Red-Black Half-Sweep Iterative Method Using Triangle Finite Element Approximation for 2D Poisson Equations

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Abstract. This paper investigates the application of the Red-Black Half-Sweep Gauss-Seidel (HSGS-RB) method by using the half-sweep triangle finite element approximation equation based on the Galerkin scheme to solve two-dimensional Poisson equations. Formulations of the full-sweep and half-sweep triangle finite element approaches in using this scheme are also derived. Some numerical experiments are conducted to show that the HSGS-RB method is superior to the Full-Sweep method.

Keywords: Half-sweep Iteration, Red-Black Ordering, Galerkin Scheme, Triangle Element.

1 Introduction

By using the finite element method, many weighted residual schemes can be used by researchers to gain approximate solutions such as the subdomain, collocation, least-square, moments and Galerkin (Fletcher [4,5]). In this paper, by using the first order triangle finite element approximation equation based on the Galerkin scheme, we apply the Half-Sweep Gauss-Seidel (HSGS) method with the Red-Black ordering strategy for solving the two-dimensional Poisson equation.

To show the efficiency of the HSGS-RB method, let us consider the two-dimensional Poisson equation defined as

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(x,y), \quad (x,y) \in D = [a,b] \times [a,b] \tag{1}$$

subject to the Dirichlet boundary conditions

$$\begin{aligned} U(x,a) &= g_1(x), \quad a \le x \le b \\ U(x,b) &= g_2(x), \quad a \le x \le b \\ U(a,y) &= g_3(y), \quad a \le y \le b \\ U(b,y) &= g_4(y), \quad a \le y \le b \end{aligned}$$

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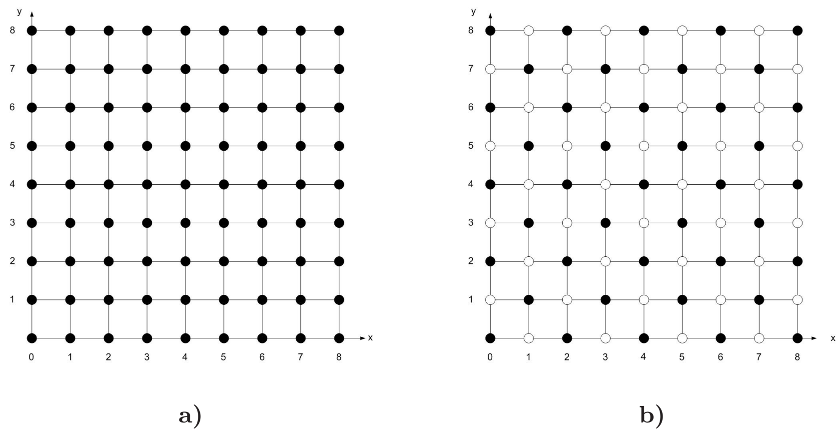


Fig. 1. a) and b) show the distribution of uniform node points for the full- and half-sweep cases respectively at $n = 7$

To facilitate in formulating the full-sweep and half-sweep linear finite element approximation equations for problem (1), we shall restrict our discussion onto uniform node points only as shown in Figure 1. Based on the figure, it has been shown that the solution domain, D is discretized uniformly in both x and y directions with a mesh size, h which is defined as

$$h = \frac{b-a}{m}, \quad m = n + 1 \tag{2}$$

Based on Figure 1, we need to build the networks of triangle finite elements in order to facilitate us to derive triangle finite element approximation equations for problem (1). By using the same concept of the half-sweep iterative applied to the finite difference method (Abdullah [1], Sulaiman *et al.* [13], Othman & Abdullah [8]), each triangle element will involves three node points only of type $\bullet$ as shown in Figure 2. Therefore, the implementation of the full-sweep and half-sweep iterative algorithms will be applied onto the node points of the same type until the iterative convergence test is met. Then other approximate solutions at remaining points (points of the different type) are computed directly (Abdullah [1], Abdullah & Ali [2], Ibrahim & Abdullah [6], Sulaiman *et al.* [13,14], Yousif & Evans [17]).

2 Formulation of the Half-Sweep Finite Element Approximation

As mentioned in the previous section, we study the application of the HSGS-RB method by using the half-sweep linear finite element approximation equation based on the Galerkin scheme to solve two-dimensional Poisson equations. By considering three node points of type $\bullet$ only, the general approximation of the

Fig. 2. a) and b) show the networks of triangle elements for the full- and half-sweep cases respectively at $n = 7$

function, $U(x,y)$ in the form of interpolation function for an arbitrary triangle element, e is given by (Fletcher [4], Lewis & Ward [7], Zienkiewicz [19])

$$\widetilde{U}^{[e]}(x,y) = N_1(x,y)U_1 + N_2(x,y)U_2 + N_3(x,y)U_3 \tag{3}$$

and the shape functions, $N_k(x,y),\ k = 1,2,3$ can generally be stated as

$$N_k(x,y) = \frac{1}{\det A}(a_k + b_k x + c_k y), \quad k = 1,2,3 \tag{4}$$

where,

$$\det A = x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2),$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

Beside this, the first order partial derivatives of the shape functions towards x and y are given respectively as

$$\left.\begin{array}{l} \frac{\partial}{\partial x}\left(N_k(x,y)\right) = \frac{b_k}{\det A} \\ \frac{\partial}{\partial y}\left(N_k(x,y)\right) = \frac{c_k}{\det A} \end{array}\right\} \quad k = 1,2,3 \tag{5}$$

Again based on the distribution of the hat function, $R_{r,s}(x,y)$ in the solution domain, the approximation of the functions, $U(x,y)$ and $f(x,y)$ in case of the full-sweep and half-sweep cases for the entire domain will be defined respectively as (Vichnevetsky [16])

$$\widetilde{U}(x,y) = \sum_{r=0}^{m}\sum_{s=0}^{m} R_{r,s}(x,y)U_{r,s} \tag{6}$$

$$\widetilde{f}(x,y) = \sum_{r=0}^{m}\sum_{s=0}^{m} R_{r,s}(x,y)f_{r,s} \tag{7}$$

and

$$\widetilde{U}(x,y) = \sum_{r=0,2,4}^{m}\sum_{s=0,2,4}^{m} R_{r,s}(x,y)U_{r,s} + \sum_{r=1,2,5}^{m-1}\sum_{s=1,3,5}^{m-1} R_{r,s}(x,y)U_{r,s} \tag{8}$$

$$\widetilde{f}(x,y)=\sum_{r=0,2,4}^{m}\sum_{s=0,2,4}^{m}R_{r,s}(x,y)f_{r,s}+\sum_{r=1,3,5}^{m-1}\sum_{s=1,3,5}^{m-1}R_{r,s}(x,y)f_{r,s} \tag{9}$$

Thus, Eqs. (6) and (8) are approximate solutions for problem (1).

To construct the full-sweep and half-sweep linear finite element approximation equations for problem (1), this paper proposes the Galerkin finite element scheme. Thus, let consider the Galerkin residual method (Fletcher [4,5], Lewis & Ward [7]) be defined as

$$\int\int_D R_{i,j}(x,y)E(x,y)\ dxdy=0,\quad i,j=0,1,2,...,m \tag{10}$$

where, $E(x,y)=\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}-f(x,y)$ is a residual function. By applying the Green theorem, Eq. 10 can be shown in the following form

$$\begin{aligned}&\oint_\lambda\left(-R_{i,j}(x,y)\tfrac{\partial U}{\partial y}\ dx+R_{i,j}(x,y)\tfrac{\partial U}{\partial x}\ dy\right)\\&-\int_a^b\int_a^b\left(\frac{\partial R_{i,j}(x,y)}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial R_{i,j}(x,y)}{\partial y}\frac{\partial U}{\partial y}\right)\ dxdy=F_{i,j}\end{aligned} \tag{11}$$

where,

$$F_{i,j}=\int_a^b\int_a^b R_{i,j}(x,y)f(x,y)\ dxdy$$

By applying Eq. (5) and substituting the boundary conditions into problem (1), it can be shown that Eq. (11) will generate a linear system for both cases. Generally both linear systems can be stated as

$$-\sum\sum K^*_{i,j,r,s}U_{r,s}=\sum\sum C^*_{i,j,r,s}f_{r,s} \tag{12}$$

where,

$$K^*_{i,j,r,s}=\int_a^b\int_a^b\left(\frac{\partial R_{i,j}}{\partial x}\frac{\partial R_{r,s}}{\partial x}\right)\ dxdy+\int_a^b\int_a^b\left(\frac{\partial R_{i,j}}{\partial y}\frac{\partial R_{r,s}}{\partial y}\right)\ dxdy$$

$$C^*_{i,j,r,s}=\int_a^b\int_a^b\left(R_{i,j}(x,y)R_{r,s}(x,y)\right)\ dxdy$$

Practically, the linear system in Eq. (12) for the full-sweep and half-sweep cases will be easily rewritten in the stencil form respectively as follows:

1. Full-sweep stencil (Zienkiewicz [19], Twizell [15], Fletcher [5])

$$\begin{bmatrix}0&1&0\\1&-4&1\\0&1&0\end{bmatrix}U_{i,j}=\frac{h^2}{12}\begin{bmatrix}0&1&1\\1&6&1\\1&1&0\end{bmatrix}f_{i,j} \tag{13}$$

2. Half-sweep stencil

$$\begin{bmatrix} 1\,0 & & 1\,0 \\ 0 & -4 & 0\,0 \\ 1\,0 & & 1\,0 \end{bmatrix} U_{i,j} = \frac{h^2}{6} \begin{bmatrix} 1\,0\,1\,0 \\ 0\,5\,0\,1 \\ 1\,0\,1\,0 \end{bmatrix} f_{i,j}, \quad i = 1 \tag{14}$$

$$\begin{bmatrix} 0\,1\,0 & & 1\,0 \\ 0\,0 & -4 & 0\,0 \\ 0\,1\,0 & & 1\,0 \end{bmatrix} U_{i,j} = \frac{h^2}{6} \begin{bmatrix} 0\,1\,0\,1\,0 \\ 1\,0\,6\,0\,1 \\ 0\,1\,0\,1\,0 \end{bmatrix} f_{i,j}, \quad i \neq 1, n \tag{15}$$

$$\begin{bmatrix} 0\,1\,0 & & 1 \\ 0\,0 & -4 & 0 \\ 0\,1\,0 & & 1 \end{bmatrix} U_{i,j} = \frac{h^2}{6} \begin{bmatrix} 0\,1\,0\,1 \\ 1\,0\,5\,0 \\ 0\,1\,0\,1 \end{bmatrix} f_{i,j}, \quad i = n \tag{16}$$

The stencil forms in Eqs. (13) till (16), which are based on the first order triangle finite element approximation equation, can be used to represent as the full-sweep and half-sweep computational molecules.

Actually, the computational molecules involve seven node points in formulating their approximation equations. However, two of its coefficients are zero. Apart of this, the form of the computational molecules for both triangle finite element schemes is the same compared to the existing five points finite difference scheme, see Abdullah [1], Abdullah and Ali [2], Yousif and Evans [17].

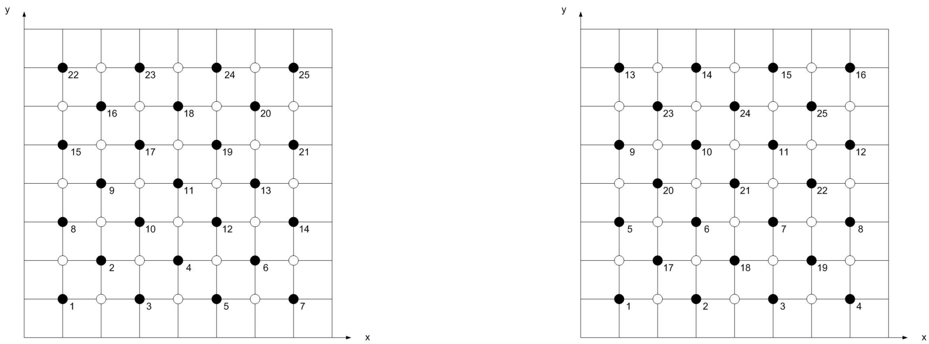
3 Implementation of the HSGS-RB

According to previous studies on the implementation of various orderings, it is obvious that combination of iterative schemes and ordering strategies which have been proven can accelerate the convergence rate, see Parter [12], Evans and Yousif [3], Zhang [18]. In this section, however, there are two ordering strategies considered in this paper such as the lexicography (NA) and red-black (RB) being applied to the HSGS iterative methods, called as HSGS-NA and HSGS-RB methods respectively. In comparison, the Full-Sweep Gauss-Seidel (FSGS) method with NA ordering, namely FSGS-NA, acts as the control of comparison of numerical results.

It can be seen from Figure 3 by using the half-sweep triangle finite element approximation equations in Eqs. (14) till (16), the position of numbers in the solution domain for $n = 7$ shows on how both HSGS-NA and HSGS-RB methods will be performed by starting at number 1 and ending at the last number.

4 Numerical Experiments

To study the efficiency of the HSGS-RB scheme by using the half-sweep linear finite element approximation equation in Eqs. [14] till [16] based on the Galerkin scheme, three items will be considered in comparison such as the number of



a) b)

Fig. 3. a) and b) show the NA and RB ordering strategies for the half-sweep case at $n = 7$

Table 1. Comparison of number of iterations, execution time (in seconds) and maximum errors for the iterative methods

Number of iterations				
Methods	Mesh size			
	32	64	128	256
FSGS-NA	1986	7368	27164	99433
HSGS-NA	1031	3829	14159	52020
HSGS-RB	1027	3825	14152	52008
Execution time (seconds)				
Methods	Mesh size			
	32	64	128	256
FSGS-NA	0.14	2.08	30.51	498.89
HSGS-NA	0.03	0.63	9.08	218.74
HSGS-RB	0.03	0.56	8.19	215.70
Maximum absolute errors				
Methods	Mesh size			
	32	64	128	256
FSGS-NA	1.4770e-4	3.6970e-5	9.3750e-6	2.8971e-6
HSGS-NA	5.7443e-4	1.6312e-4	4.4746e-5	1.1932e-5
HSGS-RB	5.7443e-4	1.6312e-4	4.4746e-5	1.1932e-5

iterations, execution time and maximum absolute error. Some numerical experiments were conducted in solving the following 2D Poisson equation (Abdullah [1])

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \left(x^2 + y^2\right) \exp(xy), \quad (x, y) \in D = [a, b] \times [a, b] \tag{17}$$

Then boundary conditions and the exact solution of the problem (17) are defined by

$$U(x, y) = exp(xy), \quad (x, y) = [a, b] \times [a, b] \tag{18}$$

All results of numerical experiments, obtained from implementation of the FSGS-NA, HSGS-NA and HSGS-RB methods, have been recorded in Table 1. In the implementation mentioned above, the convergence criteria considered the tolerance error, $\epsilon = 10^{-10}$.

5 Conclusion

In the previous section, it has shown that the full-sweep and half-sweep triangle finite element approximation equations based on the Galerkin scheme can be easily represented in Eqs. (13) till (16). Through numerical results collected in Table 1, the findings show that number of iterations have declined approximately $47.70 - 48.29\%$ and $47.68 - 48.09\%$ correspond to the HSGS-RB and HSGS-NA methods compared to FSGS-NA method. In fact, the execution time versus mesh size for both HSGS-RB and HSGS-NA methods are much faster approximately $56.76 - 78.57\%$ and $56.15 - 78.57\%$ respectively than the FSGS-NA method. Thus, we conclude that the HSGS-RB method is slightly better than the HSGS-NA method. In comparison between the FSGS and HSGS methods, it is very obvious that the HSGS method for both ordering strategies is far better than the FSGS-NA method in terms of number of iterations and the execution time. This is because the computational complexity of the HSGS method is nearly 50% of the FSGS-NA method. Again, approximate solutions for the HSGS method are in good agreement compared to the FSGS-NA method. For our future works, we shall investigate on the use of the HSGS-RB as a smoother for the halfsweep multigrid (Othman & Abdullah [8,9]) and the development and implementation of the Modified Explicit Group (MEG) (Othman & Abdullah [10], Othman *et al.* [11])and the Quarter-Sweep Iterative Alternating Decomposition Explicit (QSIADE) (Sulaiman *et al.* [14]) methods by using finite element approximation equations.

References

1. Abdullah, A.R.: The Four Point Explicit Decoupled Group (EDG) Method: A Fast Poisson Solver, Intern. Journal of Computer Mathematics, **38**(1991) 61-70.
2. Abdullah, A.R., Ali, N.H.M.: A comparative study of parallel strategies for the solution of elliptic pde's, Parallel Algorithms and Applications, **10**(1996) 93-103.
3. Evan, D.J., Yousif, W.F.: The Explicit Block Relaxation method as a grid smoother in the Multigrid V-cycle scheme, Intern. Journal of Computer Mathematics, **34**(1990) 71-78.
4. Fletcher, C.A.J.: The Galerkin method: An introduction. In. Noye, J. (pnyt.). Numerical Simulation of Fluid Motion, North-Holland Publishing Company,Amsterdam (1978) 113-170.

5. Fletcher, C.A.J.: Computational Galerkin method. Springer Series in Computational Physics. Springer-Verlag, New York (1984).
6. Ibrahim, A., Abdullah, A.R.: Solving the two-dimensional diffusion equation by the four point explicit decoupled group (EDG) iterative method. Intern. Journal of Computer Mathematics, **58**(1995) 253-256.
7. Lewis, P.E., Ward, J.P.: The Finite Element Method: Principles and Applications. Addison-Wesley Publishing Company, Wokingham (1991)
8. Othman, M., Abdullah, A.R.: The Halfsweeps Multigrid Method As A Fast Multigrid Poisson Solver. Intern. Journal of Computer Mathematics, **69**(1998) 219-229.
9. Othman, M., Abdullah, A.R.: An Effcient Multigrid Poisson Solver. Intern. Journal of Computer Mathematics, **71**(1999) 541-553.
10. Othman, M., Abdullah, A.R.: An Efficient Four Points Modified Explicit Group Poisson Solver, Intern. Journal of Computer Mathematics, **76**(2000) 203-217.
11. Othman, M., Abdullah, A.R., Evans, D.J.: A Parallel Four Point Modified Explicit Group Iterative Algorithm on Shared Memory Multiprocessors, Parallel Algorithms and Applications, **19(1)**(2004) 1-9 (On January 01, 2005 this publication was renamed International Journal of Parallel, Emergent and Distributed Systems).
12. Parter, S.V.: Estimates for Multigrid methods based on Red Black Gauss-Seidel smoothers, Numerical Mathematics, **52**(1998) 701-723.
13. Sulaiman. J., Hasan, M.K., Othman, M.: The Half-Sweep Iterative Alternating Decomposition Explicit (HSIADE) method for diffusion equations. LNCS 3314, Springer-Verlag, Berlin (2004)57-63.
14. Sulaiman, J., Othman, M., Hasan, M.K.: Quarter-Sweep Iterative Alternating Decomposition Explicit algorithm applied to diffusion equations. Intern. Journal of Computer Mathematics, **81**(2004) 1559-1565.
15. Twizell, E.H.: Computational methods for partial differential equations. Ellis Horwood Limited, Chichester (1984).
16. Vichnevetsky, R.: Computer Methods for Partial Differential Equations, Vol I. New Jersey: Prentice-Hall (1981)
17. Yousif, W.S., Evans, D.J.: Explicit De-coupled Group iterative methods and their implementations, Parallel Algorithms and Applications, **7**(1995) 53-71.
18. Zhang, J.: Acceleration of Five Points Red Black Gauss-Seidel in Multigrid for Poisson Equations, Applied Mathematics and Computation, **80(1)**(1996) 71-78.
19. Zienkiewicz, O.C.: Why finite elements?. In. Gallagher, R.H., Oden, J.T., Taylor, C., Zienkiewicz, O.C. (Eds). Finite Elements In Fluids-Volume, John Wiley & Sons,London **1**(1975) 1-23