Open Rectangle-of-Influence Drawings of Inner Triangulated Plane Graphs

Kazuyuki Miura[1], Tetsuya Matsuno[2], and Takao Nishizeki[3]

[1]Faculty of Symbiotic Systems Science
Fukushima University, Fukushima 960-1296, Japan
[2,3]Graduate School of Information Sciences
Tohoku University, Sendai 980-8579, Japan
miura@sss.fukushima-u.ac.jp
matsuno@nishizeki.ecei.tohoku.ac.jp
nishi@ecei.tohoku.ac.jp

Abstract. A straight-line drawing of a plane graph is called an open rectangle-of-influence drawing if there is no vertex in the proper inside of the axis-parallel rectangle defined by the two ends of every edge. In an inner triangulated plane graph, every inner face is a triangle although the outer face is not always a triangle. In this paper, we first obtain a sufficient condition for an inner triangulated plane graph G to have an open rectangle-of-influence drawing; the condition is expressed in terms of a labeling of angles of a subgraph of G. We then present an $O(n^{1.5}/\log n)$-time algorithm to examine whether G satisfies the condition and, if so, construct an open rectangle-of-influence drawing of G on an $(n-1) \times (n-1)$ integer grid, where n is the number of vertices in G.

1 Introduction

Recently automatic aesthetic drawing of graphs has created intense interest due to their broad applications, and as a consequence, a number of drawing methods have come out [1,3,4,5,6,12,13,14,15]. The most typical drawing of a plane graph G is a *straight-line drawing* in which all vertices of G are drawn as points and all edges are drawn as straight-line segments without any edge-intersection. A straight-line drawing is called a *grid drawing* if all vertices are put on grid points of integer coordinates. Figure 1 depicts three grid drawings of the same graph.

In this paper, we deal with a type of a grid drawing under an additional constraint, known as a "*rectangle-of-influence drawing*" [11]. A *rectangle-of-influence* of an edge e is an axis-parallel rectangle having e as one of its diagonals. In each of Figs. 1(a)–(c) a rectangle-of-influence is shaded for an edge $e = (u, v)$ drawn by a thick line. We call a grid drawing a *rectangle-of-influence drawing* (or simply an *RI-drawing*) if there is no vertex in a rectangle-of-influence of any edge. Figures 1(a) and (b) depict RI-drawings, while Fig. 1(c) depicts a grid drawing which is not an RI-drawing. An RI-drawing often looks pretty, since vertices tend to be separated from edges.

M. Kaufmann and D. Wagner (Eds.): GD 2006, LNCS 4372, pp. 138–149, 2007.

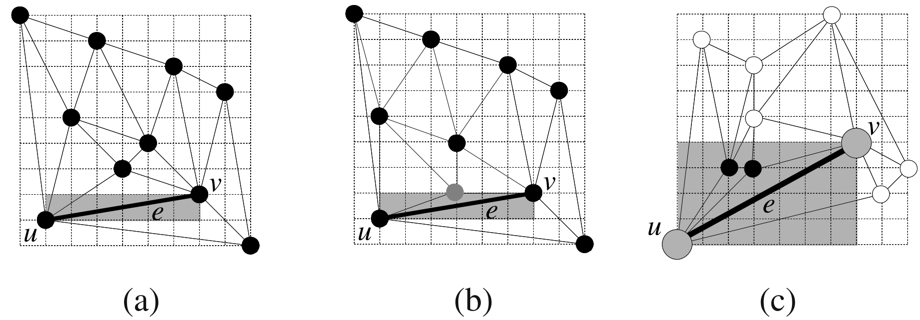

Fig. 1. (a) A closed RI-drawing, (b) an open RI-drawing, and (c) a non-RI-drawing of an inner triangulated plane graph without filled 3-cycles

A rectangle-of-influence of an edge e is *closed* if it contains the boundary of a rectangle, and is *open* if it does not contain the boundary. In a *closed RI-drawing* every rectangle-of-influence is regarded as a closed one, while in an *open RI-drawing* every rectangle-of-influence is regarded as an open one. In a closed RI-drawing, there is no vertex except the ends not only in the proper inside of a rectangle-of-influence of each edge but also on the boundary, as illustrated in Fig. 1(a). In an open RI-drawing, there may be a vertex other than the ends on the boundary of a rectangle, as illustrated in Fig. 1(b). Thus a closed RI-drawing is an open RI-drawing, but an open RI-drawing is not always a closed RI-drawing.

Biedl *et al.* [1] showed that a plane graph G has a closed RI-drawing if and only if G has no filled 3-cycle, that is, a cycle of three vertices such that there is a vertex in the proper inside. They also presented a linear-time algorithm to find a closed RI-drawing of G on an $(n-1) \times (n-1)$ grid if G has no filled 3-cycle, where n is the number of vertices in G. It is also known that every 4-connected plane graph with four or more vertices on the outer facial cycle has an open RI-drawing on a smaller grid, that is, a $W \times H$ grid with $W + H \leq n$, and such a drawing can be found in linear time [12], where W and H are the width and height of an integer grid, respectively. A plane graph G may have an open RI-drawing even if G has a filled 3-cycle. However, a necessary and sufficient condition for an open RI-drawing has not been known.

In a *triangulated* plane graph, all facial cycles are 3-cycles. In an *inner triangulated* plane graph, all inner facial cycles are 3-cycles although the outer facial cycle is not necessarily a 3-cycle, as illustrated in Fig. 2(a). Every plane graph can be augmented to an inner triangulated plane graph under some constraint [2].

In this paper we deal with open RI-drawings of triangulated plane graphs and inner triangulated plane graphs. We first show that one can decide in linear time whether a given triangulated plane graph G has an open RI-drawing, and that if G has such a drawing then it can be constructed in linear time on a $W \times H$

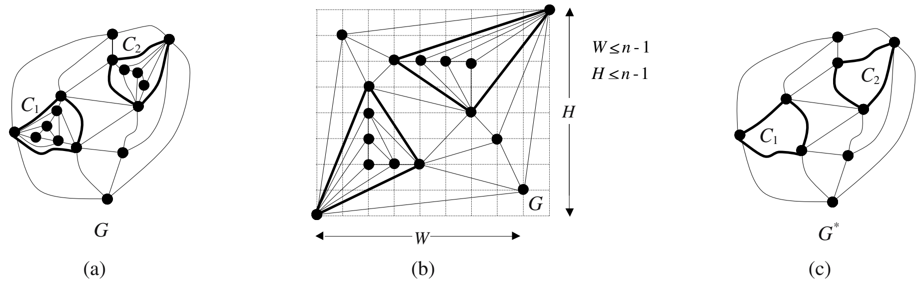

Fig. 2. (a) An inner triangulated plane graph G with two maximal filled 3-cycles C_1 and C_2, (b) an open RI-drawing of G, and (c) a graph G^* without filled 3-cycles

grid with $W + H = n$, where n is the number of vertices in G. (See Fig. 3.) We then obtain a sufficient condition for an inner triangulated plane graph G to have an open RI-drawing. (See Figs. 2(a) and (b).) Our condition is expressed in terms of a labeling of angles of a subgraph G^* of G with integers 0, 1, 2, 3 and 4, where G^* is obtained from G by removing all vertices and edges in the proper inside of every maximal filled 3-cycle of G. Figure 2(c) depicts G^* for G in Fig. 2(a). Note that G^* is an inner triangulated plane graph. We also present an $O(n^{1.5}/\log n)$-time algorithm to examine whether G satisfies the condition and, if so, construct an open RI-drawing of G on an $(n-1) \times (n-1)$ grid. The complexity $O(n^{1.5}/\log n)$ is due to a step where the algorithm finds a perfect matching in a bipartite graph. It would be interesting to know if the complexity can be improved. In the case where G has no filled 3-cycle, our algorithm provides a closed RI-drawing of G. It is an alternative algorithm to the algorithm of Biedl *et al.* [1] for the family of inner triangulated plane graphs with no filled 3-cycle.

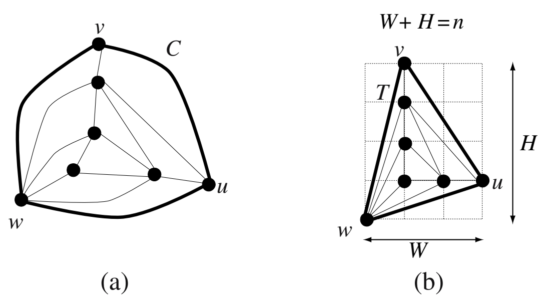

Fig. 3. (a) A triangulated plane graph G, and (b) an open RI-drawing D of G

2 Drawing Triangulated Plane Graphs

Suppose that G is a triangulated plane graph with four or more vertices as illustrated in Fig. 3(a), and that G has an open RI-drawing D as illustrated in

Fig. 3(b). The outer facial cycle $C = uvw$ of G is a filled 3-cycle, and is drawn as a triangle T in D. A straight-line segment is *oblique* if it is neither horizontal nor vertical. Two or three sides of T are oblique; otherwise, T has exactly one oblique side, and hence T is a right-angled triangle having both a vertical side and a horizontal side; since the proper inside of such a triangle T is covered by the open rectangle-of-influence of the oblique side, the inner vertices of G could not be drawn. Thus there are the following three cases to consider.

(a) Two sides of T are oblique and the other side is horizontal, as illustrated in Fig. 4(a);
(b) Two sides of T are oblique and the other side is vertical, as illustrated in Fig. 4(b); and
(c) all the three sides of T are oblique, as illustrated in Fig. 4(c).

Only the line segments in T drawn by thick lines in Figs. 4(a)–(c) are not covered by the open rectangle-of-influences of three edges of C. Therefore, all inner vertices of G must be located on the thick line segments in Figs. 4(a)–(c). Thus one can know that the graph G and the drawing D must have the structure illustrated in Fig. 4(f). More precisely, one of the three vertices u, v and w of C, say w, is adjacent to all the other vertices $z_1, z_2, \cdots, z_{n-1}$ in G. One may assume that $z_1 = v$, $z_{n-1} = u$, and $z_1, z_2, \cdots, z_{n-1}$ is a path in the triangulated plane graph G. Then, for some index c, $2 \leq c \leq n-2$, every edge of G, that is neither incident to w nor on the path $z_1, z_2, \cdots, z_{n-1}$, joins vertices z_i and z_j with $1 \leq i < c < j \leq n-1$. The drawings in Figs. 4(d) and (e) are particular cases in which exactly two of the three outer vertices, say v and w, are adjacent to all the other vertices in G and hence $c = 2$. Note that $G = K_4$ if each of u, v and w is adjacent to all the other vertices in G.

Conversely, if G has the structure above, illustrated in Fig. 4(f), then G has an open RI-drawing on a $W \times H$ grid such that $W + H = n$. Note that $W = n - c$ and $H = c$ for the index c above.

We thus have the following theorem.

Theorem 1. One can decide in linear time whether a given triangulated plane graph G has an open RI-drawing or not. If G has such a drawing, then it can be constructed in linear time on a $W \times H$ grid such that $W + H = n$.

3 Drawing Inner Triangulated Plane Graphs

In this section, we first present a sufficient condition for an inner triangulated plane graph G to have an open RI-drawing, and then give an algorithm to examine whether G satisfies the condition and, if so, construct an open RI-drawing of G. We may assume that G is 2-connected.

3.1 Sufficient Condition

If G has no filled 3-cycle, then G has a closed RI-drawing [1], which is an open RI-drawing. Therefore, we may assume without loss of generality that G has

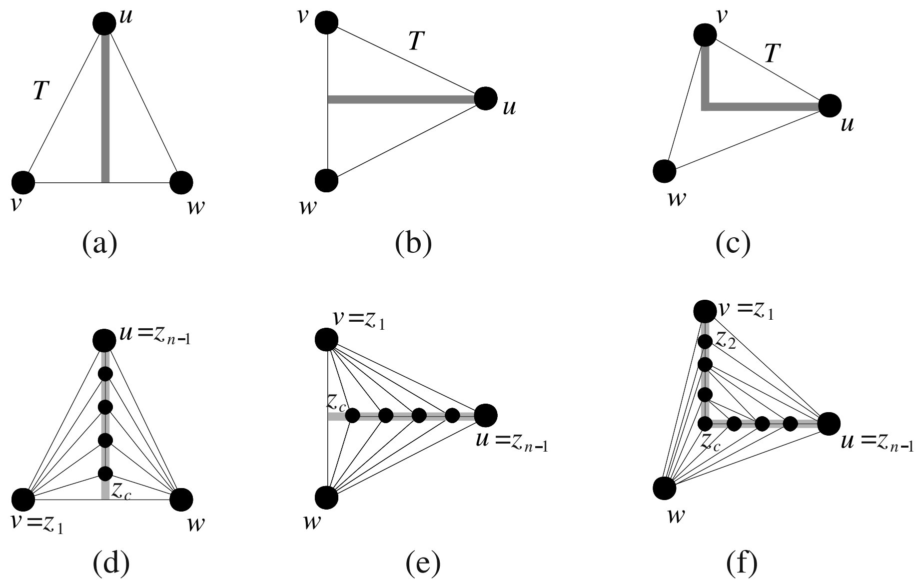

Fig. 4. (a)–(c) Three shapes of triangle T, and (d)–(f) graphs G and drawings D

filled 3-cycles. Let $C_1, C_2, \cdots, C_k$, $k \geq 1$, be the maximal filled 3-cycles of G. The plane graph G in Fig. 2(a) has two maximal filled 3-cycles C_1 and C_2 drawn by thick lines, and hence $k = 2$. We denote by $G(C_i)$ the *inside graph* induced by the vertices of C_i and the vertices inside C_i. $G(C_i)$ is a triangulated plane graph. (Figure 3(a) depicts $G(C_1)$ for the graph G and a maximal filled 3-cycle C_1 in Fig. 2(a).) One may assume without loss of generality that the inside graph $G(C_i)$ for every maximal filled 3-cycle C_i has an open RI-drawing; otherwise, G has no open RI-drawing.

One can transform an arbitrary open RI-drawing of G in a way that every edge of G^* is oblique. (The proof is omitted in this extended abstract.) Thus, one may assume without loss of generality that, in an open RI-drawing D of G, every edge of G^* is oblique, as illustrated in Fig. 2(b). A vertex on the outer facial cycle of G^* is called an *outer vertex*, while a vertex not on the outer facial cycle is called an *inner vertex*. An angle of (a polygonal drawing of) a face of G^* is called an *angle* of G^*. (See Fig. 7.) An angle of an inner face is called an *inner angle*, while an angle of the outer face is called an *outer angle*. At each vertex v in G^*, draw two lines, one with slope 0 and one with slope ∞, as illustrated in Fig. 5. These two lines define four half-lines at v. We say that an angle at v contains a number i of the four half-lines, $0 \leq i \leq 4$, if the region of the plane defined by that angle contains i half-lines at v. Thus, in Fig. 5, angles $\alpha_0, \alpha_1, \alpha_2$ and α_3 contain 0,1,2 and 1 half-lines, respectively. In Fig. 6, the outer angles of outer vertices v_i, $0 \leq i \leq 4$, contains i half-lines.

Our condition is expressed in terms of a labeling of G^*. A *labeling* L^* of G^* is an assignment of label 0,1,2,3 or 4 to each angle of G^*, as illustrated in Fig. 7(a).

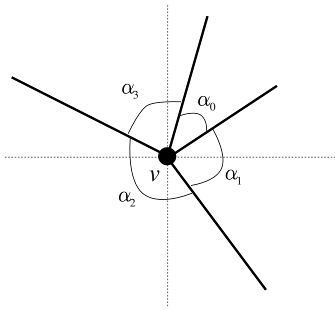

Fig. 5. Angles $\alpha_0, \alpha_1, \alpha_2$ and α_3

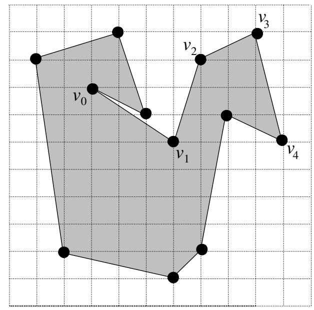

Fig. 6. Non-convex outer polygon

Label i, $0 \leq i \leq 4$, means that the angle with label i contains i half-lines. We say that a grid drawing D^* of G^* *realizes the labeling* L^* if every angle labeled i by L^* contains i half-lines in D^* for each i, $0 \leq i \leq 4$. For a grid drawing D^* of G^*, we denote by $L(D^*)$ the labeling of G^* induced by D^*.

Let D^* be a drawing of G^* in an open RI-drawing D of G, and let $L(D^*)$ be a labeling of G^* induced by D^*. Clearly $L(D^*)$ satisfies the following condition:

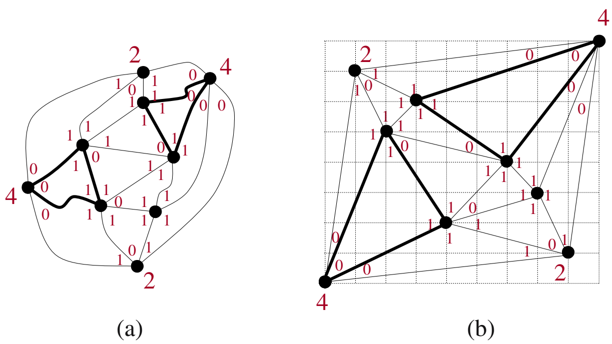

Fig. 7. (a) A good labeling L^* of G^*, and (b) a good open RI-drawing D^* of G^* realizing L^*

(a) For each vertex v of G^*, the labels around v total to 4.

We now claim that $L(D^*)$ satisfies the following condition:

(b) Every inner facial 3-cycle C of G^* has labels $0, 1$ and 1. If C is a maximal filled 3-cycle in G, then the vertex labeled 0 in C is adjacent to all the other vertices of the inside graph $G(C)$ of C; (See Fig. 8.)

Since every edge of G^* is oblique in D^*, every inner facial 3-cycle C of G^* is drawn as a triangle T having three oblique sides. Furthermore, two angles in C

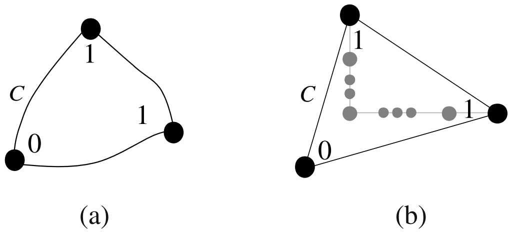

Fig. 8. Labelings of (a) a non-filled 3-cycle C and (b) a filled 3-cycle C

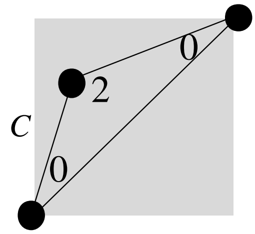

Fig. 9. A triangle such that an angle contain two half-lines

contain exactly one half-line and the other angle does not contain any half-line as illustrated in Fig. 8(b); if an angle in C contains two half-lines, then the vertex of the angle would be in the proper inside of the rectangle-of-influence of the longest edge of T as illustrated in Fig. 9. Hence C has labels $0, 1$ and 1 in the labeling $L(D^*)$. If C is a maximal filled 3-cycle in G, then $G(C)$ is a triangulated plane graph and the vertex of C labeled 0 is adjacent to all the other vertices of $G(C)$ as shown in Section 2. Thus $L(D^*)$ satisfies Condition (b).

Thus it is necessary for G to have an open RI-drawing that G^* has a labeling satisfying Conditions (a) and (b). However, the converse is not true. We will show in Section 3.2 that G has an open RI-drawing if G^* has a labeling satisfying Conditions (a), (b) and the following additional condition:

(c) Every outer angle has label $2, 3$ or 4.

A labeling of G^* satisfying Conditions (a)–(c) is called a *good labeling*. (The good labeling has a close relation with the regular edge-labeling of Kant and He [10].) We thus have the following theorem.

Theorem 2. An inner triangulated plane graph G has an open RI-drawing if G^* has a good labeling.

One may prefer to draw the outer facial cycle of G as a convex polygon, for which each outer angle contains two, three or four half-lines. We say that an open RI-drawing D of G is *good* if each outer angle contains two, three or four half-lines. For example, the drawings in Figs. 1(a), 1(b), 2(b) and 3(b) are good open RI-drawings, while an open RI-drawing having the non-convex outer facial polygon in Fig. 6 is not good. It should be noted that the outer facial polygon of a good open RI-drawing is not necessary a convex polygon. Indeed our result implies that G has a good open RI-drawing if and only if G^* has a good labeling.

3.2 Computing an Open RI-Drawing from a Good Labeling

Suppose that G^* has a good labeling L^* as illustrated in Fig. 7(a). Remember that we assume that each triangulated plane graph $G(C_i)$ has an open

RI-drawing. We first obtain an open RI-drawing D_i of each $G(C_i)$ as in Section 2. We then construct an open RI-drawing D^* of G^* from L^*, as illustrated in Fig. 7(b). We finally embed in D^* each drawing D_i after adjusting the size of D_i to the triangular drawing of C_i in D^*, as illustrated in Fig. 2(b). We claim that the resulting drawing is an open RI-drawing of G.

Our algorithm for constructing D^* from L^* consists of the following three steps.

(Step 1) Directing each edge (u, v) of G^*, we construct a directed graph G_x as illustrated in Fig. 10(a); $u \rightarrow v$ if $x(u) < x(v)$ must hold in an open RI-drawing D^* of G^* realizing the labeling L^*, where $x(u)$ and $x(v)$ are x-coordinates of u and v, respectively. Similarly, we construct a directed graph G_y as illustrated in Fig. 10(c). More precisely, we construct G_x and G_y as follows.

Let v_1 and v_2 be any two outer vertices consecutively appearing clockwise on the outer facial cycle of G^*. A drawing obtained from an open RI-drawing D^* of G by rotating it 90°, 180° or 270° is also an open RI-drawing. Therefore, one may assume without loss of generality that $x(v_1) < x(v_2)$ and $y(v_1) < y(v_2)$ in D^*. Let $C = v_1v_2v_3$ be the inner facial 3-cycle of G^* having the edge (v_1, v_2). Then the good labeling L^* assigns label 0 to one of the vertices v_1, v_2 and v_3 of C and assigns label 1 to the other two vertices. If v_1 has label 0, then we decide that $x(v_1) < x(v_2) < x(v_3)$ and $y(v_1) < y(v_3) < y(v_2)$ and hence $v_1 \rightarrow v_2$, $v_1 \rightarrow v_3$ and $v_2 \rightarrow v_3$ in G_x and $v_1 \rightarrow v_2$, $v_1 \rightarrow v_3$ and $v_3 \rightarrow v_2$ in G_y, as illustrated in Fig. 11(a). If v_2 has label 0, then we decide that $v_1 \rightarrow v_2$, $v_1 \rightarrow v_3$ and $v_3 \rightarrow v_2$ in G_x and $v_3 \rightarrow v_1$, $v_3 \rightarrow v_2$ and $v_1 \rightarrow v_2$ in G_y. If v_3 has label 0, then we decide that $v_1 \rightarrow v_2$, $v_1 \rightarrow v_3$ and $v_2 \rightarrow v_3$ in G_x and $v_3 \rightarrow v_1$, $v_3 \rightarrow v_2$ and $v_1 \rightarrow v_2$ in G_y. Thus we direct each edge of C for G_x and G_y. We then direct the edges of each inner facial 3-cycle sharing an edge with C for G_x and G_y. Repeating the operation for each inner facial 3-cycle of G, we obtain a directed graph G_x and G_y. One can show that each of G_x and G_y is acyclic and has exactly one vertex of in-degree zero, and every other vertex has in-degree one or more. (Condition (c) is crucial in this proof, which is omitted in this extended abstract, due to the page limitation.)

(Step 2) For each edge $e = u \rightarrow v$ of G_x, we assign an integer weight $w(e)$ to e. The weight $w(e)$ implies that $x(u) + w(e) \leq x(v)$ in D^*. We decide $w(e)$ as follows. If an inner facial cycle C of G^* is not filled in G, then we give, as a weight $w(e)$, either 1 or 2 to each edge e of C, as illustrated in Fig. 11(a). If C is filled in G, then we assign a weight $w(e)$ to each edge e of C, as illustrated in Fig. 11(b); the value $w(e)$ depends on both the number of vertices in $G(C)$ and the index c in Section 2. Since each inner edge e receives two weights from the two facial cycles containing e, we assign e the larger one as $w(e)$.

(Step 3) Let s_x be the *source* of G_x, that is, the vertex having in-degree zero. Since G_x is acyclic and every vertex u other than s_x has in-degree one or more, one can find in linear time the longest path from s_x to each vertex u in G_x. We decide the x-coordinate $x(u)$ of u to be the length of the longest path. Similarly, we compute the y-coordinate $y(u)$. Thus we obtain a drawing D^* of G^*, as illustrated in Fig. 7(b).

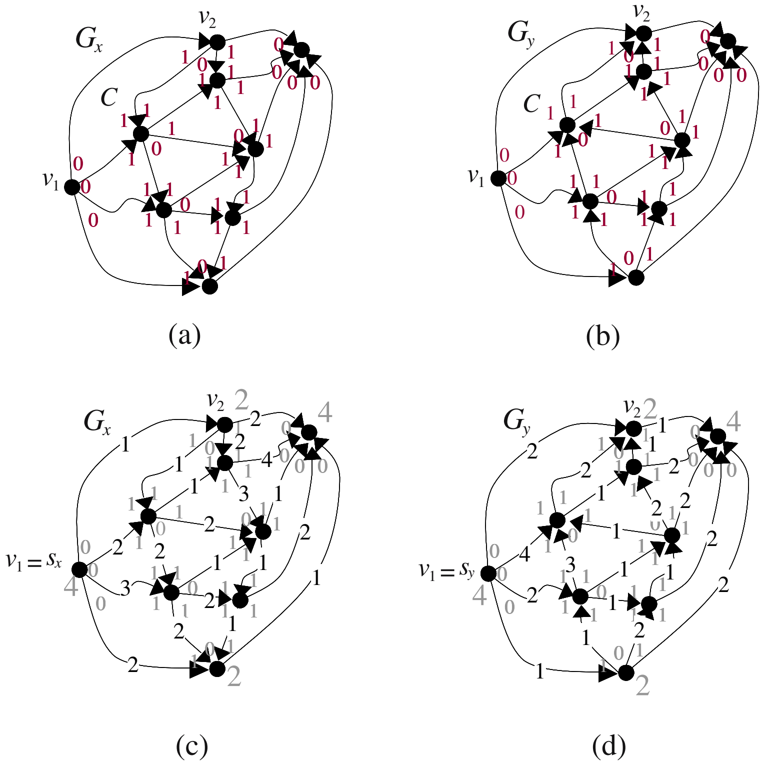

Fig. 10. (a) Directed graph G_x, (b) directed graph G_y, (c) weights in G_x, and (d) weights in G_y

In order to verify Theorem 2, it suffices to prove that the drawing D^* realizes a given good labeling L^* of G^*, that is, $L(D^*) = L^*$, and that the drawing D obtained from D^* and D_i is a good open RI-drawing of G. The proof is omitted in this extended abstract, due to the page limitation. One can easily show that D is drawn on an $(n-1) \times (n-1)$ grid.

One can construct in linear time a good open RI-drawing D^* of G^* from a given good labeling L^* of G^*. Therefore, one can construct a good open RI-drawing D of G from L^* in linear time.

3.3 Algorithm for Computing a Good Labeling

In this subsection we show how to find a good labeling of G^*.

We assign each angle of G^* with label $0, 1, x$ or y as illustrated in Fig. 12(a). Labels x and y are undecided at this moment; x will be decided to be 0 or 1 and y to be $2, 3$ or 4. For every inner facial 3-cycle C of G^* that is not filled in G, we assign a label x to each of the three angles in C. For every inner facial 3-cycle $C = uvw$ of G^* that is filled in G, we assign labels as follows: if exactly

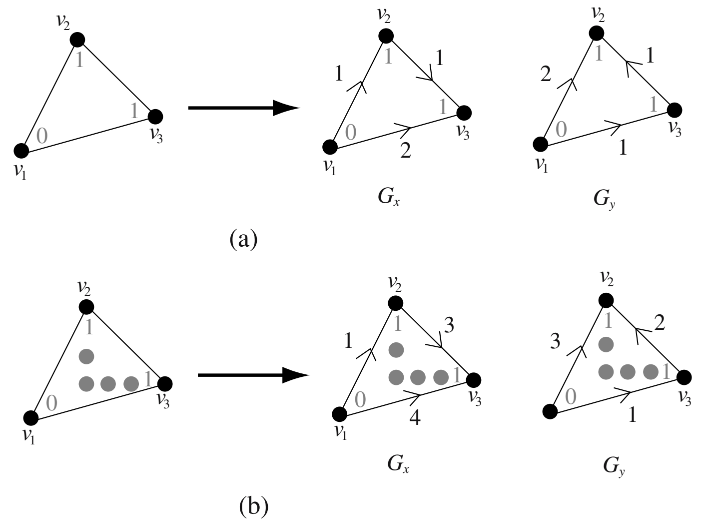

Fig. 11. Directions and weights $w(e)$ of edges e in G_x and G_y; (a) non-filled 3-cycle C, and (b) filled 3-cycle C

one of u, v and w is adjacent to all the other vertices of $G(C)$ as the case of C_1 in Fig. 2(a), then we assign 0 to the vertex and assign 1 to each of the other two vertices; if exactly two are adjacent to all the other vertices of $G(C)$ as the case of C_2 in Fig. 2(a), then we assign label x to each of them and assign 1 to the other; if each of u, v and w is adjacent to all the other vertices of $G(C)$, then $G(C) = K_4$ and hence we assign a label x to each of u, v and w. We finally assign a label y to each of the outer angles. Our problem is to determine values of all x's and y's so that the resulting labeling of G^* satisfies Conditions (a)–(c).

Let G_f be a new graph constructed from G^* as illustrated in Fig. 12(b). (The detailed construction is omitted in this extended abstract.) Let f be an appropriately chosen function $V(G_f) \to \{0, 1, \cdots, 4\}$; $f(v)$ is attached to each vertex v in Fig. 12(b). An *f-factor* of G_f, drawn by solid lines in Fig. 12(b), is a spanning subgraph of G_f in which each vertex v has degree $f(v)$ [7]. We can show that G^* has a good labeling, as illustrated in Fig. 12(d), if and only if G_f has an f-factor.

Let G_d be a new graph constructed from G_f and f, as illustrated in Fig. 12(c). We can show that G_f has an f-factor if and only if G_d has a perfect matching. A perfect matching of G_d is drawn by thick lines in Fig. 12(c). Since G_d is a bipartite graph and has $O(n)$ vertices and edges, one can determine in time $O(n^{1.5}/\log n)$ whether G_d has a perfect matching [8,9].

One can construct a good labeling of G^* from an f-factor of G_f or a perfect matching of G_d in linear time. We thus have the following theorem.

Theorem 3. For an inner triangulated plane graph G, one can determine whether G^* has a good labeling and, if so, compute a good labeling L^* of G^* in time $O(n^{1.5}/\log n)$. From L^* one can construct an open RI-drawing of G on an $(n-1) \times (n-1)$ grid in linear time.

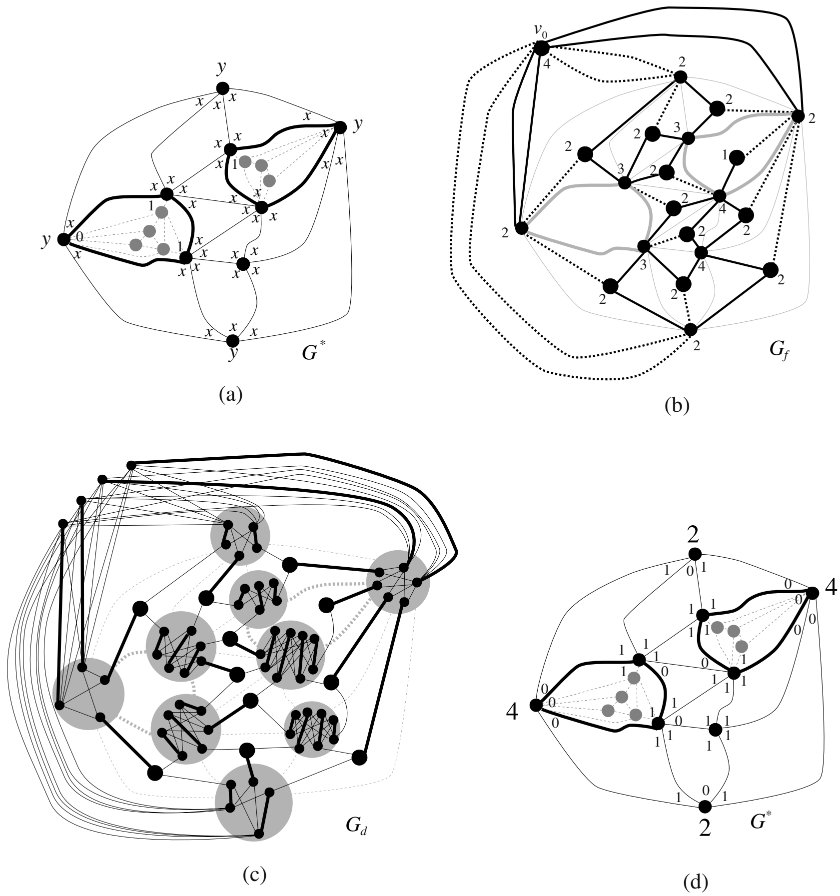

Fig. 12. (a) A labeling of G^* by labels 0,1,x and y, (b) an f-factor of G_f, (c) a perfect matching of a decision graph G_d, and (d) a good labeling of G^*

References

1. T. Biedl, A. Bretscher and H. Meijer, *Rectangle of influence drawings of graphs without filled 3-cycles*, Proceedings of Graph Drawing 1999, LNCS 1731, Springer, pp. 359-368, 2000.
2. T. Biedl, G. Kant, and M. Kaufmann, *On triangulating planar graphs under the four-connectivity constraint*, Algorithmica, 19, 4, pp. 427-446, 1997.
3. M. Chrobak and G. Kant, *Convex grid darwings of 3-connected planar graphs*, Int. J. Comput. Geom Appl., 7, 3, pp. 211-223, 1997.
4. G. Di Battista, P. Eades, R. Tamassia and I. G. Tollis, *Graph Drawing: Algorithms for the Visualization of Graphs*, Prentice-Hall, Upper Saddle River, NJ 07458, 1999.

5. G. Di Battista, W. Lenhart and G. Liotta, *Proximity drawability:A survey*, Proc. Graph Drawing 1994, Lect. Note in Computer Science, LNCS, Number 894, pp. 328-339, Springer-Verlag, 1995.
6. H. de Fraysseix, J. Pach, R. Pollack, *How to draw a graph on a grid*, Combinatorica, 10, pp. 41-51, 1990.
7. F. Harary, *Graph Theory*, Addison-Wesley, Reading, Massachusetts, 1969.
8. D. S. Hochbaum, *Faster pseudoflow-based algorithms for the bipartite matching and the closure problems*, Abstract, CORS/SCRO-INFORMS Joint Int. Meeting, Banff, Canada, p. 46, May 16-19, 2004.
9. D. S. Hochbaum and B. G. Chandran, *Further below the flow decomposition barrier of maximum flow for bipartite matching and maximum closure*, Working paper, 2004.
10. G. Kant and X. He, *Regular edge-labeling of 4-connected plane graphs and its applications in graph drawing problems*, Theoretical Computer Science, 172, pp.175-193, 1997.
11. G. Liotta, A. Lubiw, H. Meijer and S. H. Whitesides, *The rectangle of influence drawability problem*, Comput. Geom. Theory and Applications, 10, 1, pp.1-22, 1998.
12. K. Miura and T. Nishizeki, *Rectangle-of-influence drawings of four-connected plane graphs*, Proc. of Information Visualisation 2005, Asia-Pacific Symposium on Information Visualisation (APVIS2005), ACS Vol 45, pp.71-76, 2005.
13. K. Miura, S. Nakano and T. Nishizeki, *Grid drawings of four-connected plane graphs*, Discrete & Computational Geometry, 26, 1, pp.73-87, 2001.
14. T. Nishizeki and Md. S. Rahman, *Planar Graph Drawing*, World Scientific, Singapore, 2004.
15. W. Schnyder, *Embedding planar graphs in the grid*, Proc. 1st Annual ACM-SIAM Symp. on Discrete Algorithms, San Francisco, pp.138-147, 1990.