If we consider this additional requirement we obtain the following result, which deals with the case of a k-simple pair made of k-simple points, also proved in [6] (note that a pair of 4-adjacent k-simple points needs not be a k-simple pair as shown by the example in Figure 4).

Theorem 3. *Consider $X \subset \mathbb{Z}^2$ k-connected and consider a pair $\{p,q\}$ of 4-adjacent points of X. Suppose that there exists a k-continuous multivalued function $F : X \longrightarrow X \setminus \{p,q\}$ such that $F(x) = \{x\}$ if $x \neq p,q$, $F(p) \subset \mathcal{N}(p)$ and $F(q) \subset \mathcal{N}(q)$. Then $\{p,q\}$ is a k-simple pair of k-simple points.*

The converse is true under the following conditions:

a) for $k = 8$ it is always true and, moreover, we can impose that $F(p) \subset \mathcal{N}_4(p)$ and $F(q) \subset \mathcal{N}_4(q)$,

b) for $k = 4$ it is true if and only p or q is not 8-interior to X.

5 Thinning Algorithms as Multivalued $(\mathcal{N}, k)$-Retractions

It is well known that the parallel deletion of simple points needs not to preserve topology. The simplest example is given by the following figure.

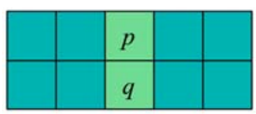

Fig. 4. A not k-deletable pair of k-simple points

Observe that p and q are both simple points but if we delete both then the connectedness (and hence the topology) of the set changes.

Two well known conditions that guarantee the preservation of topology in a parallel deletion of simple points are the following:

i) We can delete only north (or west or south or east) boundary simple points (a north boundary point p of X is any point of X such that $p_2 \notin X$ where p_2 is the point of $\mathcal{N}(p)$ just above p).
ii) We can delete only simple points of a subfield, for example, points whose integer coordinates add to an even number.

In both cases the connected components formed by only two points have to be considered independently.

In general thinning algorithms are divided in subiterations [8]. For example, in the first case, at odd subiterations a point is deleted if it is k-simple and north or west boundary while at even subiterations a point is deleted if it is south or east boundary (deleting simultaneously points with two different orientations, for example north and west, obliges for $k = 8$, to consider independently the connected components formed by two or three points mutually 8-adjacent). In the second case, at odd subiterations we delete all 8-simple points whose integer coordinates add to an even number, while in the even subiterations we delete all 8-simple points whose integer coordinates add to an odd number.

The basic notion behind thinning algorithms based on the parallel deletion of simple points is the following.

Definition 5. *Let $X \subset \mathbb{Z}^2$ and $D \subset X$. D is called k-deletable ($k = 4, 8$) in X if its deletion does not change the topology of X in the sense that after deleting D:*

- *no k-connected component of X vanishes or is split in several components,*
- *no $\bar{k}$-connected component of $\mathbb{Z}^2 \setminus X$ is created or is merged with the background or merged with another such component,*

Ronse [13] called these sets strongly k-deletable.

The following theorem relates deletable sets and multivalued retractions.

Theorem 4. *Let $X \subset \mathbb{Z}^2$ be k-connected and consider $D \subset X$. If D is deletable such that D does not contain 8-interior points, then there exists a multivalued k-retraction from X to $X \setminus D$.*

Proof. If D is k-deletable, then its points can be sequentially deleted in such a way that any of these points is a simple point of the remainder [13]. Therefore, by Corollary 1, there exists a k-continuous multivalued function $F : X \longrightarrow X \setminus D$ such that $F(x) = \{x\}$ if $x \in X \setminus D$.

Although the deletion of a k-simple point is a $(\mathcal{N}, k)$-retraction, when deleting simple points sequentially, the resulting composition of all these $(\mathcal{N}, k)$-retractions needs not to be a $(\mathcal{N}, k)$-retraction as the following example shows.

Example 1. *Consider X and $D = \{p_1, p_2, p_3\}$ are in Figure 5.*

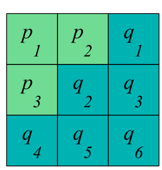

Fig. 5. A sequential deletion of simple points which is not a $\mathcal{N}$-retraction

Then, since p_1 *is* 4*-simple we may delete it by a* $(\mathcal{N}, 4)$*-retraction taking it to* $\{p_2\}$ *or* $\{p_3\}$*. Suppose we take it to* p_2 *(if we choose* p_3 *the result is analogous). Since* p_2 *is simple in the remainder, we may delete it by a* $(\mathcal{N}, 4)$*-retraction taking* p_2 *to* $\{q_1, q_3\}$ *or* $\{q_2, q_3\}$ *(even if we choose to delete* p_3 *before). But then, the composition of these two retractions would take* p_1 *to* $\{q_1, q_3\}$ *or* $\{q_2, q_3\}$ *and, although it is a* 4*-retraction it is not a* $(\mathcal{N}, 4)$*-retraction.*

On the other hand, if we started by deleting p_2 *instead of* p_1 *(starting with* p_3 *would be analogous), we may delete it by a* $(\mathcal{N}, 4)$*-retraction taking it to* $\{q_2\}$*. Now,* p_3 *is not 4-simple in the remainder so we should delete* p_1 *by a* $(\mathcal{N}, 4)$*-retraction taking it to* $\{p_3\}$*. Finally,* p_3 *would be taken to* $\{q_2, q_5\}$ *or* $\{q_4, q_5\}$*. But then, the composition of these three retractions would take* p_1 *to* $\{q_4, q_5\}$ *and, although it is a* 4*-retraction it is not a* $(\mathcal{N}, 4)$*-retraction.*

However, it is possible to define a $(\mathcal{N}, 4)$*-retraction* F *which deletes* p_1, p_2, p_3 *by defining* $F(p_1) = \{q_2\}$*,* $F(p_2) = \{q_3\}$ *and* $F(p_3) = \{q_5\}$*.*

In the next two results we show that the usual thinning algorithms based on conditions (i) and (ii) stated at the beginning of this section have subiterations which can be modeled by $(\mathcal{N}, k)$-retractions which can be constructed explicitly, using only the first subdivision. As a consequence, being the composition of all their subiterations, each of these thinning algorithms can be modeled as a digitally continuous multivalued function.

Theorem 5. *Let* $X \subset \mathbb{Z}^2$ *be a* k*-connected set with more than 2 points and let* D *be a subset of north boundary* k*-simple points of* X*. Then there exists a multivalued* $(\mathcal{N}, k)$*-retraction* $F : X \longrightarrow X \setminus D$*.*

Proof. We show first how to define the images of all the points of D in such a way that their images are contained in $X \setminus D$. To do that, consider $p \in D$ and label the points of $\mathcal{N}(p)$ as in figure 1.
<u>k=4</u>: If $p_6 \in X$, if we define $F(p)$ as in Theorem 1 then it is easy to see that $F(p)$ does not include any north boundary point. On the other hand, if $p_6 \notin X$, since p is 4-simple, $p_4 \notin X$ or $p_8 \notin X$. Suppose $p_4 \notin X$ (the case $p_8 \notin X$ is symmetric). Then $p_8 \in X \setminus D$, because p_8 can not be simple, and we define, according to Theorem 1, $F(p) = p_8$.
<u>k=8</u>: If $p_6 \in X$, if we define $F(p)$ as in Theorem 1 then it is easy to see that $F(p)$ does not include any north boundary point. On the other hand, if $p_6 \notin X$, since p is 4-simple, $\{p_3, p_4, p_5\} \cap X = \emptyset$ or $\{p_7, p_8, p_1\} \cap X = \emptyset$. Suppose $\{p_3, p_4, p_5\} \cap X = \emptyset$ (the other case is symmetric). Then, we define $F(p) = p_8$ if $p_8 \notin D$ and $F(p) = p_7$ if $p_8 \in D$.

We see now that the function F defined as above is k-continuous. Suppose $D = \{d_1, d_2, \ldots, d_n\}$. Consider, for every $i = 1, 2, \ldots, n$, the multivalued k-continuous map $F_i : X \longrightarrow X \setminus \{d_i\}$ which leaves fixed all points in $X \setminus \{d_i\}$ and such that $F_i(d_i)$ is defined as above. Then the above defined $F : X \longrightarrow X \setminus D$ agrees with the composition $F = F_n \circ F_{n-1} \circ \cdots \circ F_2 \circ F_1$ and hence is a k-continuous multivalued function. Therefore, F is a multivalued $(\mathcal{N}, k)$-retraction from X to $X \setminus D$.

Theorem 6. *Let $X \subset \mathbb{Z}^2$ be a k-connected set with more than 2 points and let D be a subset of 8-simple points of X such that all the points of D have coordinates with even sum. Then there exists a multivalued $(\mathcal{N}, k)$-retraction $F : X \longrightarrow X \setminus D$.*

Proof. We have to show how to define, in a coherent way, the images of all the points of D. To do that, consider $p \in D$. By the algorithm in Theorem 1, we have to consider two cases:

– $\mathcal{N}(p) \cap X \subset \{p_1, p_3, p_5, p_7\}$, then and $\mathcal{N}(p) \cap X$ consists on just one of those points, say p_i, and we define $F(p) = p_i$. Moreover, since p_i is the only neighbor of p, then p_i can not be 8-simple and hence $p_i \notin D$.
– $\mathcal{N}_4(p) \cap X \neq \emptyset$, in which case $F(p) \subset \mathcal{N}_4(p)$ where $\mathcal{N}_4(p) \cap D = \emptyset$.

Therefore, we can define the images of all the points of D according the algorithm in Theorem 1.

The proof that the function F defined as above is k-continuous is similar to that of the previous theorem.

The proof of the continuity of F in the previous two results suggests the following result.

Proposition 3. *Let $X \subset \mathbb{Z}^2$ be a k-connected set and let $D = \{d_1, d_2, \ldots, d_n\} \subset X$ be a set of k-simple points of X. Suppose that, for every $i = 1, 2, \ldots, n$, there exists a multivalued k-continuous map $F_i : X \longrightarrow X \setminus \{d_i\}$ which leaves fixed all points in $X \setminus \{d_i\}$ and such that $F_i(d_i) \subset \mathcal{N}(p) \cap (X \setminus D)$. Then the composition $F = F_n \circ F_{n-1} \circ \cdots \circ F_2 \circ F_1$ is a multivalued $(\mathcal{N}, k)$-retraction from X to $X \setminus D$.*

Consider now the following result, proved in [6], which can be seen as the converse of some results presented in this paper.

Theorem 7. *Let $X \subset \mathbb{Z}^2$ be a k-connected set and let D be a set of k-simple points of X such that there exists a multivalued $(\mathcal{N}, k)$-retraction $F : X \longrightarrow X \setminus D$. Then D is k-deletable.*

The condition of D formed by k-simple points is necessary and, on the other hand, the condition of the points in D being k-simple can not be deduced from the rest of the hypothesis, as shown by the examples in Figure 6.

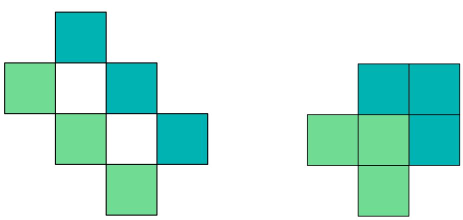

Fig. 6. Left: A $(\mathcal{N}, 8)$-retractable but not 8-deletable set of non 8-simple points. Right: A $(\mathcal{N}, 8)$-retractable and 8-deletable set of points not all of them 8-simple

The proof of this result is based on Ronse [8] sufficient conditions for the parallel deletion of the points of a subset D of a set X to preserve topology.

If we consider Proposition 3 and Theorem 7 together we would obtain the following local criterion for the parallel deletion of the points of a subset D of a set X to preserve topology.

Theorem 8. *Let $X \subset \mathbb{Z}^2$ be a k-connected set and let $D = \{d_1, d_2, \ldots, d_n\} \subset X$ be a set of k-simple points of X. Suppose that, for every $i = 1, 2, \ldots, n$, there exists a multivalued k-continuous map $F_i : X \longrightarrow X \setminus \{d_i\}$ which leaves fixed all points in $X \setminus \{d_i\}$ and such that $F_i(d_i) \subset \mathcal{N}(p) \cap (X \setminus D)$. Then D is k-deletable.*

6 Conclusion

In this paper we have continued the program started in [5] on digitally continuous multivalued maps, now focusing on retractions, and in particular, on the notion, introduced here, of $(\mathcal{N}, k)$-retraction. These types of retractions have the property that each point is retracted to its neighbors. We have modeled the deletion of simple points, one of the most important processing operations in digital topology, as a $(\mathcal{N}, k)$-retraction, and we have given a simple algorithm, requiring only the first subdivision, to define explicitly this retraction. Moreover, we have extended this algorithm to characterize some well known thinning algorithms.

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