Constrained Simultaneous and Near-Simultaneous Embeddings

Fabrizio Frati[1], Michael Kaufmann[2], and Stephen G. Kobourov[3]

[1] Dipartimento di Informatica e Automazione – Università Roma Tre, Italy
[2] Wilhelm-Schickard-Institut für Informatik – Universität Tübingen, Germany
[3] Department of Computer Science – University of Arizona, U.S.A.

Abstract. A *geometric simultaneous embedding* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with a bijective mapping of their vertex sets $\gamma : V_1 \to V_2$ is a pair of planar straight-line drawings Γ_1 of G_1 and Γ_2 of G_2, such that each vertex $v_2 = \gamma(v_1)$ is mapped in Γ_2 to the same point where v_1 is mapped in Γ_1, where $v_1 \in V_1$ and $v_2 \in V_2$.

In this paper we examine several constrained versions and a relaxed version of the geometric simultaneous embedding problem. We show that if the input graphs are assumed to share no common edges this does not seem to yield large classes of graphs that can be simultaneously embedded. Further, if a prescribed combinatorial embedding for each input graph must be preserved, then we can answer some of the problems that are still open for geometric simultaneous embedding. Finally, we present some positive and negative results on the near-simultaneous embedding problem, in which vertices are not mapped exactly to the same but to "near" points in the different drawings.

1 Introduction

Graph drawing techniques are commonly used to visualize relationships between objects, where the objects are the vertices of the graph and the relationships are captured by the edges in the graph. Simultaneous embedding is a problem that arises when visualizing two or more relationships defined on the same set of objects. If the graphs corresponding to these relationships are planar, the aim of simultaneous embedding is to find point locations in the plane for the vertices of the graphs, so that each of the graphs can be realized on the same point-set without edge crossings. To ensure good readability of the drawings, it is preferable if the edges are drawn as straight-line segments. This problem is known as *geometric simultaneous embedding*. It has been shown that only a few classes of graphs can be embedded simultaneously with straight-line segments. Brass *et al.* [1], Erten and Kobourov [5], and Geyer *et al.* [8] showed that three paths, a planar graph and a path, and two trees do not admit geometric simultaneous embeddings. On the positive side, an algorithm for geometric simultaneous embedding of two caterpillars [1] is the strongest known result.

As geometric simultaneous embedding turns out to be very restrictive, it is natural to relax some of the constraints of the problem. Not insisting on straight-line edges led to positive results such as a linear-time algorithm by Erten and

S.-H. Hong, T. Nishizeki, and W. Quan (Eds.): GD 2007, LNCS 4875, pp. 268–279, 2007.

Table 1. Known results and our contribution on geometric simultaneous embedding (Geometric), geometric simultaneous embedding with no common edges (Disj. Edges), geometric simultaneous drawing with fixed embedding (Fixed Embedding), geometric simultaneous drawing with fixed embedding and no common edges (Disj. Edges, Fixed Embedding).

	Geometric	*Disj. Edges*	*Fixed Emb.*	*Disj. Edges, Fixed Emb.*
path + path	YES [1]	YES [1]	YES [1]	YES [1]
star + path	YES [1]	YES [1]	YES *Sec. 4.1*	YES *Sec. 4.1*
double-star + path	YES [1]	YES [1]	?	YES *Sec. 4.1*
caterpillar + path	YES [1]	YES [1]	?	?
caterpillar + caterpillar	YES [1]	YES [1]	NO *Sec. 4.2*	NO *Sec. 4.2*
3 *paths*	NO [1]	?	NO [1]	?
tree + path	?	?	?	?
tree + cycle	?	?	?	?
tree + caterpillar	?	?	NO *Sec. 4.2*	NO *Sec. 4.2*
outerplanar + path	?	?	NO *Sec. 4.3*	NO *Sec. 4.3*
outerplanar + caterpillar	?	?	NO *Sec. 4.2*	NO *Sec. 4.2*
outerplanar + cycle	?	?	NO *Sec. 4.3*	NO *Sec. 4.3*
tree + tree	NO [8]	?	NO [8]	NO *Sec. 4.2*
outerplanar + tree	NO [8]	?	NO [8]	NO *Sec. 4.2*
outerplanar + outerplanar	NO [1]	?	NO [1]	NO *Sec. 4.2*
planar + path	NO [5]	NO *Sec. 3*	NO [5]	NO *Sec. 3*
planar + tree	NO [5]	NO *Sec. 3*	NO [5]	NO *Sec. 3*
planar + planar	NO [5]	NO *Sec. 3*	NO [5]	NO *Sec. 3*

Kobourov for embedding any pair of planar graphs with at most three bends per edge, or any pair of trees with at most two bends per edge [5]. In such results it is allowed for an edge connecting a pair of vertices to be represented by different Jordan curves in different drawings. As this can be detrimental to the readability of the drawings, several papers considered a slightly more constrained version of this problem, namely, *simultaneous embedding with fixed edges*, in which bends are allowed, however, an edge connecting the same pair of vertices must be drawn in exactly the same way in all drawings. Di Giacomo and Liotta [4] showed that outerplanar graphs can be simultaneously embedded with fixed edges with paths or cycles using at most one bend per edge. Frati [6] showed that a planar graph and a tree can also be simultaneously embedded with fixed edges.

Studying the existing variants of simultaneous embedding led to practical embedding algorithms for some graph classes and techniques for simultaneous embedding have been used in visualizing evolving and dynamic graphs [2]. However, many problems remain theoretically open and in practice algorithms applying these ideas to evolving and dynamic graphs do not provide any guarantees on the quality of the resulting layouts. With this in mind, we consider three further variants of the geometric simultaneous embedding problem.

Most of the proofs about the non-existence of simultaneous embeddings exploit the presence of common edges between the input graphs. Hence, it is natural ask if larger classes of graphs have geometric simultaneous embeddings when no edges are shared. In Section 3 we answer in the negative for planar graph-path pairs, generalizing the result in [5], where it is shown that a planar graph and a path that share edges do not admit a geometric simultaneous embedding.

In Section 4 we consider the problem of geometric simultaneous embedding in which the embeddings for the graphs are fixed. We call this setting *geometric simultaneous embedding with fixed embeddings.* Clearly, negative results known for geometric simultaneous embedding remain valid here. We show that some classes of graphs that have geometric simultaneous embeddings do not admit one with individually fixed embeddings. In particular, we prove such a negative result for caterpillar-caterpillar pairs. Moreover, in the fixed embedding setting we are able to solve problems that are still open for geometric simultaneous embedding. Namely, we provide an outerplanar-path pair that has no geometric simultaneous drawing with fixed embedding. All the negative results claimed are still valid if the input graphs are assumed to not share edges. On the other hand, we partially cover the known positive results for geometric simultaneous embedding, by showing that a star and a path can always be realized and that a double-star and a path can always be realized if they do not share edges.

In the quest for more practical setting where we can guarantee some properties of the layouts, in Section 5 we study a variant we call *geometric near-simultaneous embedding*. In this setting edges are straight lines and vertices representing the same entity in different graphs can be placed not exactly in the same point but just in "near" points. Assuming vertices are placed on the grid, we show that there exist pairs of n-vertex planar graphs in which vertices that represent the same entity in different graphs must be placed at distance $\Omega(n)$. We then consider graphs "similar" in their combinatorial structure, describing algorithms which guarantee that vertices representing the same entity have only constant displacement from one drawing to the next. Such algorithms can be used to guarantee limited displacement in dynamic graph drawings.

Due to space limitations, we leave out some proofs, that can be found in [7].

2 Preliminaries

We summarize basic terminology used in this paper; for more details see [3,11]. A *straight-line drawing* of a graph is a mapping of each vertex to a unique point in the plane and of each edge to a segment between the endpoints of the edge. A *planar drawing* is one in which no two edges intersect. A *planar graph* is a graph that admits a planar drawing. A *grid drawing* is one in which every vertex is placed at a point with integer coordinates in the plane. An *embedding* of a graph is a circular ordering of the edges incident on each vertex of G. An embedding of a graph specifies the faces in any drawing respecting such an embedding, even though the embedding does not determine which one is the *external face*. A graph is *triconnected* if for every pair of distinct vertices there exist three vertex-disjoint paths connecting them. A triconnected graph has an unique embedding, up to a reversal of its adjacency lists.

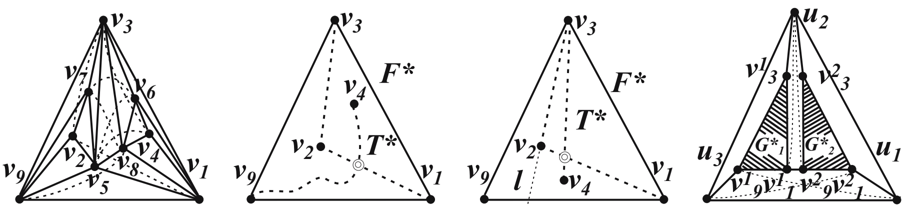
An *outerplanar graph* is a graph that admits a drawing in which all the vertices are incident to the same face. A *caterpillar* is a tree in which the removal of all the leaves and their incident edges yields a path. A *star* (*double-star*) is a caterpillar with only one vertex (two vertices) of degree greater than one.

A *geometric simultaneous embedding* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with a bijective mapping γ of their vertex sets is a pair of planar straight-line drawings Γ_1 of G_1 and Γ_2 of G_2, such that each vertex $v_2 = \gamma(v_1)$ is mapped in Γ_2 to the same point where v_1 is mapped in Γ_1, where $v_1 \in V_1$ and $v_2 \in V_2$.

3 Simultaneous Embedding without Common Edges

We consider the geometric simultaneous embedding of graphs not sharing common edges, exhibiting a planar graph and a path that cannot be drawn simultaneously. We revisit the problem of embedding simultaneously graphs not sharing edges in the conclusions (Section 6).

Let G^* be the planar graph on vertices v_1, v_2, ..., v_9 shown in Fig. 1(a). Since G^* is triconnected, it has the same faces in any planar embedding. Let F^* denote the triangular face $\Delta v_1 v_3 v_9$ and P^* be the path $(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)$.



(a) (b) (c) (d)

Fig. 1. (a) Planar graph G^* drawn with solid edges and path P^* drawn with dashed edges; (b)–(c) Illustrations for the proof of Lemma 1; (d) Planar graph G drawn with solid edges and path P drawn with dashed edges.

Lemma 1. *There does not exist a geometric simultaneous embedding of G^* and P^* in which the external face of G^* is F^*.*

Proof: All vertices of G^*, other than v_1, v_3 and v_9, are inside F^* as F^* is the external face of G^*. Consider the triangle T^* formed by edges (v_1, v_2), (v_2, v_3) of P^*, and by edge (v_1, v_3) of G^*. Since v_9 is incident to F^*, it must lie outside T^*. Let l be the line passing through v_2 and v_3; l separates the plane in two open half-planes, one containing v_9, called the *exterior part* of l, and one not containing v_9, called the *interior part* of l. Consider the possible placements of v_4. If v_4 is placed inside T^* then the subpath of P^* composed of edges (v_1, v_2) and (v_2, v_3) crosses the subpath of P^* connecting v_4, that lies inside T^*, and v_9, that lies outside T^*; see Fig. 1(b). Suppose v_4 is placed outside T^*. Since vertex v_4 (vertex v_2) must lie inside triangle $\Delta v_1 v_3 v_5$ (inside triangle $\Delta v_3 v_5 v_9$), the clockwise order of edges $(v_3, v_1), (v_3, v_5), (v_3, v_9)$ of G^* and edges $(v_3, v_4), (v_3, v_2)$ of P^* around v_3 must be $(v_3, v_1), (v_3, v_4), (v_3, v_5), (v_3, v_2), (v_3, v_9)$. Therefore v_4 is in the *interior part* of l and hence edge (v_1, v_2) crosses edge (v_3, v_4) in P^*; see Fig. 1(c). □

Theorem 1. *There exist a planar graph G, a path P, and a mapping between their vertices such that: (i) G and P do not share edges, and (ii) G and P have no geometric simultaneous embedding.*

Proof: We construct G and P out of two copies of G^* and P^* described above. Let G_1^* and G_2^* be two copies of G^*. Denote by v_i^j the vertex of G_j^* that corresponds to the vertex v_i in G^*, where $j = 1, 2$ and $i = 1, \ldots, 9$. Let G be the graph composed of G_1^* and G_2^* together with three additional vertices u_1, u_2, and u_3 and eight additional edges (u_1, u_2), (u_1, u_3), (u_2, u_3), (u_1, v_1^2), (u_2, v_3^1), (u_2, v_3^2), (u_3, v_9^1), and (v_1^1, v_9^2); see Fig. 1(d). Graph G is triconnected and therefore it has exactly one planar embedding and it has the same faces in any plane drawing. Let P be the path $(u_1, v_9^1, v_8^1, v_7^1, v_6^1, v_5^1, v_4^1, v_3^1, v_2^1, v_1^1, u_2, v_9^2, v_8^2, v_7^2, v_6^2, v_5^2, v_4^2, v_3^2, v_2^2, v_1^2, u_3)$. It is easy to verify that G and P do not share edges. Note that the subpaths of P induced by the vertices of G_1^* and by the vertices of G_2^* play the same role that path P^* plays for graph G^* in Lemma 1.

Let F_1^* and F_2^* denote cycles (v_1^1, v_3^1, v_9^1) and (v_1^2, v_3^2, v_9^2); these cycles are faces of G_1^* and G_2^*. We now show that every plane drawing Γ of G determines a non-planar drawing of P. Consider the embedding $\mathcal{E}_G$ of G obtained by choosing $\Delta u_1 u_2 u_3$ as external face; see Fig. 1(d). Choosing any face external to F_1^* (F_2^*) in $\mathcal{E}_G$ as external face of Γ leaves G_1^* (G_2^*) embedded with external face F_1^* (F_2^*). Hence, we can apply Lemma 1 and conclude that there does not exist a simultaneous embedding of G and P. □

4 Simultaneous Drawing with Fixed Embedding

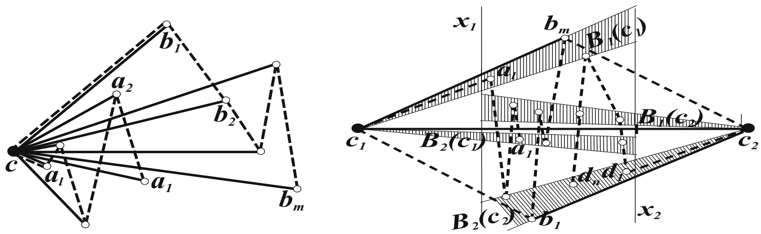
Next, we examine the possibility of embedding graphs simultaneously with straight-line edges and with fixed embeddings.

4.1 Simultaneous Drawing of Stars, Double-Stars and Paths with Fixed Embedding

Let P be an n-vertex path and let S be an n-vertex star with *center* c and embedding $\mathcal{E}$. Let $P = (a_1, a_2, \ldots, a_l, c, b_1, b_2, \ldots, b_m)$, where one among sequences $(a_1, a_2, \ldots, a_l)$ and $(b_1, b_2, \ldots, b_m)$ could be empty. Draw S with c as leftmost point and with all edges in an order around c consistent with $\mathcal{E}$, so that edge (c, b_1), if it exists, is the *uppermost* edge of S. This can be done so that the x-coordinate of a vertex b_i is greater than the one of a vertex a_j, with $1 \leq i \leq m$ and $1 \leq j \leq l$, the x-coordinate of a vertex b_i is greater than the one of a vertex b_j, with $1 \leq j < i \leq m$, and the x-coordinate of a vertex a_i is greater than the one of a vertex a_j, with $1 \leq i < j \leq l$; see Fig. 2(a). The resulting drawing of S is clearly planar. Further, P is not self-intersecting as it is realized by two x-monotone curves joined by an edge that is higher than every other edge of P. This yields the following result:

Theorem 2. *An n-vertex star and an n-vertex path admit a geometric simultaneous embedding in which the star has a fixed prescribed embedding.*

Now let P be an n-vertex path and let D be an n-vertex double-star with *centers* c_1 and c_2 and with embedding $\mathcal{E}$. Suppose D and P do not share edges. Let $P = (a_1, a_2, \ldots, a_l, c_1, b_1, b_2, \ldots, b_m, c_2, d_1, d_2, \ldots, d_p)$. Sequences $(a_1, a_2, \ldots, a_l)$



(a) (b)

Fig. 2. (a) Simultaneous embedding of a star and a path; (b) Simultaneous embedding of a double-star and a path not sharing edges

and $(d_1, d_2, \ldots, d_p)$ could be empty, while $m \geq 2$. Further, b_1 is neighbor of c_2 and b_m is neighbor of c_1 in D; see Fig. 2(b). Group the edges incident to c_1 (incident to c_2), except for (c_1, c_2), in two *bundles* $B_1(c_1)$ and $B_2(c_1)$ (resp. $B_1(c_2)$ and $B_2(c_2)$). $B_1(c_1)$ is made up of the edges starting from (c_1, b_m) until, but not including, (c_1, c_2) in the clockwise order of the edges incident to c_1. $B_2(c_1)$ is made up of the edges starting from (c_1, c_2) until, but not including, (c_1, b_m) in the clockwise order of the edges incident to c_1. The other two bundles $B_1(c_2)$ and $B_2(c_2)$ are defined analogously. P is divided into three subpaths, $P_1 = (c_1, a_l, a_{l-1}, \ldots, a_2, a_1)$, $P_2 = (c_1, b_1, b_2, \ldots, b_m, c_2)$, and $P_3 = (c_2, d_1, d_2, \ldots, d_p)$.

Draw (c_1, c_2) as an horizontal segment, with c_1 on the left. $B_1(c_1)$ and $B_2(c_1)$ ($B_1(c_2)$ and $B_2(c_2)$) are drawn inside *wedges* centered at c_1 (at c_2) and directed rightward (leftward), with $B_1(c_1)$ above (c_1, c_2) and $B_2(c_1)$ below (c_1, c_2) (with $B_1(c_2)$ above (c_2, c_1) and $B_2(c_2)$ below (c_2, c_1)). Such wedges are disjoint and they share an interval $[x_1, x_2]$ of the x-axis, where $[x_1, x_2]$ is a sub-interval of the x-extension of the edge (c_1, c_2). Draw each edge inside the wedge of its bundle, respecting $\mathcal{E}$ and so that the following rules are observed: the x-coordinate of a vertex b_i is greater than the one of a vertex a_j, with $1 \leq i \leq m$ and $1 \leq j \leq l$; the x-coordinate of a vertex d_k is greater than the one of a vertex b_i, with $1 \leq k \leq n$ and $1 \leq i \leq m$; the vertices of P_1, of P_2, and of P_3 have increasing, increasing, and decreasing x-coordinates, respectively. Each vertex has an x-coordinate in the open interval (x_1, x_2). Edge (c_1, b_m) ((c_2, b_1)) of D is drawn so high (so low) that edge (c_2, b_m) ((c_1, b_1)) of P does not create crossings with other edges of the path. The drawing of D is planar since the edges of D are drawn inside disjoint regions of the plane. The absence of crossings in the drawing of P follows from (1) the planarity of the drawings of its subpaths, which in turn follows from the strictly increasing or decreasing x-coordinate of its vertices; and (2) from the fact that the subpaths occupy disjoint regions, except for edges (c_1, b_1) and (c_2, b_m) which do not create crossings, as already discussed. Thus, we have:

Theorem 3. *An n-vertex double-star and an n-vertex path not sharing edges admit a geometric simultaneous embedding in which the double-star has a fixed prescribed embedding.*

4.2 Simultaneous Drawing of Two Caterpillars with Fixed Embedding

Insisting on a fixed embedding when simultaneously embedding planar graphs is a very restrictive requirement as shown by the following theorem:

Theorem 4. *It is not always possible to find a geometric simultaneous embedding for two caterpillars with fixed embeddings.*

Proof: Let C_1 and C_2 be the two caterpillars with fixed embeddings $\mathcal{E}_1$ and $\mathcal{E}_2$ and a bijective mapping $\gamma(x) = x$ between their vertices; see Fig. 3(a-b). We now show that there does not exist a geometric simultaneous embedding of C_1 and C_2 in which C_1 and C_2 respect $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively.

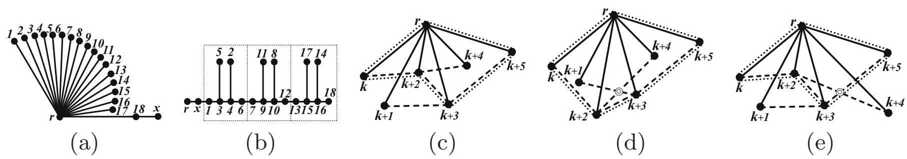


Fig. 3. (a)–(b) Caterpillars C_1 and C_2; (c)–(e) Illustrations for the proof of Theorem 4

Construct a straight-line drawing Γ_1 of C_1. The embedding $\mathcal{E}_1$ of C_1 forces the vertices $1, 2, \ldots, 18$ to appear in this order around r in Γ_1. Consider the subtrees of C_1 induced by the vertices $r, 1, 2, \ldots, 6$, by the vertices $r, 7, 8, \ldots, 12$, and by the vertices $r, 13, 14, \ldots, 18$. Since such subtrees appear consecutively around r, then at least one of them must be drawn in a wedge rooted at r and with angle less than π. Let C_S be such a subtree and let $k, k+1, \ldots, k+5$ be the vertices of C_S, with $k = 1, 7$ or 13. Without loss of generality, let r be the uppermost point of this wedge. It follows that C_S must be drawn *downward*. Denote by P the polygon composed of the edges (r, k) and $(r, k+5)$ of C_1 and of the edges $(k, k+2)$, $(k+2, k+3)$, and $(k+3, k+5)$ of C_2. Note that vertices $k+1$ and $k+4$ must be either both inside or both outside P. In fact, placing one of these vertices inside and the other outside P is not consistent with the embedding constraints of $\mathcal{E}_2$; see Fig. 3(c). If both vertices $k+1$ and $k+4$ are placed inside P, then the embedding constraints of $\mathcal{E}_1$ and $\mathcal{E}_2$ and the upwardness of C_S imply that edge $(k+2, k+4)$ must cut edge $(r, k+3)$ and that edge $(k+1, k+3)$ must cut edge $(r, k+2)$. It follows that there is an intersection between edges $(k+2, k+4)$ and $(k+1, k+3)$, both belonging to C_S; see Fig. 3(d). Similarly, if both vertices $k+1$ and $k+4$ are placed outside P, then by the embedding constraints of $\mathcal{E}_1$ and $\mathcal{E}_2$ vertex $k+2$ is placed inside the polygon formed by the edges $(r, k+1)$, $(r, k+5)$ of C_1 and by the edges $(k+1, k+3)$, $(k+3, k+5)$ of C_2. Hence, edge $(k+2, k+4)$ cuts such a polygon either in edge $(k+1, k+3)$ or in edge $(k+3, k+5)$; see Fig. 3(e) and this concludes the proof. □

4.3 Simultaneous Drawing of Outerplanar Graphs and Paths with Fixed Embedding

Let O^* be the outerplanar graph on vertices v_1, v_2, ..., v_7 shown in Fig. 4(a) and $\mathcal{E}^*$ be the embedding of O^* shown in Fig. 4(b). Let F^* be the face of $\mathcal{E}^*$ with incident vertices v_1, v_3, and v_7 and let P^* be the path $(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$.

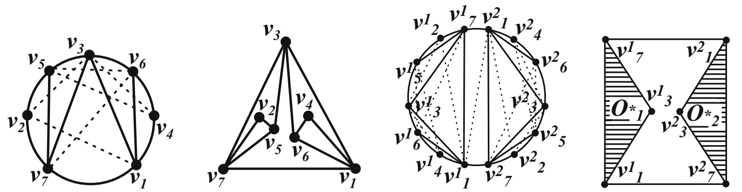


Fig. 4. (a) Outerplanar graph O^*, drawn with solid edges, and path P^*, drawn with dashed edges. (b) Embedding $\mathcal{E}^*$ of O^*. (c) Outerplanar graph O, drawn with solid edges, and path P, drawn with dashed edges. (d) Embedding $\mathcal{E}$ of O.

Lemma 2. *There does not exist a geometric simultaneous embedding of O^* and P^* in which the embedding of O^* is $\mathcal{E}^*$ and the external face of O^* is F^*.*

Theorem 5. *There exist an outerplanar graph O, an embedding $\mathcal{E}$ of O, a path P, and a mapping between their vertices such that: (i) O and P do not share edges, and (ii) O and P have no geometric simultaneous embedding.*

Proof: Let O_1^* and O_2^* be two copies of the outerplanar graph O^* defined above. Denote by v_i^j, with $j = 1, 2$ and $i = 1, \ldots, 7$, the vertex of O_j^* that corresponds to vertex v_i of O^* in O. Let $\mathcal{E}_1^*$ and $\mathcal{E}_2^*$ be the embeddings of O_1^* and O_2^* corresponding to the embedding $\mathcal{E}^*$ of O^*. Let O be the graph composed of O_1^*, of O_2^*, and of edges (v_7^1, v_1^2), (v_1^1, v_7^2); see Fig. 4(c). Let the embedding $\mathcal{E}$ for O be defined as follows: (i) each vertex of O_1^* (of O_2^*) but for v_1^1 and v_7^1 (but for v_1^2 and v_7^2) has the same adjacency list as in $\mathcal{E}_1^*$ (in $\mathcal{E}_2^*$); (ii) the adjacency lists of the remaining vertices are as follows: $v_1^1 \rightarrow (v_7^1, v_6^1, v_4^1, v_3^1, v_7^2)$, $v_7^1 \rightarrow (v_1^2, v_3^1, v_2^1, v_5^1, v_1^1)$, $v_1^2 \rightarrow (v_7^2, v_6^2, v_4^2, v_3^2, v_7^1)$, $v_7^2 \rightarrow (v_1^1, v_3^2, v_2^2, v_5^2, v_1^2)$. Let P be the path $(v_7^1, v_6^1, v_5^1, v_4^1, v_3^1, v_2^1, v_1^1, v_1^2, v_2^2, v_3^2, v_4^2, v_5^2, v_6^2, v_7^2)$. O and P do not share edges, and the subpaths of P induced by the vertices of O_1^* (O_2^*) play for O_1^* (O_2^*) the same role that path P^* plays for graph O^* in Lemma 2.

Let F_1^* and F_2^* denote cycles (v_1^1, v_3^1, v_7^1) and (v_1^2, v_3^2, v_7^2), respectively. These cycles are faces of O_1^* and O_2^*. We now show that every plane drawing $\Gamma_{\mathcal{E}}$ of O with embedding $\mathcal{E}$ determines a non-planar drawing of P. Consider the embedding $\mathcal{E}_O$ of O obtained by choosing $(v_1^1, v_7^1, v_1^2, v_7^2)$ as external face; see Fig. 4(d). Choosing any face external to F_1^* (F_2^*) in $\mathcal{E}_O$ as external face of $\Gamma_{\mathcal{E}}$ leaves O_1^* (O_2^*) embedded with external face F_1^* (F_2^*). Hence, we can apply Lemma 2 and conclude that there is no simultaneous embedding of O and P. □

5 Near-Simultaneous Embedding

In this section we study the variation of geometric simultaneous embedding in which vertices representing the same entity in different graphs can be placed in different points in different drawings. However, in order to preserve the viewer's "mental map" corresponding vertices should be placed as close as possible. This turns out to be impossible for general planar graphs, as the first lemma of this section shows. First, define the *displacement* of a vertex v between two drawings Γ_1 and Γ_2 as the distance between the location of v in Γ_1 and the location of v in Γ_2. Second, we show that there exist two n-vertex planar graphs G_1 and G_2 with a bijection γ between their vertices such that for any two planar straight-line grid drawings Γ_1 and Γ_2 of G_1 and G_2, respectively, there exists a vertex v that has a displacement $\Omega(n)$ between Γ_1 and Γ_2.

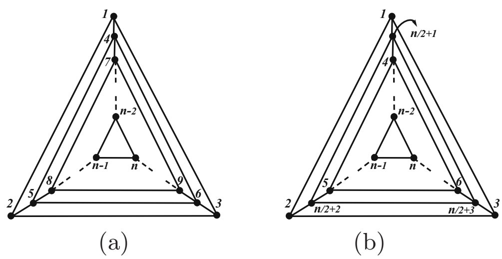


Fig. 5. (a) Nested triangle graph G_1; (b) Nested triangle graph G_2

Let G_1 and G_2 be two n-vertex *nested triangle* graphs; see Fig. 5. A nested triangle graph G is a triconnected planar graph with a triangular face $F(G)$ such that removing the vertices of $F(G)$ and their incident edges leaves a smaller nested triangle graph or an empty vertex set. Suppose the mapping $\gamma(v_1) = v_2$ between vertices $v_1 \in V(G_1)$ and vertices $v_2 \in V(G_2)$ is the one shown in Fig. 5 and defined by the following procedure: embed G_1 and G_2 with external faces $F(G_1)$ and $F(G_2)$, respectively. Starting from G_1 (G_2), for $i = 1, \ldots, n/3$, remove from the current graph the three vertices of the external face and label them $3i-2, 3i-1$, and $3i$ ($3(i+1)/2-2, 3(i+1)/2-1$, and $3(i+1)/2$ if i is odd, or $(n+3i)/2-2, (n+3i)/2-1$, and $(n+3i)/2$ if i is even). Then, for any two planar straight-line grid drawings Γ_1 of G_1 and Γ_2 of G_2 and G_2, we have:

Lemma 3. *There exists a vertex representing the same entity in G_1 and G_2 that has displacement $\Omega(n)$ between Γ_1 and Γ_2.*

The lower bound in Lemma 3 is easily matched by an upper bound obtained by independently drawing each planar graph in $O(n) \times O(n)$ area: Each vertex is displaced by at most the length of the diagonal of the drawing's bounding box. Clearly, such a diagonal has length $O(n)$.

The above result shows that we cannot hope to guarantee near-simultaneous embeddings for arbitrary pairs of planar graphs. It is possible, however, that for graphs that are "similar", near-simultaneous embeddings might exist. Similarity between graphs could be defined and regarded in several different ways, by minding both the combinatorial structure of the graphs and the mapping between the vertices of the graphs. With this in mind, in the following we look for near-simultaneous embeddings of similar paths and similar trees.

5.1 Near-Simultaneous Drawings of Similar Paths

Recall that two paths always have a geometric simultaneous embedding, while three of them might not have one [1]. Therefore, in order to represent a sequence of paths using a sequence of planar drawings, vertices that are in correspondence under the mapping must be displaced from one drawing to the next.

Observing that a path induces an ordering of the vertices, call two n-vertex paths P_1 and P_2 with orderings π_1 and π_2 of their vertices and with a bijective mapping γ between their vertices *k-similar* if for each vertex $v_1 \in P_1$ the position of v_1 in π_1 differs by at most k positions from the one of $v_2 = \gamma(v_1)$ in π_2. Drawing the paths as horizontal polygonal lines with uniform horizontal distances between adjacent vertices gives a near-simultaneous drawing. As any vertex v_i if P_1 occurs within k positions in P_2 (compared with its position in P_1) then the extent of the displacement of the vertex from one drawing to the next is limited by exactly k units. More generally, this idea can be summarized as follows:

Theorem 6. *A sequence of n-vertex paths $P_0, P_1, ..., P_m$, where each two consecutive paths are k-similar, can be drawn so that the displacement of any vertex in a pair of paths that are consecutive in the sequence is at most k.*

5.2 Near-Simultaneous Drawings of Similar Trees

Generalizing the idea of k-similarity to trees, call two rooted arbitrarily ordered trees T_1 and T_2 with vertex sets V_1 and V_2 and with bijective mapping γ between their vertices, k-similar if: (i) The depths of any vertex $v_1 \in V_1$ and of its corresponding vertex $\gamma(v_1) \in V_2$ differ by at most k; (ii) The positions of any two corresponding vertices in any pre-established traversal of the tree among pre-, in-, post-order, or breadth-first-search traversal differ by at most k.

Given two trees T_1 and T_2 that are k-similar with respect to a pre-established traversal order π, we can draw each of T_1 and T_2 as follows: (1) Assign to each vertex v_i its position $\pi(v_i)$ as an x-coordinate; (2) Assign to each vertex v_i its depth as a y-coordinate.

Such an algorithm produces layouts that are planar and *layered*. A drawing is layered if (i) each vertex is assigned to a *layer*, (ii) for each layer an order of its vertices is specified, and (iii) there are only edges joining vertices on consecutive layers. Since subsequent trees are k-similar, the depth of any vertex and its position in a tree traversal changes only by k in two consecutive trees; hence, we have that the displacement of a vertex representing the same entity in different drawings is at most $\sqrt{k^2 + k^2} = k\sqrt{2}$. More generally, we have the following:

Theorem 7. *A sequence of n-vertex trees $T_0, T_1, ..., T_m$, where each two consecutive trees are k-similar, can be drawn such that the displacement of any vertex in a pair of trees that are consecutive in the sequence is at most $k\sqrt{2}$.*

Observe that an analogous definition of similarity between two graphs and the same layout algorithm work more generally for *level planar graphs* [9,10] (and hence for *outerplanar graphs*). Finally, the area requirement of the drawings produced by the described algorithm is worst-case quadratic in the number of vertices of a tree (or of a level planar graph).

6 Conclusions

In this paper we have considered some variations of the well-known problem of embedding graphs simultaneously.

Concerning the geometric simultaneous embedding without common edges, we provided a negative result that seems to show that the geometric simultaneous embedding is not more powerful by assuming the edge sets of the input graphs to be disjoint. Further, we believe that there exist two trees not sharing common edges that do not admit a geometric simultaneous embedding. This would extend the result in [8] where two trees that do not admit a simultaneous embedding and that do share edges are shown. Consider two isomorphic rooted trees $T_1(h,k)$ and $T_2(h,k)$ a mapping γ between their vertices defined as follows (see Fig. 6): (i) the root of $T_1(h,k)$ (of $T_2(h,k)$) has k children; (ii) each vertex of $T_1(h,k)$ (of $T_2(h,k)$) at distance i from the root, with $1 \leq i < h$, has a number of children one less than the number of vertices at distance i from the root in $T_1(h,k)$ (in $T_2(h,k)$); (iii) one vertex of $T_1(h,k)$ (of $T_2(h,k)$) at distance h from the root has one child; (iv) each child of the root of $T_1(h,k)$ is mapped to a distinct child of the root of $T_2(h,k)$; (v) for each pair of vertices v_1 of $T_1(h,k)$ and v_2 of $T_2(h,k)$ that are at distance i from the root of their own tree and that are such that $v_2 \neq \gamma(v_1)$, there exists a child of v_1 that is mapped to a child of v_2; (vi) the only vertex of $T_1(h,k)$ (of $T_2(h,k)$) that is at distance $h+1$ from the root is mapped to the root of $T_2(h,k)$ (to the root of $T_1(h,k)$).

Conjecture 1. For sufficiently large h and k, $T_1(h,k)$ and $T_2(h,k)$ do not admit a geometric simultaneous embedding with mapping γ between their vertices.

For the problem of drawing graphs simultaneously with fixed embedding, we provided more negative results than in the usual setting for geometric simultaneous

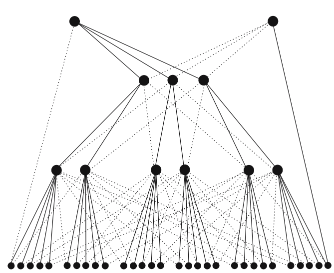


Fig. 6. Trees $T_1(3,3)$ and $T_2(3,3)$ with the mapping γ between their vertices. $T_1(3,3)$ has solid edges and $T_2(3,3)$ has dashed edges.

embedding, while providing only two positive results partially covering the ones already known for geometric simultaneous embedding. We believe that understanding the possibility of obtaining a simultaneous embedding of a tree and a path in which the tree has a fixed embedding could be useful for the same problem in the non-fixed embedding setting.

Even in the more relaxed near-simultaneous setting, we have shown that without assuming a similarity in the sequence of graphs to be drawn, it is difficult to limit the displacement of a vertex from a drawing to the next. We have shown that for paths, for trees, and for level planar graphs there exist reasonable similarity measures that allow us to obtain near-simultaneous drawings. However, in the case of general planar graphs it is not yet clear what kind of similarity metric can be defined and how well can such graphs be drawn.

Acknowledgments

Thanks to Markus Geyer for useful discussions.

References

1. Braß, P., Cenek, E., Duncan, C.A., Efrat, A., Erten, C., Ismailescu, D., Kobourov, S.G., Lubiw, A., Mitchell, J.S.B.: On simultaneous planar graph embeddings. Comput. Geom. 36(2), 117–130 (2007)
2. Collberg, C., Kobourov, S.G., Nagra, J., Pitts, J., Wampler, K.: A system for graph-based visualization of the evolution of software. In: SoftVis, pp. 377–386 (2003)
3. Di Battista, G., Eades, P., Tamassia, R., Tollis, I.G.: Graph Drawing. Prentice Hall, Upper Saddle River, NJ (1999)
4. Di Giacomo, E., Liotta, G.: A note on simultaneous embedding of planar graphs. In: EWCG, pp. 207–210 (2005)
5. Erten, C., Kobourov, S.G.: Simultaneous embedding of planar graphs with few bends. In: Pach, J. (ed.) GD 2004. LNCS, vol. 3383, pp. 195–205. Springer, Heidelberg (2005)
6. Frati, F.: Embedding graphs simultaneously with fixed edges. In: Kaufmann, M., Wagner, D. (eds.) GD 2006. LNCS, vol. 4372, pp. 108–113. Springer, Heidelberg (2007)
7. Frati, F., Kaufmann, M., Kobourov, S.: Constrained simultaneous and near-simultaneous embeddings. Tech. Report RT-DIA-120-2007, University of Roma Tre (2007), `http://dipartimento.dia.uniroma3.it/ricerca/rapporti/rt/2007-120.pdf`
8. Geyer, M., Kaufmann, M., Vrto, I.: Two trees which are self-intersecting when drawn simultaneously. In: Healy, P., Nikolov, N.S. (eds.) GD 2005. LNCS, vol. 3843, pp. 201–210. Springer, Heidelberg (2006)
9. Healy, P., Kuusik, A., Leipert, S.: A characterization of level planar graphs. Discr. Math. 280(1-3), 51–63 (2004)
10. Jünger, M., Leipert, S.: Level planar embedding in linear time. Jour. of Graph Alg. and Appl. 6(1), 67–113 (2002)
11. Kaufmann, M., Wagner, D. (eds.): Drawing Graphs. LNCS, vol. 2025. Springer, Heidelberg (2001)