On Planar Polyline Drawings

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Abstract. We present a linear time algorithm that produces a planar polyline drawing for a plane graph with n vertices in a grid of size bounded by $(p+1) \times (n-2)$, where $p \leq (\lfloor \frac{2n-5}{3} \rfloor)$. It uses at most $p \leq \lfloor \frac{2n-5}{3} \rfloor$ bends, and each edge uses at most one bend. Compared with the area optimal polyline drawing algorithm in [3], our algorithm uses a larger grid size bound in trade for a smaller bound on the total number of bends. Their bend bound is $(n-2)$. Our algorithm is based on a transformation from Schnyder's realizers [6,7] of maximal plane graphs to transversal structures [4,5] for maximal internally 4-connected plane graphs. This transformation reveals important relations between the two combinatorial structures for plane graphs, which is of independent interest.

1 Introduction

We focus on planar graph drawings. Such graphs can be drawn without any edge crossings. Several styles of drawings [1] have been introduced. Common objectives include small area, few bends and good angular resolution. We deal with polyline drawings [1]. A polyline drawing is a drawing of a graph in which each edge is represented by a polygonal chain and every vertex is placed on a grid point. Bonichon et al. [3] presented a linear time algorithm that produces polyline drawings for a graph with n vertices within a grid of area $(n - \lfloor \frac{p}{2} \rfloor - 1) \times (p+1)$, where $p \leq \frac{2n-5}{3}$. It is area optimal and each edge has at most one bend. However the total number of bends used by this algorithm could be $(n-2)$.

Our goal is to have a tradeoff between the grid size and the number of bends. We present a linear time algorithm that produces a polyline drawing in a grid with size bounded by $(p+1) \times (n-2)$, where $p \leq \lfloor \frac{2n-5}{3} \rfloor$, and each edge uses at most one bend. Although the grid size is not as good as the algorithm in [3], our algorithm only needs at most $p \leq \lfloor \frac{2n-5}{3} \rfloor$ bends.

A maximal plane graph G is associated with *realizers* $\mathcal{R}$ [6,7], which is a partition of the set of interior edges into three particular trees. A maximal internally 4-connected plane graph G' with four exterior vertices is associated with *transversal structures* $\mathcal{T}$ [4,5], which is a partition of the set of interior edges into two st-graphs. In this paper, we introduce a transformation from a maximal plane graph G to a maximal internally 4-connected plane graph G' with

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four exterior vertices by a certain number of operations. These operations are determined by a realizer of G and can be done in linear time. Then our algorithm uses the derived G' and its transversal structure to obtain the polyline drawing.

The present paper is organized as follows. In Section 2, we recall a few definitions. In Section 3, we present the transformation from a realizer to a transversal structure. Then we present our drawing algorithm.

2 Preliminaries

The graphs are simple graphs. We abbreviate "counter clockwise" and "clockwise" as ccw and cw respectively.

Definition 1. *[6,7] Let G be a maximal plane graph of n vertices with three exterior vertices v_1, v_2, v_3 in ccw order. A* realizer $\mathcal{R}(G) = \{T_1, T_2, T_3\}$ *of G is a partition of its interior edges into three sets T_1, T_2, T_3 of directed edges such that the following holds: (1) for each $i \in \{1, 2, 3\}$, the interior edges incident to v_i are in T_i and directed toward v_i; (2) for each interior vertex of G, v has exactly one edge leaving v in each of T_1, T_2, T_3. The ccw order of the edges incident to v is: leaving in T_1, entering in T_3, leaving in T_2, entering in T_1, leaving in T_3 and entering in T_2. Each entering block could be empty.*

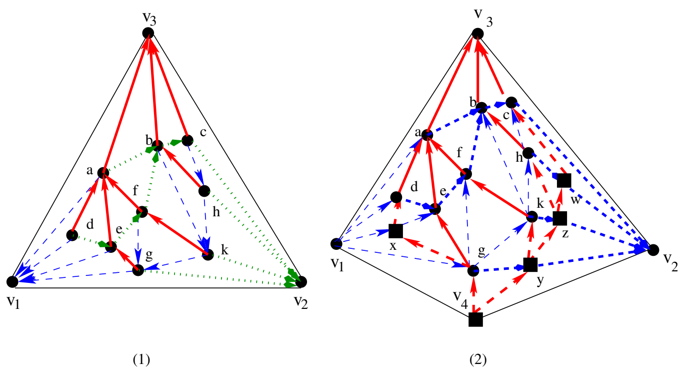

Fig. 1. (1) A maximal plane graph G and a realizer $\mathcal{R}(G)$ of G. (2) A maximal internally 4-connected plane graph G' with four exterior vertices and a transversal structure $\mathcal{T}(G')$ for G'.

Schnyder presented a linear time algorithm to construct a realizer for G. An example of a maximal plane graph G, and one of its realizers is given in Fig. 1 (1). Next, we introduce the concept of *transversal structures* [4,5].

Definition 2. *let G' be a maximal internally 4-connected plane graph with four exterior vertices v_1,v_4,v_2, and v_3 in ccw order. A* transversal structure $\mathcal{T}(G')$ *of G' is a partition of its interior edges into two sets, say in red and blue edges, such that the following conditions are satisfied:*

1. *In cw order around each interior vertex v, its incident edges form: a non empty interval of red edges entering v, a non empty interval of blue edges entering v, a non empty interval of red edges leaving v, and a non empty empty interval of blue edges leaving v.*
2. *All interior edges incident to v_3 are red edges entering v_3, all interior edges incident to v_4 are red edges leaving v_4, all interior edges incident to v_1 are blue edges leaving v_1, and all interior edges incident to v_2 are blue edges entering v_2. Each such block is non empty.*

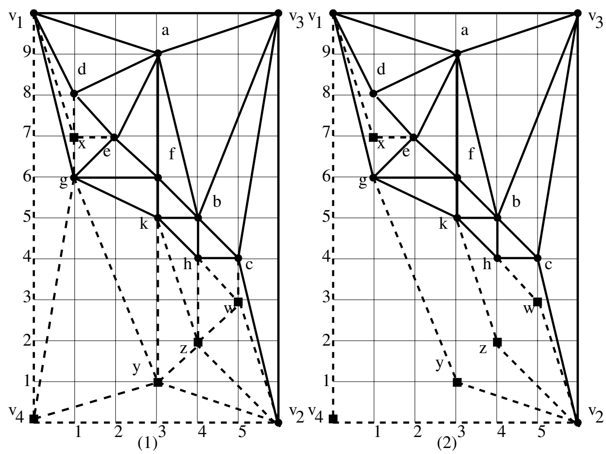

Fig. 2. (1) A straight-line grid drawing of the graph G' in Fig. 1 (2). (2) A polyline drawing of G in Fig. 1 (1).

Fig. 1 (2) shows an example of a transversal structure $\mathcal{T}(G')$ for a maximal internally 4-connected plane graph G' with four exterior vertices. The subgraph of G' with all its red-colored edges (blue colored edges respectively) and all its four exterior edges is called a *red map* (*blue map* respectively) of G', it is denoted by G'_r (G'_b respectively). For any interior vertex v of G', let $P_b(v)$ be the unique path from v_1 to v_2 in G'_b such that, the subpath of $P_b(v)$ from v_1 to v is the rightmost one before arriving at v, and the subpath of $P_b(v)$ from v to v_2 is the leftmost one after leaving v. Let $y(v)$ be the number of faces in G'_b enclosed by the path (v_1, v_4, v_2) and $P_b(v)$. Similarly, for any interior vertex v of G', let $P_r(v)$ be the unique path from v_4 to v_3 in G'_r such that, the subpath of $P_r(v)$ from v_4 to v is the rightmost one before arriving at v, and the subpath of $P_r(v)$ from v to v_3 is the leftmost one after leaving v. Let $x(v)$ be the number of faces in G'_r enclosed by the path (v_4, v_1, v_3) and $P_r(v)$.For example, vertex k in Fig. 1 (2) satisfies $P_r(k) = (v_4, y, k, f, a, v_3)$, so that $x(k) = 3$; and $P_b(k) = (v_1, g, k, b, c, v_2)$, so that $y(k) = 5$. Let $x(\mathcal{T}(G'))$ be the number of interior faces of G'_r. $y(\mathcal{T}(G'))$ be the number of interior faces of G'_b. For the vertices v_1, v_2, v_3, v_4, we define $x(v_1) = 0, y(v_1) = y(\mathcal{T}(G'))$, $x(v_4) = 0, y(v_4) = 0$,

$x(v_3) = x(\mathcal{T}(G')), y(v_3) = y(\mathcal{T}(G'))$, and $x(v_2) = x(\mathcal{T}(G')), y(v_2) = 0$. We have the following lemma from [4]:

Lemma 1. *Let G' be a maximal internally 4-connected plane graph with 4 exterior vertices v_1, v_4, v_2, v_3 in ccw order. Then:*

1. *G' admits a transversal structure $\mathcal{T}(G')$, which is computable in linear time.*
2. *Applying $\mathcal{T}(G')$, for each vertex v, embed it in the grid point $(x(v), y(v))$. For each edge of G', simply connect its end vertices by a straight line. The drawing is a straight-line grid drawing for G'. Its drawing size is $x(\mathcal{T}(G')) \times y(\mathcal{T}(G'))$. This drawing is computable in linear time.*

Fig. 2 (1) presents a straight-line grid drawing of the graph of G' in Fig. 1 (2), by applying Lemma 1 to $\mathcal{T}(G')$ in Fig. 1 (2).

3 Transformation from Realizers to Transversal Structures and Its Application in Planar Polyline Drawing

Let G be a maximal plane graph with 3 exterior vertices v_1, v_2, v_3 in ccw order. Let $\mathcal{R}(G) = \{T_1, T_2, T_3\}$ be one of its realizers. T_i is rooted at v_i. Next, we illustrate how to transform a realizer for G to a transversal structure for a targeted maximal internally 4-connected plane graph G' with 4 exterior vertices. Our transformation uses a tree from $\mathcal{R}(G)$. Subject to a color and index rotation, we only need to show the case of using T_3. Let v be a leaf node of T_3. v is an interior vertex of G. Let $p_1(v)$ and $p_2(v)$ be its parents in T_1 and T_2 respectively. The face f enclosed by $\{v, p_1(v), p_2(v)\}$ is an interior face of G. Consider the edge $e = (p_1(v), p_2(v))$. According to the property of realizer, e cannot be in T_3. Furthermore, e cannot be (v_1, v_3), neither can it be (v_2, v_3).

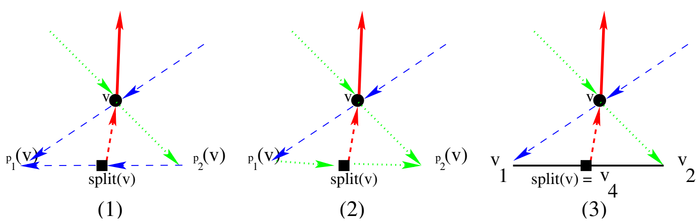

Fig. 3. Step 1

We complete the transformation in the following three steps. We will use G' to denote both the target graph and the intermediate forms.

Step 1: For every leaf node v of T_3, insert a vertex $split(v)$ in the middle of $e = (p_1(v), p_2(v))$. $split(v)$ splits e into two edges. Let the two edges keep the original color and directions as e in G. Add a directed edge from $split(v)$ to v, and color it by red. We have three different cases, as illustrated in (1), (2) and (3) of Fig. 3. Note that, in Fig. 3 (3), for the case where $e = (v_1, v_2)$, we denote the inserted vertex by v_4. v_4 is an exterior vertex of G'. G' has 4 exterior vertices v_1, v_4, v_2, v_3 in ccw order.

Step 2: For each leaf v of T_3, still consider the edge $e = (p_1(v), p_2(v))$, as if it were not split. There are three cases to consider:

Case 1: $e = (v_1, v_2)$. No additional operation needed.

Case 2: e is in T_1. e is adjacent to another triangle g. Let u be the vertex $\notin \{p_1(v), p_2(v)\}$ in g. According to the properties of realizer, only five scenarios are possible. They are shown in Fig. 4. In Fig. 4 (1) or (2), we add a directed edge from u to $split(v)$, and color it by red. In Fig. 4 (3) or (4), consider $p_2(v)$, it must also be a leaf in T_3. Therefore, in Step 1, a vertex $split(p_2(v))$ and an edge $(split(p_2(v)), p_2(v))$ have been inserted for it already. In this step, we further add an edge, directed from $split(p_2(v))$ to $split(v)$, and color it by red. Fig. 4 (5) is similar to Fig. 4 (3) or (4) except that $u = v_2$.

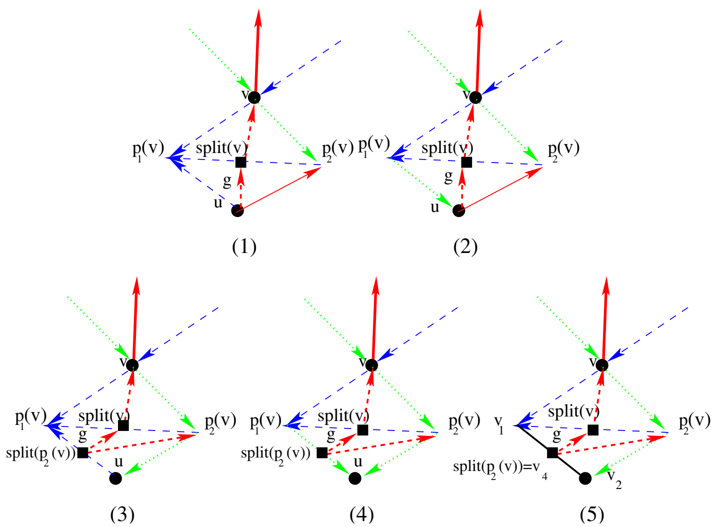

Fig. 4. Case 2 of Step 2

Case 3: e is in T_2. This case is similar to Case 2.

Step 3: Reverse the direction of the blue-colored edges in G'. Recolor the green-colored edges by blue.

The above coloring and directions of the edges of G' is denoted by $\mathcal{T}_3(G')$. (If we use T_1, T_2 instead, then we denote it by $\mathcal{T}_1(G'), \mathcal{T}_2(G')$ instead). The proof of the following lemma is omitted here due to space limitation.

Lemma 2. *Let G be a maximal plane graph with n vertices. v_1, v_2, v_3 be its exterior vertices in ccw order. $\mathcal{R}(G) = \{T_1, T_2, T_3\}$ be one of its realizers. T_i is* *rooted at* v_i. *Let* l_i *be the number of leaves of* T_i, $i \in \{1, 2, 3\}$. *Then for the above introduced transformation:*

1. $\mathcal{T}_i(G')$ *is a transversal structure of* G'. $x(\mathcal{T}_i(G')) = l_i + 1$ *and* $y(\mathcal{T}_i(G')) = (n - 2)$, *where* $i \in \{1, 2, 3\}$.
2. *The transformation from the realizer* $\mathcal{R}(G)$ *to* $\mathcal{T}_i(G')$, $i \in \{1, 2, 3\}$ *can be done in linear time.*

For the maximal plane graph G in Fig. 1 (1), Fig. 1 (2) shows a transversal structure $\mathcal{T}_3(G')$, constructed as above by using T_3 in the realizer $\mathcal{R}(G) = \{T_1, T_2, T_3\}$. The inserted vertices are represented by black squares. The inserted red-colored edges are drawn in dashed lines.

Applying Lemma 1 to $\mathcal{T}_i(G')$, we obtain a straight-line grid drawing of G'. By removing the inserted edges and the inserted vertices, but keeping the split edges in the drawing of G', it becomes a polyline drawing of G. It is easy to see that, only an edge in G which has had a vertex inserted in it during the transformation maybe drawn as two-segment polylines. The total number of such edges is l_i, i.e., the number of leaves in T_i. In [2], Bonichon et al. proved that in any realizer, $l_1 + l_2 + l_3 \leq (2n - 5)$. Combined with Lemma 2, we have the following theorem:

Theorem 1. *A plane graph* G *with* n *vertices admits a polyline drawing in a grid with size bounded by* $(p + 1) \times (n - 2)$, *where* $p \leq \lfloor \frac{2n-5}{3} \rfloor$. *The number of bends is at most* p, *and each edge has at most one bend. The drawing can be constructed in linear time.*

Fig. 2 (2) shows a polyline drawing of the original graph G in Fig. 1 (1), where the edges represented as two-segment polylines are drawn in dashed lines.

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