A New Fuzzy Connectivity Class Application to Structural Recognition in Images

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Abstract. Fuzzy sets theory constitutes a poweful tool, that can lead to more robustness in problems such as image segmentation and recognition. This robustness results to some extent from the partial recovery of the continuity that is lost during digitization. Here we deal with fuzzy connectivity notions. We show that usual fuzzy connectivity definitions have some drawbacks, and we propose a new definition, based on the notion of hyperconnection, that exhibits better properties, in particular in terms of continuity. We illustrate the potential use of this definition in a recognition procedure based on connected filters. A max-tree representation is also used, in order to deal efficiently with the proposed connectivity.

1 Introduction

Connectivity is a key concept in image segmentation, filtering, and pattern recognition, where objects of interest are often constrained to be connected according to some definition of connectivity. This definition depends on the selected representation of objects. The binary representation on a discrete grid remains the most widespread, and the connectivity is then generally derived from an elementary connectivity, such as 4- or 8-connectivity in 2D. The axiomatization of classes of connectivity [1,2] provides a rigorous framework to handle the concept of connectivity, which leads to the design of connected filters (e.g. [3]). These definitions were further extended to general complete lattices [2,4,5,6] and to the notion of hyperconnectivity (i.e. based on a different definition of overlapping).

In this paper we deal with connectivity of fuzzy objects. Object representation using fuzzy sets theory [7] makes it possible to model various types of imperfections, in particular related to the imprecision in images, and constitutes a powerful tool, that can lead to more robustness in problems such as image segmentation and recognition. This robustness results to some extent from the partial recovery of the continuity that is lost during the digitization process. The initial definition of fuzzy connectivity [8] provides a crisp characterization of the connectivity of a fuzzy set. Its later extension [5] leads to a characterization of the connectivity as a degree. This degree is however not continuous with respect

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to the membership function. Therefore we propose a new definition, based on the notion of hyperconnection, that exhibits better properties, in particular in terms of continuity.

We first recall in Section 2 some previous definitions on fuzzy sets and fuzzy connectivity, and we illustrate some of their drawbacks. Section 3 is the core of the paper. We introduce a new measure of connectivity, and we show that it leads to a hyperconnection and to nice continuity properties. Hyperconnected components are defined, and an efficient representation as a max-tree is proposed. These notions allow us to build connected filters. In Section 4, we illustrate the proposed approach on an example for brain imaging.

2 Preliminaries

Fuzzy sets – Let X be a set (typically the spatial domain). A fuzzy set on X is defined as $\tilde{A} = \{(x, \mu_A(x)) | x \in X\}$, where μ_A is the membership function, which quantifies the membership degree of x to $\tilde{A}$, and takes values in $[0,1]$. In the following, X is the digital space $\mathbb{Z}^n$, and we restrict ourselves to fuzzy sets having a bounded support. A fuzzy set is entirely characterized by the set of its α-cuts, denoted by $(\mu_A)_\alpha$: $(\mu_A)_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$. We denote by $\mathcal{F}$ the set of fuzzy sets defined on X. The binary relation $\leq$ on $\mathcal{F}$, defined by $\tilde{A} \leq \tilde{B} \Leftrightarrow \forall x \in X, \mu_A(x) \leq \mu_B(x)$, is a partial order, and $(\mathcal{F}, \leq)$ is a complete lattice. The supremum $\tilde{A} \vee \tilde{B}$ and infimum $\tilde{A} \wedge \tilde{B}$ are defined by their membership functions, as $\forall x \in X, \mu_{A \vee B}(x) = \max(\mu_A(x), \mu_B(x))$ and $\forall x \in X, \mu_{A \wedge B}(x) = \min(\mu_A(x), \mu_B(x))$, respectively. The smallest element is denoted by $0_\mathcal{F}$ and the largest element by $1_\mathcal{F}$. They are fuzzy sets with constant membership functions, equal to 0 and 1, respectively.

A family Δ of fuzzy sets on X is said sup-generating if $\forall \tilde{A} \in \mathcal{F}, \tilde{A} = \bigvee\{\delta \in \Delta \mid \delta \leq \tilde{A}\}$. We will consider in particular the family $\{\delta_{x,t}\}$ defined as $\delta_{x,t}(y) = t$ if $y = x$ and $\delta_{x,t}(y) = 0$ otherwise, which is sup-generating in the lattice $(\mathcal{F}, \leq)$.

As a metric on $\mathcal{F}$ we use: $d_\infty(\tilde{A}, \tilde{B}) = \sup_{x \in X} |\mu_A(x) - \mu_B(x)|$, and $(\mathcal{F}, d_\infty)$ is a metric space, inducing a definition of continuity.

Usual fuzzy connectivity – The first definition of fuzzy connectivity was proposed by Rosenfeld [8]. More precisely, a degree of connectivity between two points in a fuzzy set was defined, from which the connectivity of a fuzzy set was derived.

Definition 1. *[8] The degree of connectivity between two points x and y of X in a fuzzy set $\tilde{A}$ ($\tilde{A} \in \mathcal{F}$) is defined as:*

$$c^1_{\tilde{A}}(x, y) = \max_{\substack{l \in L_{x,y} \\ l = \{x_0 = x, x_1, \ldots, x_n = y\}}} \min_{0 \leq i \leq n} \mu_A(x_i)$$

where $L_{x,y}$ denotes the set of digitial paths from x to y, according to the underlying digital connectivity defined on X.

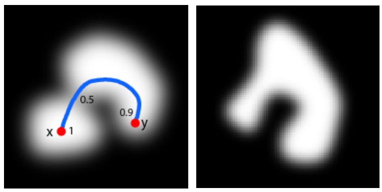
This degree of connectivity is symmetrical in x and y, weakly reflexive (i.e. $\forall (x, y) \in X^2, c^1_{\tilde{A}}(x, x) \geq c^1_{\tilde{A}}(x, y)$), and max-min transitive (i.e. $\forall (x, y, z)$

$\in X^3, c^1_{\tilde{A}}(x,z) \geq \min(c^1_{\tilde{A}}(x,y), c^1_{\tilde{A}}(y,z)))$. We have $c^1_{\tilde{A}}(x,x) = \mu_A(x)$ and $c^1_{\tilde{A}}(x,y) \leq \min(\mu_A(x), \mu_A(y))$.

Based on this definition segmentation processes were designed in [9,10]. An affinity based on adjacency and grey level similarity was proposed, and the induced notion of connectivity was used to perform image segmentation, initialized with a set of seed points.

Definition 2. *[8] A fuzzy set is said connected if all its α-cuts are connected (in the sense of the connectivity on X).*

Proposition 1. *[8] A fuzzy set $\tilde{A}$ is connected iff $\forall(x,y) \in X^2, c^1_{\tilde{A}}(x,y) = \min(\mu_A(x), \mu_A(y))$.*



(a) (b)

Fig. 1. Examples of non-connected (a) and connected (b) fuzzy sets according to Def. 2

These definitions are illustrated in Figure 1. One of the optimal paths between x and y (achieving the max-min criterion of the definition) is displayed in (a), and the minimal value on this path is 0.5, which provides the degree of connectivity between x and y. The fuzzy set in (a) is non-connected since $c^1_{\tilde{A}}(x,y) = 0.5$, which is stricly less than the membership degrees of x and y ($\mu_A(x) = 1$ and $\mu_A(y) = 0.9$). On the contrary, the fuzzy set in Figure 1(b) is connected.

Connection and hyperconnection – Definition 2 provides a crisp notion of the connectivity of a fuzzy set. However, if a set is fuzzy, it may be intuitively more satisfactory to consider that its connectivity is also a matter of degree. The notions of connection and hyperconnection [1,2,4] provide an appropriate framework to this aim.

Definition 3. *[2] Let $(L, \leq)$ be a lattice. A connected class, or connection, $\mathcal{C}$ is a family of elements of L such that:*

1. *$0_L \in \mathcal{C}$,*
2. *$\mathcal{C}$ is sup-generating,*
3. *for any family $\{C_i\}$ of elements of $\mathcal{C}$ such that $\bigwedge_i C_i \neq 0_L$, then $\bigvee_i C_i \in \mathcal{C}$.*

Let us first consider the lattice $(\mathcal{P}(X), \subseteq)$. On this lattice, we use the usual connection $\mathcal{C}_d$ induced by a digital connectivity c_d on X (in the sense of the graph of digital points). An element of $\mathcal{C}_d$ is then simply a subset A of X that is connected in the sense of c_d (i.e. $\forall(x, y) \in A^2, \exists x_0 = x, x_1, ..., x_n = y, \forall i < n, x_i \in A$, and $c_d(x_i, x_{i+1}) = 1$).

Now, on the lattice $(\mathcal{F}, \leq)$, let us consider the binary definition of connectivity in Definition 2, and the 1D examples in Figure 2. In (a), each fuzzy set is connected, and so is there union. However, in (b), the union is not connected, while each fuzzy set is connected and their intersection is not equal to $0_{\mathcal{F}}$. Therefore Definition 3 cannot account for this type of situation on the lattice of fuzzy sets. Dealing with such cases require to replace the infimum ($\bigwedge$) in condition 3 by another overlap mapping $\perp$ [2], leading to the notion of hyperconnection.

Definition 4. *[2,5] Let $(L, \leq)$ be a lattice. A hyperconnection $\mathcal{H}$ is a family of elements of L such that:*

1. *$0_L \in \mathcal{H}$,*
2. *$\mathcal{H}$ is sup-generating,*
3. *for any family $\{H_i\}$ of elements of $\mathcal{H}$ such that $\perp_i H_i \neq 0_L$, then $\bigvee_i H_i \in \mathcal{H}$.*

Note that it is sufficient to have $\Delta \subseteq \mathcal{H}$, for a sup-generating family Δ, in order to achieve condition 2.

On the lattice $(\mathcal{F}, \leq)$, the hyperconnection $\mathcal{H}^1$ containing the connected fuzzy sets according to Definition 2 is obtained for the following overlap mapping $\perp$ [5]:

$$\perp^1(\{\tilde{A}_i\}) = \begin{cases} 1 \text{ if } \forall\alpha \in [0, 1],\ \bigcap_i \{(\mu_{A_i})_\alpha \mid (\mu_{A_i})_\alpha \neq \emptyset\} \neq \emptyset \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

For the sake of simplicity, we denote the values taken by $\perp$ as 1 and 0 (instead of $1_{\mathcal{F}}$ and $0_{\mathcal{F}}$). It is easy to check that the union of connected fuzzy sets such that their non empty α-cuts intersect is connected in the sense of Definition 2.

This overlap mapping was extended in [5] to the following family:

$$\perp^1_\tau(\{\tilde{A}_i\}) = \begin{cases} 1 \text{ if } \forall\alpha \leq \tau,\ \bigcap_i \{(\mu_{A_i})_\alpha \mid (\mu_{A_i})_\alpha \neq \emptyset\} \neq \emptyset \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

Let us define $\mathcal{H}^1_\tau = \{\tilde{A} \in \mathcal{F}, \forall\alpha \leq \tau, (\mu_A)_\alpha \in \mathcal{C}_d\}$.

Proposition 2. *[5] Each $\mathcal{H}^1_\tau$ is an hyperconnection, i.e. verifies all items of Definition 4, for the overlap mapping $\perp^1_\tau$. It contains in particular the sup-generating family $\Delta = \{\delta_{x,t}, x \in X, t \in [0, 1]\}$. The family $\{\mathcal{H}^1_\tau, \tau \in [0, 1]\}$ is decreasing with respect to τ: $\tau_1 \leq \tau_2 \Rightarrow \mathcal{H}^1_{\tau_2} \subseteq \mathcal{H}^1_{\tau_1}$.*

Now the connectivity of a fuzzy set can be defined as a degree, instead as a crisp notion, as follows: $c^1(\tilde{A}) = \sup\{\tau \in [0, 1] \mid \tilde{A} \in \mathcal{H}^1_\tau\}$. This definition is equivalent to applying the extension principle [11] to the crisp connectivity: $c^1(\tilde{A}) = \sup\{\alpha \in [0, 1] \mid (\mu_A)_\alpha \in \mathcal{C}_d\}$.

As an illustration, the fuzzy sets in Figure 2(c) and (d) have a degree of connectivity of 0.25 and 0.05, respectively. However, intuitively we would rather say that the example in (d) is more connected than the one in (c), which seems to have two very distinct parts. The degree of connectivity depends on the height of

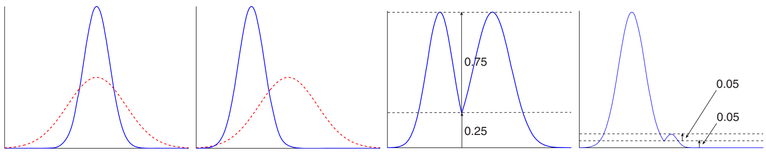


Fig. 2. Examples of fuzzy sets on $\mathbb{Z}$. (a) The union is connected in the sense of Definition 2, while in (b) it is not. The degree of connectivity of the fuzzy set in (c) is equal to 0.25, and in (d) to 0.05, although it seems to be more connected.

the lowest minimum or saddle point, and not on its depth. A small modification in (d) would make the fuzzy set fully connected, illustrating that this definition is not continuous.

The aim of this paper is to propose a new definition that overcomes these drawbacks.

3 A New Class of Connectivity

3.1 Connectivity Measure

In this section, we introduce a new measure of connectivity of a fuzzy set, with better properties than $c^1(\tilde{A})$. The proposed construction is based on the fact that, since it always holds that $c^1_{\tilde{A}}(x,y) \leq \min(\mu_A(x), \mu_A(y))$, then the condition for a fuzzy set to be connected, in the sense of Proposition 1, is equivalent to: $\forall (x,y) \in X^2,\ \min(\mu_A(x), \mu_A(y)) \leq c^1_{\tilde{A}}(x,y)$. We propose to replace the inequality by a degree of satisfaction of this inequality, based on Lukasiewicz' implication [12]: $\forall (a,b) \in [0,1]^2$, $\mu_{\leq}(a,b) = \min(1, 1-a+b)$. Rewriting this expression for $a = \min(\mu_A(x), \mu_A(y))$ and $b = c^1_{\tilde{A}}(x,y)$ leads to the following definition.

Definition 5. *The connectivity degree between two points x and y in a fuzzy set $\tilde{A}$ is defined by:*

$$\begin{aligned} c^2_{\tilde{A}}(x,y) &= \min(1, 1 - \min(\mu_A(x), \mu_A(y)) + c^1_{\tilde{A}}(x,y)) \\ &= 1 - \min(\mu_A(x), \mu_A(y)) + c^1_{\tilde{A}}(x,y). \end{aligned} \quad (3)$$

This measure takes its values in $[0,1]$, it is symmetrical and reflexive ($c^2_{\tilde{A}}(x,x) = 1$). It is not transitive. From this degree of connectivity between two points we derive the following definition of the connectivity degree of a fuzzy set.

Definition 6. *The connectivity degree of a fuzzy set $\tilde{A}$ is defined as:* $c^2(\tilde{A}) = \min_{(x,y)\in X^2} c^2_{\tilde{A}}(x,y)$.

It is easy to show that, for given x and y, $c^1_{\tilde{A}}(x,y)$ and $c^2_{\tilde{A}}(x,y)$ are achieved for the same point on the same path from x to y, and that $c^2(\tilde{A})$ is achieved for

x such that $\mu_A(x) = \max_{x' \in X} \mu_A(x')$ (i.e. x is a global maximum), and for y belonging to a regional maximum (hence $c^2(\tilde{A}) = 1 - \mu_A(y) + c^1_{\tilde{A}}(x, y)$). Note that $c^1(\tilde{A})$ and $c^2(\tilde{A})$ are not achieved for the same points. Roughly speaking, the connectivity degree of a fuzzy set now depends on the depth of the deepest saddle point in the fuzzy set. On the examples illustrated in Figure 2, it can be observed that the fuzzy set in (c) is $0.25-$connected $(1-0.75)$, while the fuzzy set in (d) is $0.95-$connected. In the later case, if one of the modes is progressively shrinking to 0, the degree of connectivity will evolve smoothly towards 1. This is expressed formally by the following result, using as a distance between two function f_1 and f_2 from X^2 into $[0, 1]$: $d_\infty(f_1, f_2) = \sup_{(x,y) \in X^2} |f_1(x, y) - f_2(x, y)|$.

Proposition 3. *For fixed x and y, the mapping associating $\tilde{A}$ to $c^1_{\tilde{A}}(x, y)$ is continuous and Lipschitz, and the mapping associating $\tilde{A}$ to $c^2_{\tilde{A}}(x, y)$ is continuous and 2-Lipschitz. The mapping associating $\tilde{A}$ to $c^2(\tilde{A})$ is continuous and 2-Lipschitz.*

3.2 Link with a Hyperconnection

We propose a new overlap measure, considering that two fuzzy sets do not overlap if they "do not significantly overlap" (i.e. only low α-cuts can overlap), as follows:

$$\perp^2_\tau(\{\tilde{A}_i\}) = \begin{cases} 1 \text{ if } \forall \alpha \in [0,1],\ \bigcap_i \{(\mu_{\gamma_{x_i}(\tilde{A}_i)})_\alpha \mid \alpha \leq h_i - 1 + \tau\} \neq \emptyset \\ 0 \text{ otherwise} \end{cases} \tag{4}$$

where $h_i = \max_{x \in X} \mu_{A_i}$ (the height of $\tilde{A}_i$), x_i is a point of X such that $\mu_{A_i}(x_i) = h_i$, and $\gamma_{x_i}(\tilde{A}_i)$ denotes the geodesic reconstruction by dilation of $\tilde{A}_i$ from the marker δ_{x_i,h_i} (i.e. $\gamma_{x_i}(\tilde{A}_i) = (\delta_c(\delta_{x_i,h_i}) \wedge \mu_A)^\infty$, where δ_c denotes the elementary dilation on X, according to c_d).

Let us now define $\mathcal{H}^2_\tau$ as: $\mathcal{H}^2_\tau = \{\tilde{A} \in \mathcal{F} \mid c^2(\tilde{A}) \geq \tau\}$.

Proposition 4. *$\mathcal{H}^2_\tau$ defines a hyperconnection for the overlap mapping $\perp^2_\tau$.*

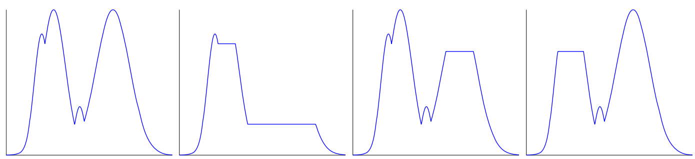
These definitions lead to connected components that are more interesting than using $\mathcal{H}^1_\tau$, as seen next.

3.3 Connected Components

In the general framework of connections, connected components of an element A of a lattice $(L, \leq)$, relatively to a connection $\mathcal{C}$ on L, are the elements C_i of $\mathcal{C}$ such that: $C_i \leq A$ and $\nexists C \in \mathcal{C}, C_i < C \leq A$ (i.e. the largest elements of $\mathcal{C}$ that are smaller than A) [2].

This definition extends to hyperconnections [6]. Let $\mathcal{H}$ be a hyperconnection on L. The hyperconnected components of $A \in L$ are the elements H_i of $\mathcal{H}$ such that: $H_i \leq A$ and $\nexists H \in \mathcal{H}, H_i < H \leq A$. For any two hyperconnected components H_i and H_j of A, either $H_i = H_j$ or $H_i \perp H_j = 0$. Moreover, $\bigvee_i H_i = A$, where the supremum is taken over all connected components of A. If the overlap is taken as $\perp_\tau^2$, we will speak of $\tau-$hyperconnected component. In particular the $1-$hyperconnected components are exactly the reconstructions $\gamma_{x_i}(\tilde{A})$ where each x_i is a representative point of a regional maximum of μ_A.

These notions are illustrated in Figure 3, for the hyperconnection $\mathcal{H}_\tau^2$. Let $\tilde{A}$ be the fuzzy set in (a). It has four $1-$hyperconnected components, corresponding to each regional maximum of $\tilde{A}$, one of them being displayed in (b), two $0.5-$hyperconnected components (c and d), and one $0.1-$hyperconnected component, equal to $\tilde{A}$. The computation of the hyperconnected components will be explained in Section 3.4. The degree of connectivity of $\tilde{A}$ is $c^2(\tilde{A}) = 0.2$, hence $\tilde{A}$ is a connected component in the sense of $\mathcal{H}_\tau^2$ for $\tau \leq 0.2$. If we denote by $\tilde{A}_1$ and $\tilde{A}_2$ the two $0.5-$hyperconnected components in (c) and (d), it is easy to check that $c^2(\tilde{A}_1) = c^2(\tilde{A}_2) = 0.5$, hence they are elements of $\mathcal{H}_{0.5}^2$ ($\tau = 0.5$). Let x_1 be the maximum of $\tilde{A}_1$ and x_2 the maximum of $\tilde{A}_2$. We have $h_1 = \mu_{A_1}(x_1) = h_2 = \mu_{A_2}(x_2) = 1$. The two reconstructions $\gamma_{x_1}(\tilde{A}_1)$ and $\gamma_{x_2}(\tilde{A}_2)$ overlap only until level $\alpha = 0.2$, which is less than $h_i - 1 + \tau = 0.5$. This shows that they actually do not overlap in the sense of $\perp_\tau^2$.



(a) (b) (c) (d)

Fig. 3. (a) Fuzzy set (equal to its $\tau-$hyperconnected components for $\tau \leq 0.2$). (b) One of the four $1-$hyperconnected components. (c, d) The two $0.5-$hyperconnected components.

3.4 Tree Representation

From an algorithmical point of view, the obtention of the hyperconnected components and their processing can benefit from an appropriate representation. Since the α-cuts are a core component of our definitions, we suggest to rely on the usual max-tree [3] representation of a function. From now on, we assume that the values of α are quantified, in a uniform way. For each level α of the quantification, nodes of a tree are associated with the connected components (in the sense of $\mathcal{C}_d$) of the α-cut of the considered fuzzy set. Edges are induced by the inclusion relation between connected components for two successive values of α. A fuzzy set $\tilde{A}$ is then bi-univoquely represented by a tree $T(\tilde{A})$, with:

- $\mathcal{V}$ the set of vertices of the tree (v denotes an element of $\mathcal{V}$ and $h(v)$ denotes its altitude, i.e. the value of α corresponding to this node),
- R the root of the tree,
- $\mathcal{L}$ the set of leaves,

– $\mathcal{E}$ the set of edges of the tree ($\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$), defined from the inclusion relation,
– for $e = (v_1, v_2) \in \mathcal{E}$, $w(e) = |h(v_1) - h(v_2)|$.

There are several algorithms for computing the tree, a recent one being of quasi-linear complexity [13].

If it exists, the chain from v_1 to v_2 (a sub-tree of $T(\tilde{A})$) is denoted by $C_{T(\tilde{A})}(v_1, v_2)$. Its nodes are $v'_i, i = 0...n$ from v_1 to v_2 such that $\forall i, v'_i \in \mathcal{V}$, $v'_0 = v_1$, $v'_n = v_2$, $\forall i < n, (v'_i, v'_{i+1}) \in \mathcal{E}$, and $h(v'_i) - h(v'_{i+1})$ has the same sign as $h(v_1) - h(v_2)$ (i.e. $\{v'_i, i = 0...n\}$ is either a descending or an ascending path from v_1 to v_2, depending on the relative altitudes of v_1 and v_2). Its edges are $(v'_i, v'_{i+1}), i = 0...(n-1)$.

Let v_1 and v_2 two nodes such that $h(v_1) \leq h(v_2)$. We denote by $d(v_1, v_2)$ the length of the chain $C_{T(\tilde{A})}(v_1, v_2)$, expressed as the sum of $w(e)$, over all edges of this chain. If the chain does not exist (typically if v_1 and v_2 are on two different branches of the tree), then $d(v_1, v_2) = \infty$.

For any sub-tree G of $T(\tilde{A})$, we denote by $D_{T(\tilde{A})}(G, \nu)$ the dilation of G in $T(\tilde{A})$ of size ν ($\nu \in [0, 1]$), obtained by adding all ascending chains of length less or equal than ν issued from a node of G. A pseudo-erosion $E_{T(\tilde{A})}(G, \nu)$ is obtained by keeping all nodes v of G such that there exists at least one ascending chain of length ν issued from v. An important remark here is that E is not a true erosion (it does not commute with the infimum), and E and D are neither dual nor adjoint (even for the same ν), hence their composition does not have the usual property of an opening or a closing.

Proposition 5. *The set $\{\tilde{A}_i\}$ of $1-$hyperconnected components of $\tilde{A}$ is isomorphic to $\mathcal{L}$, and $T(\tilde{A}_i) = C_{T(\tilde{A})}(R, l_i)$, where l_i is the leaf associated with $\tilde{A}_i$.*

This result shows that it is possible to handle the $1-$hyperconnected components of a fuzzy set by processing the associated sub-tree.

Proposition 6. *If $G \subseteq T(\tilde{A})$ is a sub-tree representing a $\tau-$hyperconnected fuzzy subset of $\tilde{A}$, then $D_{T(\tilde{A})}(G, \nu)$ represents a $\max(0, \tau - \nu)-$hyperconnected fuzzy subset and $E_{T(\tilde{A})}(G, \nu)$ a $\min(1, \tau + \nu)-$hyperconnected fuzzy subset.*

Proposition 7. *The set of $\tau-$hyperconnected components of a fuzzy set $\tilde{A}$ is isomorphic to the set of leaves of $E_{T(\tilde{A})}(T(\tilde{A}), 1-\tau)$. A $\tau-$hyperconnected component of $\tilde{A}$ can then be obtained by a dilation of size $(1-\tau)$ of a $1-$hyperconnected component of $E_{T(\tilde{A})}(T(\tilde{A}), 1-\tau)$.*

Figure 4 illustrates in (b) the component tree $T(\tilde{A})$ of the fuzzy set shown in (a). The $1-$hyperconnected components (c–f) correspond to each regional maximum of (a). The results of a pseudo-erosion of size 0.4 of $T(\tilde{A})$ and the dilation of size 0.4 of one of its connected components are shown in (g) and (h), respectively, providing exactly the sub-tree associated with one $0.6-$hyperconnected component of $\tilde{A}$. The corresponding $0.6-$hyperconnected component in the image is displayed in (i). Another $0.6-$hyperconnected component is shown in (j)

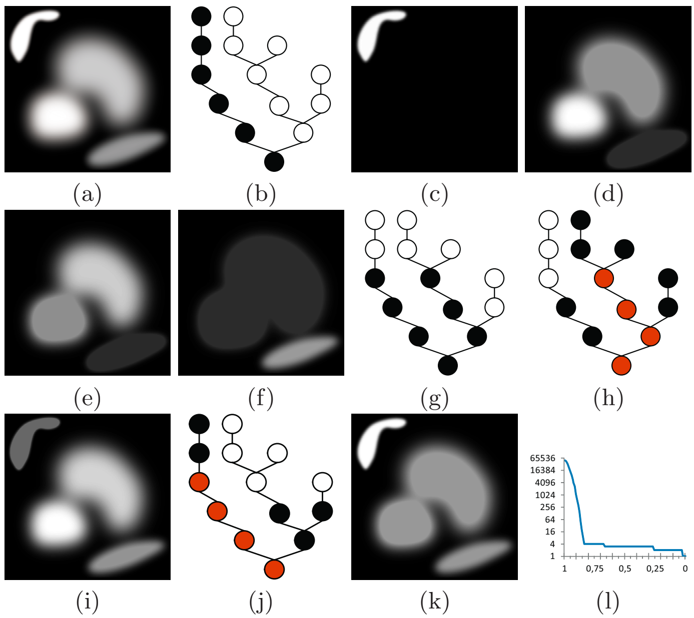


Fig. 4. (a) Fuzzy set. (b) Component tree (the α-cuts are quantizied with a step 0.2), with a chain from a leaf to the root shown in black. (c–f) $1-$hyperconnected components of the fuzzy set in (a). (g) Subtree corresponding to the pseudo-erosion of size 0.4 (in black). (h) A $0.6-$hyperconnected component (in black and red) obtained by dilation of one connected component (in red) of the pseudo-erosion and the corresponding image (i). Another $0.6-$hyperconnected component in the tree (j) and the corresponding image (k). (l) Number of τ-hyperconnected components of the noisy image as a function of τ.

(subtree) and (k) (image). If the fuzzy set in Figure 4 (a) is degraded by a Gaussian noise of variance 0.05, more than 20000 $1-$hyperconnected components are obtained. The evolution of the number of $\tau-$hyperconnected components as a function of τ is displayed in (l), showing a grouping effect.

3.5 Connected Filters

One of the main interests of the tree structure is that it allows finding efficiently the hyperconnected components, and therefore applying connected filters on the image. Let $f : \mathcal{F} \rightarrow \{0, 1\}$ be an increasing function defining a filtering criterion (e.g. on the size of the connected components). A connected filter according to criterion f is then defined as:

$$\xi(\tilde{A}) = \bigvee \{\tilde{h} \in \mathcal{H}^2_\tau \mid \tilde{h} \leq \tilde{A} \text{ and } f(\tilde{h}) = 1\}. \tag{5}$$

This defines an increasing and idempotent operator and thus a morphological filter. In this particular form, it is moreover anti-extensive, and thus a morphological opening. Such a filter can be implemented in a very efficient way based on the tree structure.

4 Illustrative Example

In order to illustrate the proposed definitions, we consider the problem of segmenting brain structures in 3D MRI images. An axial slice is displayed in Figure 6 (a). Typically, we may want to segment connected objects such as the ventricular system or internal grey nuclei. As in our previous work [14,15], we rely on anatomical knowledge expressed as spatial relations between structures [16], and on grey level information. As illustrated in Figure 6, this knowledge is translated into fuzzy representations in the image space, that drive the segmentation and the recognition. Here we show how to include additional connectivity criteria in this procedure.

For each anatomical structure S_i, we define two fuzzy sets $\tilde{P}_i$ and $\tilde{N}_i$, corresponding to an over-estimation and an under-estimation of S_i, respectively: $\tilde{N}_i \leq S_i \leq \tilde{P}_i$. This idea is close to the concept of fuzzy rough sets. At the beginning of the procedure, $\tilde{N}_i$ is empty and $\tilde{P}_i$ is the whole space. Exploiting the available knowledge then allows reducing $\tilde{P}_i$ and extend $\tilde{N}_i$ so as to get as close as possible to the structure of interest. For instance for $S_i = S_{LV}$ being the lateral ventricle, we define $\tilde{P}_{Gl_{LV}}$ representing the knowledge on grey levels, so as to have $\tilde{S}_{LV} \leq \tilde{P}_{Gl_{LV}}$ (Figure 6 (b)). Once the brain has been segmented, it becomes possible to represent the central location of the ventricules inside the brain (Figure 6 (c)), so as to guarantee $\tilde{S}_{LV} \leq \tilde{P}_{Sp_{LV}}$. The conjunctive fusion of $\tilde{P}_{Sp_{LV}}$ and $\tilde{P}_{Gl_{LV}}$ is shown in Figure 6 (d), and provides an including fuzzy set $\tilde{P}_{LV}$. Although the over-estimation has been strongly reduced, it still exhibits several connected components. A connectivity contraint can now be introduced, via a connected filter based on a marker.

The criterion f used in the filter (Equation 5) relies on the inclusion of a marker $\tilde{N}$ in h and the inclusion of h in $\tilde{P}$. Here, for the first inclusion, we consider actually a degree of inclusion (as in Section 3.1), to achieve more robustness with respect to the position of the marker. The filter then writes:

$$\xi_{\tilde{N}}(\tilde{P}) = \bigvee \{\tilde{h} \in \mathcal{H}_\tau^2 \mid \max_{x \in X} \mu_h(x) \leq \mu_{\leq}(\tilde{N}, \tilde{h}) \text{ and } \tilde{h} \leq \tilde{P}\}. \tag{6}$$

Note that the criterion is not increasing in this case. However it is increasing if $\max_{x \in X} \mu_h(x)$ is constant. Equation 6 can thus be decomposed as a supremum over all possible values of this maximum (i.e. all levels of the quantification), and each term of this supremum can then be handled efficiently using the tree representation, as explained in Sections 3.4 and 3.5.

Proposition 8. *Let $\alpha = \max_{x \in X} \mu_N(x)$. The result of the connected filter defined in Equation 6 is $(\alpha - (1 - \tau))-$hyperconnected.*

A 1D example is shown in Figure 5, where a fuzzy set is progressively filtered by a marker getting larger and larger. Intuitively, hyperconnected components

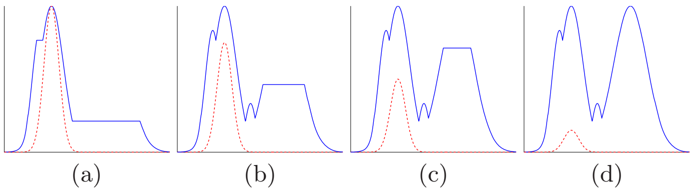


Fig. 5. Progressive filtering of the fuzzy set of Figure 3 (a) by a marker of increasing size (in red). The result is displayed in blue.

verifying the inclusion constraint are kept, while the other ones are reduced to a level corresponding to the degree of satisfaction of the constraint.

We illustrate now the effect of this connected filter, applied to $\tilde{P}_{LV}$, based on a marker $\tilde{N}$ defined as a fuzzy set having a support reduced to one point centered in the right lateral ventricle, with a membership value taking values 1, 0.75, 0.5 and 0, respectively (Figure 6 (e–h)). A potential application of this approach is to perform a filter, preserving connectivity properties, and being more or less strong depending on the confidence we may have in the marker. This may lead to more robustness and can be used as a preliminary step in a segmentation process. This will be further explored in future work.

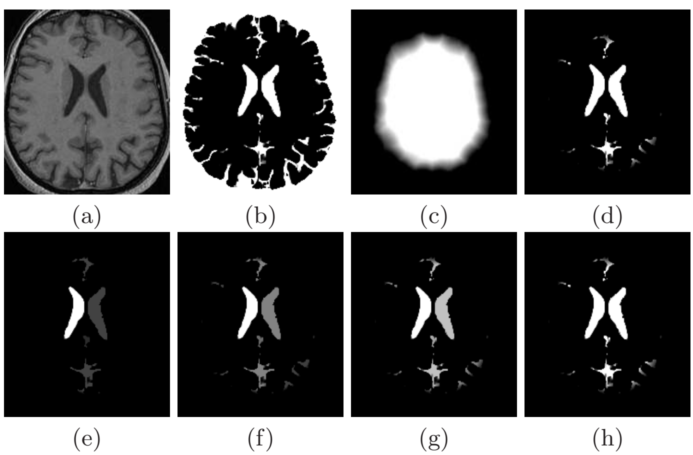


Fig. 6. (a) One axial slice of a 3D brain MRI. (b) Grey level information: $\tilde{P}_{Gl_{LV}}$. (c) Central location inside the brain: $\tilde{P}_{Sp_{LV}}$. (d) Conjunctive fusion. (e–h) Connected filter results using a marker centered in the right ventricle, with maximal value 1, 0.75, 0.5, 0, respectively.

5 Conclusion

In this paper we have introduced a new definition of fuzzy connectivity, based on the notion of hyperconnection, that overcomes some drawbacks of previous definitions, and that has in particular nice continuity properties. We have shown that a representation as a max-tree can lead to efficient extraction of hyperconnected components, and processing with connected filters. An illustrative example on a brain image has been shown. Future work aims at exploring other properties of the proposed definitions, and at developing a complete applicative framework for brain segmentation, including pathological cases.

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