Embedding Graphs Simultaneously with Fixed Edges*

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Abstract. We show that a planar graph and a tree can always be simultaneously embedded with fixed edges and that two outerplanar graphs generally cannot.

1 Introduction

A *simultaneous embedding* of two planar graphs G_1 and G_2 is a pair of drawings of G_1 and G_2 such that each drawing is planar and each vertex common to G_1 and G_2 is represented by the same point in both drawings. Unfortunately, if one wishes to visualize the edges of G_1 and G_2 as rectilinear segments (the so called *geometric simultaneous embedding*), not all pairs of graphs can be embedded simultaneously. Erten and Kobourov ([4]), Brass et al. ([1]), and Geyer et al. ([7]) have shown that it is not always possible to embed simultaneously with straight-line edges a planar graph and a path, three paths, and two trees, respectively. On the other hand, if one permits that each edge of a graph is displayed as a different Jordan curve (the so called *simultaneous embedding*), then by the results of Pach and Wenger ([9]) any number of planar graphs can be embedded simultaneously. Restricting the last constraints, one could permit that each edge is represented by a Jordan curve, but could force edges common to more graphs to have the same representation in the drawing of each graph (the so called *simultaneous embedding with fixed edges*). Di Giacomo and Liotta ([3]) showed that an outerplanar graph and a cycle can always be simultaneously embedded with fixed edges, improving the results in [4], where it is shown how to embed simultaneously with fixed ("consistent") edges a tree and a path. In [4] and [3] the problem of finding simultaneous embeddings with fixed edges of pairs of trees and of pairs of planar graphs is explicitly mentioned.

In this paper we improve the results on simultaneous embedding with fixed edges of graphs, by showing that there exist two outerplanar graphs that cannot be simultaneously embedded with fixed edges (Section 3) and that a planar graph and a tree can always be simultaneously embedded with fixed edges (Section 4). Then in Section 5 we give conclusions and suggest some open problems.

2 Preliminaries

We assume familiarity with graphs and their drawings (see e.g. [2]).

A *drawing* of a graph is a mapping of each vertex to a distinct point in the plane and of each edge to a Jordan curve between the endpoints of the edge. A *planar drawing* is

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such that no two edges intersect. A *planar graph* is a graph that admits a planar drawing. An *embedding* of a graph G is a circular ordering of the edges incident on each vertex of G. An embedding of a graph specifies what are its faces in any drawing respecting such embedding and so it specifies the *dual graph* of G that is the graph with one vertex for each face of G and with one edge between two vertices if the corresponding faces share an edge in G. A *poly-line drawing* is such that the edges are sequences of rectilinear segments. A *straight-line drawing* is such that all edges are rectilinear segments. It has been shown in [5] that every planar graph admits a planar straight-line drawing.

A *simultaneous embedding with fixed edges* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with a bijective mapping $\gamma : V_1 \rightarrow V_2$ between their vertices is a pair of drawings Γ_1 and Γ_2 of G_1 and of G_2, respectively, such that: (i) each of Γ_1 and Γ_2 is a planar drawing, (ii) each vertex $v_2 = \gamma(v_1)$, with $v_1 \in V_1$ and $v_2 \in V_2$, is mapped in Γ_2 to the same point where v_1 is mapped in Γ_1, and (iii) an edge belonging to both E_1 and E_2 is represented by the same simple Jordan curve in Γ_1 and Γ_2. In this paper we only deal with simultaneous embedding with fixed edges, so in the following, unless otherwise specified, *simultaneous embedding* will always stand for *simultaneous embedding with fixed edges*.

3 Simultaneous Embeddings of Outerplanar Graphs

In this section we show that there exist two outerplanar graphs that cannot be simultaneously embedded. This result was already obtained in [1] for geometric simultaneous embedding. Although we believe that the pair of outerplanar graphs presented in [1] can not be simultaneously embedded even in the *fixed edges* setting, this was never pointed up and also the proof in [1] exploits the fact that the edges are drawn as segments.

Theorem 1. *There exist two outerplanar graphs that can not be simultaneously embedded with fixed edges.*

To prove Theorem 1 we first show the topologies of two outerplanar graphs G_1 and G_2 and a bijective mapping γ between their vertices. We use the same name v_i for a vertex u of G_1 and a vertex v of G_2 to mean that $v = \gamma(u)$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two outerplanar graphs with six vertices each and with $E_1 = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_1, v_4), (v_2, v_4), (v_2, v_5), (v_4, v_5), (v_2, v_6), (v_3, v_6)\}$, and $E_2 = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_1, v_6), (v_2, v_6), (v_2, v_5), (v_6, v_5), (v_2, v_4), (v_3, v_4)\}$ (see Fig. 1). To show that G_1 and G_2 admit no simultaneous

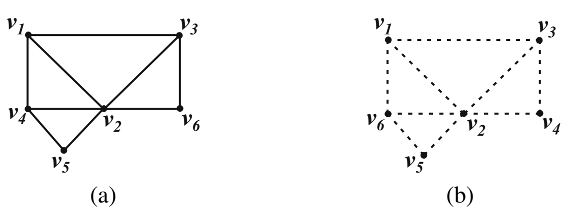


Fig. 1. Outerplanar graphs (a) G_1 and (b) G_2

embedding with mapping γ between their vertices, we try to construct embeddings E_1 of G_1 and E_2 of G_2, proving that at least one between E_1 and E_2 must be non planar.

First we embed vertices v_1, v_2, and v_3. These vertices form a cycle C_1, that is common to E_1 and E_2, and that divides the plane in two parts, one inside and one outside C_1. Then, after having drawn v_4 and its incident edges, the plane is subdivided in four regions F_1, F_2, F_3, and F_4, delimited by cycles between vertices (v_1, v_2, v_3), (v_1, v_2, v_4), (v_1, v_3, v_4), and (v_2, v_3, v_4), respectively, each one with the fourth vertex outside. This is because drawing v_4 inside or outside C_1 and changing the clockwise order of the edges incident in v_4 only permit to choose the external face of the simultaneous embedding, as shown in Fig. 2 (a) – (d). Since only the edges (v_1, v_4) and (v_3, v_4) can intersect, regions F_2, F_3 and F_4 can overlap, while region F_1 can not intersect any other region, as shown in Fig. 2 (e).

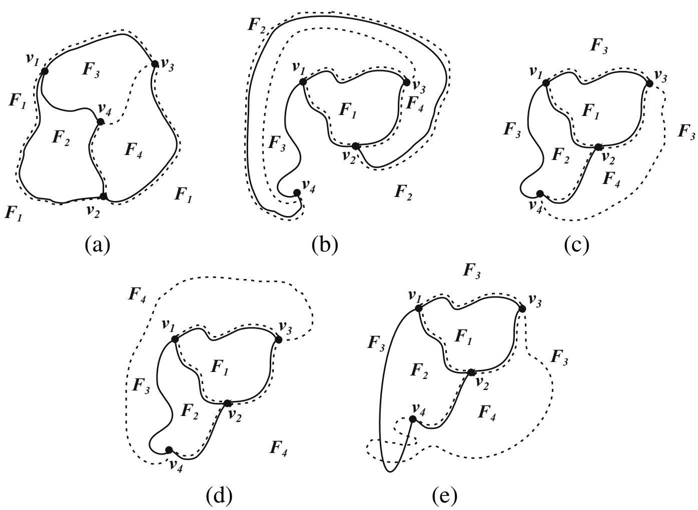


Fig. 2. Embedding vertices v_1, v_2, v_3, and v_4 with different external faces. (a) F_1. (b) F_2. (c) F_3. (d) F_4. (e) Intersection between faces of E_1 and E_2.

Now we embed v_6 and its incident edges. It is easy to observe that v_6 must be placed inside region F_1. In fact if v_6 is placed inside F_2, then (v_3, v_6) intersects the cycle (v_1, v_2, v_4) in E_1; if v_6 is placed inside F_4, then (v_1, v_6) intersects the cycle (v_2, v_3, v_4) in E_2; if v_6 is placed inside F_3, then (v_2, v_6) either intersects an edge between (v_1, v_3) and (v_1, v_4) in E_1 or intersects an edge between (v_1, v_3) and (v_3, v_4) in E_2. We have shown that v_4 and v_6 must be placed one inside and the other outside C_1. Note that v_4 is adjacent to v_5 in G_1 and v_6 is adjacent to v_5 in G_2. Hence, embedding v_5 anywhere in the plane creates an edge e from inside to outside C_1. Since e intersects C_1, that is common to G_1 and G_2, there is not a placement for v_5 preserving the planarity of both E_1 and E_2 and this concludes the proof.

4 Simultaneous Embedding of a Planar Graph and a Tree

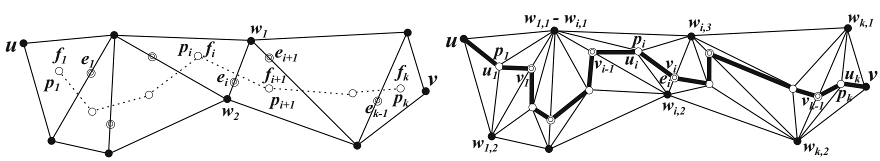
In this section we prove the following theorem:

Theorem 2. *A planar graph and a tree can be simultaneously embedded with fixed edges with any established mapping between their vertices.*

To prove Theorem 2 we show how to construct a simultaneous drawing of a planar graph G and of a tree T with any mapping γ between their vertices.

Augment G to a triangular graph G', by adding edges that will be removed later and then, by using one of the well-known methods to draw with straight-line edges a triangular graph ([5,6,10]), construct a straight-line drawing of G' with dual graph D. Note that the edges of T that are common to edges of G' have been already drawn. Let E_T denote the set of edges of T that have still to be drawn. From now on, we call Γ the current simultaneous drawing of G' and T. Let also S_N denote the set of already drawn segments representing straight-line edges of T or representing parts of poly-line edges of T. Now, until all the edges in E_T have been drawn, choose one of them, say (u, v), and draw it by using the following procedure that we call *Add Edge*.

1. Select a simple path $p = (f_1, f_2, \ldots, f_k)$ in D with the following properties: (i) f_1 is dual to a face of G' that contains u, (ii) f_k is dual to a face of G' that contains v, and (iii) each edge (f_i, f_{i+1}), $1 \leq i < k$, is not dual to an edge of G' represented by a segment in S_N. Then, for every face F_i of G' that is dual to a vertex f_i of p, $1 \leq i \leq k$, choose a point p_i in the interior of the region representing F_i in Γ. For every edge (w_1, w_2) of G' dual to an edge (f_i, f_{i+1}) of p, $1 \leq i < k$, choose a point e_i in the interior of the segment representing (w_1, w_2) in Γ (see Fig. 3 (a)).



(a) (b)

Fig. 3. (a) A path p in the dual graph D of G', as in Step 1 of *Add Edge*. The vertices of G' (of D) are the black ones (the white ones). The p_i's are choosen coincident with the vertices of D. The double circles show the e_i's. (b) G' augmented and drawn as in Steps 2 and 3 of *Add Edge*. Both the thin and the thick edges belong to G'. The thick segments show the polygonal line e_{pol}, representing the edge (u, v) of T in Γ.

2. Augment G' by adding to it a vertex u_i for each vertex f_i of p, $1 \leq i \leq k$, and a vertex v_i for each edge (f_i, f_{i+1}) of p, $1 \leq i < k$. Add also to G' the following edges: for each face F_i dual to $f_i \in p$, $1 < i < k$, let $(w_{i,1}, w_{i,2})$ and $(w_{i,2}, w_{i,3})$ be the edges of F_i dual to (f_{i-1}, f_i) and (f_i, f_{i+1}), respectively. Add to G' the edges $(w_{i,1}, u_i)$, $(w_{i,2}, u_i)$, $(w_{i,3}, u_i)$, (v_{i-1}, u_i), and (v_i, u_i). Split $(w_{i,1}, w_{i,2})$ in two edges $(w_{i,1}, v_{i-1})$ and $(w_{i,2}, v_{i-1})$ and split $(w_{i,2}, w_{i,3})$ in two edges $(w_{i,2}, v_i)$ and

$(w_{i,3}, v_i)$. For the face F_1 dual to $f_1 \in p$ let a and b be the vertices of F_1 distinct from u, and let $(w_{1,1}, w_{1,2})$ be the edge of F_1 dual to (f_1, f_2). Add to G' edges (u, u_1), (a, u_1), (b, u_1), and (v_1, u_1). Split $(w_{1,1}, w_{1,2})$ in two edges $(w_{1,1}, v_1)$ and $(w_{1,2}, v_1)$. For the face F_k dual to $f_k \in p$ let c and d be the vertices of F_k distinct from v and let $(w_{k,1}, w_{k,2})$ be the edge of F_k dual to (f_{k-1}, f_k). Add to G' edges (v, u_k), (c, u_k), (d, u_k), and (v_{k-1}, u_k). Split $(w_{k,1}, w_{k,2})$ in two edges $(w_{k,1}, v_{k-1})$ and $(w_{k,2}, v_{k-1})$.

3. As shown in Fig. 3 (b), map each u_i to p_i, $1 \leq i \leq k$ and each v_i to e_i, $1 \leq i < k$. Draw the edges added to G' as straight lines. Draw the edge (u, v) of T in Γ as a polygonal line e_{pol} passing through points $u, p_1, e_1, \ldots, p_i, e_i, \ldots, p_{k-1}, e_{k-1}, p_k, v$.
4. Remove (u, v) from E_T and add to S_N every segment of e_{pol}.

When $E_T = \varnothing$, the final drawing of G is obtained by deleting from the current G' all the edges not belonging to G. Note that it is possible that some edges of G are now represented by a polygonal line obtained by repeatedly splitting the starting straight-line edge. The final drawing of T is formed by all the segments in S_N.

We now show that the above described algorithm constructs a simultaneous embedding of G and T. To check this, we show that: (i) the construction of the simultaneous drawing of G and T starts from a partial simultaneous planar drawing, (ii) the planarity of the drawing of G is preserved after each application of *Add Edge*, (iii) the planarity of the drawing of T is preserved after each application of *Add Edge*, (iv) one can always apply *Add Edge* until all the edges in E_T have been drawn, and (v) each edge common to G and T has the same representation in both the drawings of G and of T.

(i) The planarity of the starting simultaneous drawing Γ is a consequence of the planarity of the straight-line drawings obtained by applications of the algorithms in [5,6,10].

(ii) Each time one applies *Add Edge* to draw an edge of E_T, G' is augmented by adding new vertices and new edges to it. This is done in such a way that both the triangulation of G' and the planarity of its current drawing are preserved, as can be easily checked (see Figure 3).

(iii) After each execution of *Add Edge* the subgraph of T that has been already drawn is a subgraph of G'. By construction, the last assertion is true after triangulating G and after straight-line drawing G' and it remains true also after each application of *Add Edge*. To prove this, observe that at step 2 of *Add Edge* one augments G' by adding some new vertices and edges to it, then at step 3 one draws these new vertices and edges. Then an edge of T is drawn as a polygonal line whose bends coincide with the new vertices of G' and whose edges coincide with some of the new edges of G'. So the planarity of the drawing of T is a consequence of the planarity of the drawing of G'.

(iv) Suppose that when we are starting a new execution of *Add Edge* the dual graph D of G' is not connected after the removal of the edges that are dual to edges of T represented by segments in S_N. This is equivalent of saying that the removed edges form a cutset for D. From [8] we know that:

Lemma 1. *Let G be a planar graph and D be a geometric dual of G, then a set of edges in G forms a cycle (or cutset) in G if and only if the corresponding set of edges of D forms a cutset (res. cycle) in D.*

So the set of edges of T already drawn forms a cycle and this gives us a contradiction, since T is a tree. So D is connected even after removing from it the edges that are dual to edges of T already drawn and this permits us to select a path p in D with the properties described at step 1 of *Add Edge* and to apply *Add Edge*.

(v) After triangulating G and after drawing the resulting triangular graph G', the edges that are common to both graphs are drawn as straight-line segments between the same end-points and while applying *Add Edge* they are splitted in the same way. So the edges that are common to G and T have the same drawing in Γ.

5 Conclusions

In this paper we have shown that two outerplanar graphs can not always be simultaneously embedded with fixed edges, while a planar graph and a tree can. Observe that after the last execution of *Add Edge* (see Section 4) the dual graph D of the augmented triangular graph G' is still connected even after the removal from D of the edges dual to edges of the tree. Hence an other execution of *Add Edge* is still possible and so the algorithm proposed in Section 4 works more generally when the first graph is planar and the second is a tree augmented by an edge. This straightforwardly implies that the algorithm works also for a planar graph and a cycle. A drawback of the algorithm in Section 4 is that of using a large number of bends, and so it remains an open problem to find an algorithm for drawing a planar graph and a tree with fixed edges with only a small number of bends and within a small area. As far as we know, it is also still open the geometric simultaneous embedding of a tree and a path and the simultaneous embedding without mapping ([1]) of two planar graphs.

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