Drawing Bipartite Graphs on Two Curves*

Emilio Di Giacomo, Luca Grilli, and Giuseppe Liotta

Dipartimento di Ingegneria Elettronica e dell'Informazione
Università degli Studi di Perugia
{digiacomo, liotta, luca.grilli}@diei.unipg.it

Abstract. Let G be a bipartite graph, and let λ_e, λ_i be two parallel convex curves; we study the question about whether G admits a planar straight line drawing such that the vertices of one partite set of G lie on λ_e and the vertices of the other partite set lie on λ_i. A characterization is presented that gives rise to linear time testing and drawing algorithms.

1 Introduction

Common requirements for drawing a bipartite graph are that the bipartition is highlighted in the visualization by representing the vertices on two distinct layers, the edges have as few bends as possible, and the number of edge crossings is minimized. A bipartite graph is a *biplanar graph* if it has a straight line crossing-free drawing where the vertices of the same partite set all lie along one of the horizontal layer [5]. Biplanar graphs have been independently characterized in [4,6,8]. Also, the problem of computing straight-line drawings of bipartite graphs with the vertices on two horizontal layers and minimum number of crossings has been well studied; see, e.g. [2,7] for some basic references on this topic.

This paper studies planar drawings of bipartite graphs where vertices are constrained to be on two parallel convex curves, which generalizes the case of horizontal layers. Let G be a bipartite graph, and let λ_e, λ_i be two parallel convex curves; we want to answer the question about whether G admits a planar straight line drawing such that the vertices of one partite set of G lie on λ_e and the vertices of the other partite set lie on λ_i.

Our interest in this question is in part motivated by the observation that the class of bipartite graphs that admit a planar straight line drawing on two horizontal lines is quite restricted and that one may hopefully enlarge this class by allowing some curvature on the two layers. Indeed, there is already some evidence in the literature that if the vertices in a drawing are not constrained to be collinear but instead can lie on curves, the family of representable graphs for specific drawing conventions can increase significantly; see, e.g. [3] for drawings of planar graphs with at most one bend per edge and vertices constrained to be on a given curve.

The problem addressed in this paper is also related to the study of *radial planarity testing* initiated by Bachmaier, Brandenburg and Forster [1]. In [1] the

* Research partially supported by MIUR under Project "ALGO-NEXT".

M. Kaufmann and D. Wagner (Eds.): GD 2006, LNCS 4372, pp. 380–385, 2007.

input is a k-partite graph G and k-concentric circles; the question is whether G has a crossing-free drawing where the vertices of the same partite set are points of the same radial level (circle) and the edges are simple Jordan curves in the outward direction. Here, we study radial planarity testing for bipartite graphs with the additional constraint that the edges are straight-line segments (indeed, two concentric circles are a special case of two parallel convex curves).

Our contribution is as follows. The family of bipartite graphs which admit a planar straight-line drawing with the vertices constrained to be on two parallel convex curves and with no two vertices of the same partite set on different curves is characterized. The characterization gives rise to a linear time testing algorithm. The proof of sufficiency uses a linear time (real RAM) drawing algorithm.

2 Preliminaries

A graph $G = (V, E)$ is *bipartite* if there exists a partition $V = V_0 \cup V_1$ of the vertices of G such that $E \subseteq V_0 \times V_1$. The two sets V_0 and V_1 are called *partite sets* of G. A bipartite graph with a given planar embedding is *maximal* if every internal face of G consists of four edges.

A simple curve λ in the Euclidean plane is a *closed* curve if it partitions the plane into two topologically connected regions; λ is an *open* curve otherwise. Curve λ is *convex* if any straight line intersects λ in at most two points. Note that a circle is a special case of closed convex curve.

Let p, q be two distinct points of λ. If λ is an open curve we say that p *precedes* q on λ if p is encountered before q when traversing λ in the clockwise direction. If λ is a closed curve, let p and q be two distinct points of λ such that the portion of λ traversed when going from p to q in the clockwise direction is shorter than the portion of λ traversed when going from q to p; we say that p *precedes* q and that q *follows* p on λ.

Two convex curves are *parallel* if every normal to one curve is also a normal to the other curve and the distance between the points where the normals intersect the two curves is a constant. In the rest of this paper we denote with λ_e, λ_i two parallel convex curves such that the curvature of λ_e is less than the curvature of λ_i; λ_e is the *external curve*, λ_i is the *internal curve* (in the special case of two concentric circles, λ_e is the circle with larger radius). Curves λ_e, λ_i are *paired* if there exist two points $p \in \lambda_e$ and $q \in \lambda_i$ such that the straight-line segment $\overline{pq}$ intersects λ_i twice. A straight-line segment with the property of $\overline{pq}$ is said to *cross* curve λ_i. Observe that two concentric circles are paired. Two curves will be called *non-paired* if they are parallel, convex, but are not paired.

Let λ_e, λ_i be two parallel convex curves. A bipartite graph G is *curve biplanar* on λ_e, λ_i if it admits a *curve biplanar drawing*, i.e. a planar straight-line drawing such that all vertices of a bipartite set of G are represented as points on λ_e and the vertices of the other bipartite set are represented as points on λ_i. As the next theorem shows, if λ_e, λ_i are not paired, the family of curve biplanar graphs on two non-paired curves coincides with the family of biplanar graphs characterized in [4,6,8]. The proof is an easy adaptation of the arguments in [4,6,8] and it has been omitted for space reasons. A graph is a *caterpillar* if deleting all vertices of degree one produces a (possibly empty) path.

Theorem 1. *A bipartite graph admits a curve biplanar drawing on two non-paired curves if and only if it is a forest of caterpillars.*

Motivated by Theorem 1, we will investigate the family of bipartite graphs that admit a curve biplanar drawing on two paired curves. We first show how to draw a specific family of graphs, namely *bipartite fans*, and then present a complete characterization of curve biplanar graphs on two paired curves.

3 How to Draw a Bipartite Fan

Let G be a biconnected bipartite graph with a given planar embedding. G is a *bipartite fan* if it has a vertex u, called *apex*, that is shared by all its faces (including the external one). The edges incident on u are the *radial edges* of the fan. Let $u, v_0, v_1, \ldots, v_{n-2}$ be the vertices of a fan G in the clockwise order they have on the external face. Edges (u, v_0) and (u, v_{n-2}) are called *first edge* and *last edge* of the fan, respectively. Any three vertices $v_{2j}, v_{2j+1}, v_{2j+2}$ $(0 \leq j \leq \frac{n-4}{2})$ form a *fan triplet* of G. Notice that v_{2j+1} belongs to the same partite set as u.

We show how to compute a curve biplanar drawing of a bipartite fan on two paired curves such that the drawing is contained in a suitable region of the plane called a *wedge* and defined as follows. Let λ_e, λ_i be two paired curves, let p, q, r be three points such that: (i) $p, r \in \lambda_e$ and p precedes r on λ_e, (ii) $q \in \lambda_i$, (iii) segment $\overline{pq}$ does not cross curve λ_i, (iv) segment $\overline{qr}$ crosses λ_i. Let λ_{pr} be the portion of λ_e consisting of all points $x \in \lambda_e$ such that x follows p and precedes r. The closed bounded region delimited by $\overline{pq}$, $\overline{qr}$ and λ_{pr} is a wedge of λ_e, λ_i and is denoted as $W(p, q, r)$.

Lemma 1. *Let G be a bipartite fan with n vertices and apex u. Let λ_e, λ_i be two paired curves and let $W(p, q, r)$ be a wedge of λ_e, λ_i. Fan G admits a curve biplanar drawing on λ_e, λ_i contained inside $W(p, q, r)$ such that: (i) The first and the last edge of G are represented by segments $\overline{pq}$ and $\overline{qr}$, respectively; (ii) For every fan triplet $v_{2j}, v_{2j+1}, v_{2j+2}$ of G $(0 \leq j \leq \frac{n-2}{2})$, the three points representing the triplet define a wedge of λ_e, λ_i.*

Sketch of Proof: We assume that G is maximal, i.e. that every internal face consists of four edges; if not, we can split each internal face f having more than four edges by connecting u to all vertices of f that are not adjacent to u and do not belong to the same partite set of u (it is immediate to see that the resulting augmented graph is still a bipartite fan). Let $u, v_0, v_1, \ldots, v_{n-2}$ be the vertices of fan G in the clockwise order they have on its external face.

In what follows refer to Figure 1 for an illustration. The apex of the fan u is drawn at point q; vertices v_0 and v_{n-2} are drawn at points p and r, respectively. Thus, the first edge of G is represented by segment $\overline{pq}$ and the last edge of G is represented by segment $\overline{qr}$. Let x be the point where segment $\overline{qr}$ crosses λ_i.

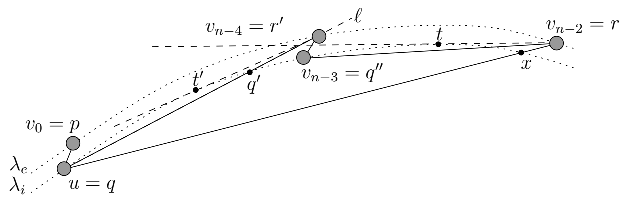

Fig. 1. An illustration of the drawing technique for a bipartite fan

For the convexity of λ_i, the straight line through r and tangent to λ_i intersects λ_i at a point t such that t follows q and precedes x on λ_i; since $\overline{pq}$ does not cross λ_i, then t is inside wedge $W(p, q, r)$. If $n = 4$, i.e. G is a 4-cycle, choose a point q' that follows q and precedes t on λ_i and such that segment $\overline{pq'}$ does not cross λ_i. Draw v_1 at point q'. Since q' precedes t, segment $\overline{q'r}$ crosses λ_i, while segment $\overline{pq'}$ does not cross λ_i and therefore points p, q', and r, which represent the vertices of the fan triplet v_0, v_1, v_2, define a wedge $W(p, q', r)$.

If $n > 4$ choose a point q' that follows q and precedes t on λ_i and let ℓ be the straight line through q and q'. Draw vertex v_{n-4} at point $r' = \ell \cap \lambda_e$. Draw vertex v_{n-3} at any point q'' of λ_i such that q'' follows q' and precedes t on λ_i. Segment $\overline{q''r}$ crosses λ_i because q'' precedes t on λ_i and λ_i is convex. Let t' be the point where the tangent to λ_i through r' intersects λ_i; t' follows q and precedes q' on λ_i. It follows that t' precedes q'' and hence segment $\overline{r'q''}$ does not cross λ_i. Therefore points r', q'', r, which represent the vertices of the fan triplet $v_{n-4}, v_{n-3}, v_{n-2}$, define a wedge $W(r', q'', r)$. In order to complete the drawing of G, we observe that segment $\overline{qr'}$ crosses λ_i, and therefore points p, q, and r' define a wedge $W(p, q, r')$. We recursively draw the fan obtained from G after removing vertices v_{n-3} and v_{n-2} inside wedge $W(p, q, r')$. □

4 Curve Biplanar Graphs

We start with a sufficient condition whose proof uses the following definition. Let $G = (V, E)$ be a connected graph. A subset of vertices $S \subset V$ is a *cut-set* if the removal of S disconnects G. Let $G_0, \ldots, G_{k-1}$ be the connected components of $G - S$ (possibly isolated vertices). The *S-components* of G are the subgraphs of G induced by sets $V(G_j) \cup S$ $(0 \leq j \leq k-1)$.

Lemma 2. *Let G be a biconnected bipartite graph with a given planar embedding such that all vertices in one partite set belong to the external face. Then G is curve biplanar on two paired curves.*

Sketch of Proof: We describe now how to compute an embedding preserving curve biplanar drawing of G. To this aim, we decompose it into subgraphs that are bipartite fans and draw each fan by using the technique described in Lemma 1.

Let V_0 and V_1 be the two partite sets of G and assume that all vertices of V_0 belong to the external face of G in the given embedding. Since G is bipartite, there exists a vertex $u \in V_1$ such that u belongs to the external face of G. Let F_u be the subgraph of G induced by all vertices that share an internal face with u (F_u exists because G is biconnected). Note that F_u is a bipartite fan, which we call *the fan of* u. Let λ_e and λ_i be two paired curves, let $W(p, q, r)$ be an arbitrarily chosen wedge of λ_e, λ_i (such a wedge always exists because λ_e and λ_i are paired). By Lemma 1, F_u can be drawn inside $W(p, q, r)$ so that its first and last edge are represented by segments $\overline{pq}$ and $\overline{qr}$, respectively.

Let $u, v_0, v_1, \ldots, v_{n-2}$ be the vertices of F_u in the clockwise order they have on the external face of F_u. Since $u \in V_1$, vertices v_{2j} ($j = 0, 1, \ldots, \frac{n-2}{2}$) belong to V_0 and are on the external face of G. This implies that every fan triplet $\tau = \{v_{2j}, v_{2j+1}, v_{2j+2}\}$ is a cut-set for G unless v_{2j+1} is on the external face. Indeed, since G is biconnected, the boundary of its external face is a cycle C, and if $v_{2j+1} \in \tau$ is not on the external face, then the vertices of τ induce a path that splits C in two paths C' and C''; since no edge of G can connect a vertex of C' to a vertex of C'' (otherwise G would not be planar), then τ is a cut-set also for G. The τ-component of G that does not contain u is a planar bipartite graph that satisfies the condition expressed by the statement and, by Lemma 1, points p', q', r' representing the vertices of τ in the drawing of F_u define a wedge $W(p', q', r')$. The τ-component of G that does not contain u can be recursively drawn inside $W(p', q', r')$. □

Theorem 2. *A bipartite graph G is curve biplanar on two paired curves if and only if it admits a planar embedding such that all vertices in one partite set belong to the external face. Also, if G is curve biplanar on two paired curves, a curve biplanar drawing of G on two paired curves can be computed in $O(n)$ time in the real RAM model of computation, where n is the number of vertices of G.*

Sketch of Proof: Sufficiency. Let V_0 be a partition set of G such that all vertices of V_0 belong to the external face of a planar embedding of G. If G is biconnected, the sufficiency follows from Lemma 2. Otherwise, G can be augmented by adding dummy vertices and edges such that the augmented graph G' is biconnected and bipartite, one of its partition sets is V_0, and has a planar embedding with the vertices of V_0 on the external face (for reasons of space we do not describe in more detail this augmentation technique). It follows that G' has a biplanar drawing on two paired curves by Lemma 2 and hence G is curve biplanar on two paired curves.

Necessity. Let Γ be a curve biplanar drawing of a graph G on two paired curves λ_e and λ_i. All vertices drawn as points of λ_e are on the external face of Γ because the curves are convex and the drawing is straight-line. Since all vertices on the same curve are in the same partite set, G admits a planar embedding such that all vertices in one of the partite set belong to the external face.

Time complexity. The augmentation technique can be performed in time proportional to the number of biconnected components of G. The fan F_u of u can be computed in time proportional to the number n_u of vertices in F_u. It can be proved that the drawing of F_u can be computed in $O(n_u)$ time with the technique of Lemma 1 if the real RAM model of computation is adopted. Since each vertex belongs to at most three fans, the time complexity is $O(n)$. □

Theorem 3. *Let G be a bipartite planar graph with n vertices. The curve biplanarity of G on two paired curves can be tested in $O(n)$ time.*

Sketch of Proof: A curve biplanarity test can be performed by executing at most two planarity tests with the additional constraint that all vertices in one of the partition sets of G belong to the external face. □

A k-partite graph $G = (V_0, \ldots, V_{k-1}, E)$ is *radial k-level planar* if it admits a planar drawing on k concentric circles $C_0, \ldots, C_{k-1}$, with the vertices of partite set V_j drawn on circle C_j $(0 \leq j \leq k-1)$ and the edges drawn as strictly monotone curves from inner to outer circles. A linear time algorithm for radial planarity testing and embedding is presented in [1]. Theorems 2 and 3 make it possible to specialize radial 2-level planarity testing to the case that the edges are straight-line segments.

Corollary 1. *A bipartite graph G is radial 2-level planar with straight-line edges if and only if it admits an embedding such that all vertices in one partite set belong to the external face. Also, there exists an $O(n)$-time algorithm that tests whether a bipartite graph G is radial 2-level planar with straight-line edges.*

5 Open Problems

- Extend the study to k-partite graphs and k parallel curves with $k > 2$. In particular, it would be interesting to study radial planarity testing with straight-line edges and more than two concentric circles.
- Study the complexity of the following problem: Let G be a planar bipartite graph and let c be a positive integer. Does G have a curve biplanar subgraph (not necessarily induced) with at least c edges?
- Study the complexity of the edge crossing minimization problem for straight-line drawings of bipartite graphs on two parallel convex curves.

References

1. C. Bachmaier, F. J. Brandenburg, and M. Forster. Radial level planarity testing and embedding in linear time. *J. Graph Algorithms Appl.*, 9(1):53–97, 2005.
2. G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing.* Prentice Hall, Upper Saddle River, NJ, 1999.
3. E. Di Giacomo, W. Didimo, G. Liotta, and S. K. Wismath. Curve-constrained drawings of planar graphs. *Comput. Geom.: Theory and Appl.*, 30:1–23, 2005.
4. P. Eades, B. D. McKay, and N. C. Wormald. On an edge crossing problem. In *Proc 9th Austr. Comp. Sci. conf.*, pages 327–334. Australian National University, 1986.
5. P. Eades and S. H. Whitesides. Drawing graphs in two layers. *Theoretical Computer Science*, 131(2):361–374, 1994.
6. F. Harary and A. Schwenk. A new crossing number for bipartite graphs. *Utilitas Mathematica*, 1:203–209, 1972.
7. M. Kaufmann and D. Wagner, editors. *Drawing Graphs*, volume 2025 of *Lecture Notes in Computer Science.* Springer-Verlag, 2001.
8. N. Tomii, Y. Kambayashi, and S. Yajima. On planarization algorithms of 2-level graphs. Technical Report EC77-38, Inst. of Elect. and Comm. Eng. Japan, 1977.