Multiset Bisimulations as a Common Framework for Ordinary and Probabilistic Bisimulations*

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Abstract. Our concrete objective is to present both ordinary bisimulations and probabilistic bisimulations in a common coalgebraic framework based on multiset bisimulations. For that we show how to relate the underlying powerset and probabilistic distributions functors with the multiset functor by means of adequate natural transformations. This leads us to the general topic that we investigate in the paper: a natural transformation from a functor F to another G transforms F-bisimulations into G-bisimulations but, in general, it is not possible to express G-bisimulations in terms of F-bisimulations. However, they can be characterized by considering Hughes and Jacobs' notion of simulation, taking as the order on the functor F the equivalence induced by the epi-mono decomposition of the natural transformation relating F and G. We also consider the case of alternating probabilistic systems where non-deterministic and probabilistic choices are mixed, although only in a partial way, and extend all these results to categorical simulations.

1 Introduction

Bisimulations are the adequate way to capture behavioural indistinguishability of states of systems. Ordinary bisimulations were introduced [11] to cope with labelled transition systems and other similar models and have been used to define the formal observational semantics of many popular languages and formalisms, such as CCS. Bisimilarity is also the natural way to express equivalence of states in any system described by means of a coalgebra over an arbitrary functor F. The general categorical definition can be presented in a more concrete way for the class of polynomial functors, that are defined by means of a simple signature of constructors and whose properties, including the definition of relation lifting, can be studied by means of structural induction. In particular, the powerset constructor is one of them, and therefore the class of labelled transition systems can be studied as a simple and illustrative example of the categorical framework.

The simplicity and richness of the theory of bisimulations made it interesting to define several extensions in which the structure on the set of labels of

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the considered systems was taken into account, instead of the plain approach made by simple (strong) bisimulations. For instance, weak bisimulation takes into account the existence of non-observable actions, while timed and probabilistic bisimulation introduce timed or probabilistic features. In particular, the original definition of probabilistic bisimulation for probabilistic transition systems had to capture the fact that one should be able to accumulate the probabilities of several transitions arriving at equivalent (bisimilar) states in order to simulate some transition or, conversely, that one should be able to distribute the probability of a transition among several others connecting the same states.

The classical definition by Larsen and Skou [9] certainly generalizes the definition of ordinary bisimulation in a nice way, although at the cost of leaving out the categorical scenario discussed above. However, Vink and Rutten proved in [17] that the definition can be reformulated in a coalgebraic way. For that, they considered a functor $\mathcal{D}$ defining probabilistic distributions, that appears as the primitive construction in the definition of the corresponding probabilistic systems. Even though this is quite an elegant characterization, it forces us to leave the realm of (probabilistic) transition systems, moving into the more abstract one of probabilistic distributions.

We would like to directly manage probabilistic transition systems in order to compare the results about ordinary transition systems and those on probabilistic systems as much as possible. We have found that multi-transition systems, where we can have several identical transitions and the number of times they appear matters, constitute the adequate framework to establish the relation between those two kinds of transition systems. As a matter of fact, we will see that the use of multisets instead of just plain sets leads us to a natural presentation of relation lifting for that construction; besides, we can add the corresponding functor to the collection defining polynomial functors, thus obtaining an enlarged class with nice properties similar to those in the original class.

Although a general theory combining non-deterministic and probabilistic choices seems quite hard to develop, since it is difficult to combine both functors in a smooth way [16], we will present the case of *alternating*[1] probabilistic systems. In those systems, the classical definitions of ordinary and probabilistic bisimulation can be combined to obtain the natural definition of alternating probabilistic bisimulation, that perfectly fits into our framework based on categorical simulations on our multi-transition systems.

The functors defining ordinary transition systems and probabilistic systems can be obtained by applying an adequate *natural transformation* to a functor defining multiset transition systems. In both cases bisimulations are preserved in both directions when applying those transformations. This leads us to the general theory that we investigate in this paper: as is well-known, any natural transformation between two functors F and G transforms F-bisimulations into G-bisimulations; in addition, and more interesting, whenever the natural transformation relating F

[1] Although we call alternating to our systems, we do not need the strict alternation between non-deterministic and probabilistic states as appears in [4], but only that these two kind of choices do not appear mixed after the same state.

and G is an epi, we can reflect G-bisimulations and express them at the level of the functor F, though this cannot be done in general just by means of F-bisimulations. However, they can be characterized by using Hughes and Jacobs' notion of simulation [6], when we consider as the order on the functor F the equivalence induced by the epi-mono decomposition of the natural transformation relating F and G. Once categorical simulations have come into play, it is nice to find that we can extend all our results to simulations based on any order. These extensions can be considered to be the main results in the paper, since all our previous results on bisimulations could be presented as particular cases of them, using the fact that bisimulations are a particular case of categorical simulations.

Although in a different direction, namely, that of exploring the relation between non-deterministic and probabilistic choices instead of the different notions of distributed bisimulations, in this paper we continue the work initiated in FORTE 2007 [3]. The goal is the exploration of ways in which the general theory of categorical bisimulations and simulations can be applied to obtain almost for free interesting results on concrete cases that, without the support of that general theory, would need different non-trivial proofs. Therefore, our work has a mixed flavour: on the one hand we develop new abstract results that extend the general theory; on the other hand we apply these results to simple but important concrete concepts, that therefore are proved to be particular cases of the rich general theory. These are only concrete examples that we hope to extend and generalize in the near future.

2 Basic Definitions

We review in this section standard material on coalgebras and bisimulations, as can be found for example in [8,12,7]. Besides, we introduce some notations on multisets and the corresponding functor $\mathcal{M}$, as well as for the functor $\mathcal{D}$ defining discrete probabilistic distributions.

An arbitrary endofunctor $F : \mathbf{Sets} \longrightarrow \mathbf{Sets}$ can be lifted to a functor in the category **Rel** of relations $\mathrm{Rel}(F) : \mathbf{Rel} \longrightarrow \mathbf{Rel}$. In set-theoretic terms, for a relation $R \subseteq X_1 \times X_2$,

$$\mathrm{Rel}(F)(R) = \{\langle u, v\rangle \in FX_1 \times FX_2 \mid \exists w \in F(R).\, F(r_1)(w) = u, F(r_2)(w) = v\}\,.$$

It is well-known that for polynomial functors F, $\mathrm{Rel}(F)$ can be equivalently defined by induction on the structure of F. Since we will be making extensive use of the powerset functor, we next present how the definition particularizes to it:

$$\begin{aligned}\mathrm{Rel}(\mathcal{P}G)(R) = \{(U, V) \mid\ & \forall u \in U.\, \exists v \in V.\, \mathrm{Rel}(G)(R)(u, v) \wedge \\ & \forall v \in V.\, \exists u \in U.\, \mathrm{Rel}(G)(R)(u, v)\}\,.\end{aligned}$$

Multisets will be represented by considering their characteristic function $\chi_M : X \longrightarrow \mathbb{N}$; similarly, discrete probabilistic distributions are represented by discrete measures $p_D : X \longrightarrow [0, 1]$, with $\sum_{x\in X} p_D(x) = 1$.

We will use along the paper several different ways to enumerate the "elements" of a multiset. We define the support of a multiset M as the set of elements that appear in it: $\{M\}_X = \{x \in X \mid \chi_M(x) > 0\}$. We are only interested in multisets having a finite support, so that in the following we will assume that every multiset is finite. Given a finite subset Y of X and an enumeration of its elements $\{y_1, \ldots y_m\}$, for each tuple of natural weights $\langle n_1, \ldots, n_m\rangle$ we will denote by $\sum_{y_i \in Y} n_i \cdot y_i$ the multiset M given by $\chi_M(y_i) = n_i$ and $\chi_M(y) = 0$ for $y \notin Y$. By abuse of notation we will sometimes consider sets as a particular case of multisets, by taking for each finite set $Y = \{y_1, \ldots y_n\}$ the canonical associated multiset $\sum_{y_i \in Y} 1 \cdot y_i$. Finally, we also enumerate the elements of a multiset by means of a generating function: given a finite set I and $x : I \longrightarrow X$, we denote by $\{x_i \mid i \in I\}$ the multiset M_I given by $\chi_{M_I}(y) = |\{i \in I \mid x_i = y\}|$. Note that in this case sets are just the multisets generated by an injective generating function.

We will denote by $\mathcal{M}(X)$ the set of multisets on X, while $\mathcal{D}(X)$ represents the set of probabilistic distributions on X. Both constructions can be naturally extended to functions, thus getting the desired functors: for $f : X \longrightarrow Y$ we define $\mathcal{M}(f) : \mathcal{M}(X) \longrightarrow \mathcal{M}(Y)$ by $\mathcal{M}(f)(\chi)(y) = \sum_{f(x)=y} \chi(x)$, and $\mathcal{D}(f) : \mathcal{D}(X) \longrightarrow \mathcal{D}(Y)$ by $\mathcal{D}(f)(p)(y) = \sum_{f(x)=y} p(x)$.

Although the multiset and the probabilistic distributions functors are not polynomial, this class can be enlarged by incorporating them since their liftings can be defined with the following equations:

$$\begin{aligned}\mathrm{Rel}(\mathcal{M}G)(R) = \{(M, N) \mid {} & \exists f : I \longrightarrow GX, g : I \longrightarrow GY, \text{generating functions of} \\ & M \text{ and } N \text{ s.t. } \forall i \in I.\, (f(i), g(i)) \in \mathrm{Rel}(G)(R)\}\,;\end{aligned}$$

$$\begin{aligned}\mathrm{Rel}(\mathcal{D}G)(R) = \{(d^x, d^y) \in \mathcal{D}(G(X)) \times \mathcal{D}(G(Y)) \mid \forall U \subseteq G(X).\, \forall V \subseteq G(Y). \\ \Pi_1^{-1}(U) = \Pi_2^{-1}(V) \Rightarrow \textstyle\sum_{x \in U} d^x(x) = \sum_{y \in V} d^y(y)\}\,,\end{aligned}$$

where Π_1 and Π_2 are the projections of $\mathrm{Rel}(G)(R)$ into GX and GY, respectively.

F-coalgebras are just functions $\alpha : X \longrightarrow FX$. For instance, plain labelled transition systems arise as coalgebras for the functor $\mathcal{P}(A \times X)$. We will also consider multitransition systems, which correspond to the functor $\mathcal{M}(A \times X)$, and probabilistic transition systems, corresponding to $\mathcal{M}_1([0, 1] \times A \times X)$, where we only allow multisets in which the sum of its associated probabilities is 1.

Then, the lifting of the functor $\mathcal{M}_1([0, 1] \times \cdot)$ is defined as a particular case of that of $\mathcal{M}$ by:

$$\begin{aligned}&\mathrm{Rel}(\mathcal{M}_1([0, 1] \times \cdot)G)(R) = \\ &\quad \{(M, N) \in \mathcal{M}_1([0, 1] \times GX) \times \mathcal{M}_1([0, 1] \times GY) \mid \\ &\quad\ \exists f : I \to [0, 1] \times GX, g : I \to [0, 1] \times GY, \text{generating functions of } M \text{ and} \\ &\quad\ N \text{ s.t. } \forall i \in I.\, \Pi_1(f(i)) = \Pi_1(g(i)) \wedge (\Pi_2(f(i)), \Pi_2(g(i))) \in \mathrm{Rel}(G)(R)\}\,.\end{aligned}$$

A bisimulation for coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ is a relation $R \subseteq X \times Y$ which is "closed under c and d": if $(x, y) \in R$ then $(c(x), d(y)) \in \mathrm{Rel}(F)(R)$. We shall use the term F-bisimulation sometimes to emphasize the functor we are working with.

Bisimulations can also be characterized by means of spans, using the general categorical definition by Aczel and Mendler [1]:

$$\begin{array}{ccccc} X & \xleftarrow{r_1} & R & \xrightarrow{r_2} & Y \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle d} \\ FX & \xleftarrow{Fr_1} & FR & \xrightarrow{Fr_2} & FY \end{array}$$

R is a bisimulation iff it is the carrier of some coalgebra e making the above diagram commute, where the r_i are the projections of R into X and Y.

We will also need the general concept of simulation introduced by Hughes and Jacobs [6] using orders on functors. Let $F : \mathbf{Sets} \longrightarrow \mathbf{Sets}$ be a functor. An order on F is defined by means of a functorial collection of preorders $\sqsubseteq_X \subseteq FX \times FX$ that must be preserved by renaming: for every $f : X \longrightarrow Y$, if $u \sqsubseteq_X u'$ then $Ff(u) \sqsubseteq_Y Ff(u')$.

Given an order $\sqsubseteq$ on F, a $\sqsubseteq$-simulation for coalgebras $c : X \longrightarrow FX$ and $d : Y \longrightarrow FY$ is a relation $R \subseteq X \times Y$ such that

$$\text{if } (x, y) \in R \text{ then } (c(x), d(y)) \in \mathrm{Rel}(F)_{\sqsubseteq}(R)\,,$$

where $\mathrm{Rel}(F)_{\sqsubseteq}(R)$ is $\sqsubseteq \circ \mathrm{Rel}(F)(R) \circ \sqsubseteq$, which can be expanded to

$$\mathrm{Rel}(F)_{\sqsubseteq}(R) = \{(u, v) \mid \exists w \in F(\mathcal{R}).\ u \sqsubseteq Fr_1(w) \wedge Fr_2(w) \sqsubseteq v\}\,.$$

One of the cases under this general notion of coalgebraic simulation is that of ordinary simulation. Also, equivalence (functorial) relations, represented by $\equiv$, are a particular class of orders on F, thus generating the corresponding class of $\equiv$-simulations. As is the case for ordinary bisimulations, $\equiv$-simulations themselves need not be equivalence relations, but once we impose to the equivalence $\equiv$ the technical condition of being stable [6] then the induced notion of $\equiv$-similarity becomes an equivalence itself.

Proposition 1. *For any stable functorial equivalence relation $\equiv_X \subseteq FX \times FX$, the induced notion of $\equiv_a$-similarity relating elements of X for a coalgebra $a : X \longrightarrow FX$ is an equivalence relation. In particular, for the plain equality relation $=_X \subseteq FX \times FX$, $=_X$-similarity coincides with plain F-bisimulation.*

3 Natural Transformations and Bisimulations

Natural transformations are the natural way to relate two functors. Given F and G, two functors on **Sets**, a natural transformation $\alpha : F \Rightarrow G$ is defined as a family of functions $\alpha_X : FX \to GX$ such that, for all $f : X \longrightarrow Y$, $Gf \circ \alpha_X = \alpha_Y \circ Ff$. We are particularly interested in the natural transformations relating $\mathcal{M}$ and $\mathcal{P}$, and those between the functors defining probabilistic transition systems and probabilistic distributions. For the sake of conciseness we will often omit the action component A when working with these functors; this does not affect the validity of the definitions nor the results.

Proposition 2. *The support of multisets, $\{\cdot\}_X : \mathcal{M}(X) \longrightarrow \mathcal{P}(X)$, gives rise to a natural transformation $\{\cdot\} : \mathcal{M} \Rightarrow \mathcal{P}$.*

Similarly, $\mathcal{D}_{M_X} : \mathcal{M}_1([0,1] \times X) \longrightarrow \mathcal{D}(X)$ given by

$$\mathcal{D}_M(\sum n_i \cdot (p_i, x_i))(x) = \sum_{x_i = x} n_i p_i$$

induces a natural transformation $\mathcal{D}_M : \mathcal{M}_1([0,1] \times \cdot) \Rightarrow \mathcal{D}(\cdot)$.

Proof. Let $f : X \longrightarrow Y$. We have $(\mathcal{P}f \circ \{\cdot\}_X)(\sum n_i \cdot x_i) = \mathcal{P}f(\{x_i\}) = \{f(x_i)\} = \{\cdot\}_Y(\sum n_i \cdot f(x_i)) = (\{\cdot\}_Y \circ \mathcal{M}f)(\sum n_i \cdot x_i)$, which proves that $\{\cdot\}$ is a natural transformation.

In the case of $\mathcal{D}_M$: $(\mathcal{D}f \circ \mathcal{D}_{M_X})(\sum n_i \cdot (p_i, x_i)) = \mathcal{D}f(\sum n_i p_i \cdot x_i)$, which is $\sum_{f(x_i)=y} n_i p_i \cdot y = \mathcal{D}_{M_Y}(\sum n_i \cdot (p_i, f(x_i))) = (\mathcal{D}_{M_Y} \circ \mathcal{M}_1 f)(\sum n_i \cdot (p_i, x_i))$; this proves that $\mathcal{D}_M$ is a natural transformation. □

Probabilistic transition systems were defined in [9] as $\mathcal{P} = (Pr, Act, Can, \mu)$, where Pr is a set of processes, Act the set of actions, $Can : Pr \longrightarrow \mathcal{P}(Act)$ indicates the initial offer of each process, and $\mu_{p,a} \in \mathcal{D}(Pr)$ for all $p \in Pr$, $a \in Can(p)$. Under this definition we cannot talk about "different probabilistic transitions" reaching the same process, that is, whenever we have a transition $p \xrightarrow{a}_\mu p'$ it "accumulates" all possible ways to go from p to p' executing a.

In our opinion this is not a purely operational way to present probabilistic systems. For instance, if we are defining the operational semantics of a process such as $p = \frac{1}{2}a + \frac{1}{2}a$, then we would intuitively have two different transitions reaching the same final state *stop*, but if we were using Larsen and Skou's original definition, we should mix them both into a single $p \xrightarrow{a}_1 stop$. Certainly, we could keep these two transitions separated under that definition if, for some reason, we decided to introduce in the set Pr two different states $stop_1$ and $stop_2$, thus obtaining $p \xrightarrow{a}_{1/2} stop_1$ and $p \xrightarrow{a}_{1/2} stop_2$. But then we observe that whether our model captures or not the existence of two different transitions depends on the way we define our set of processes Pr.

In order to get a more natural operational representation of probabilistic systems we define them[2] as $\mathcal{M}_1([0,1] \times A \times \cdot)$-coalgebras. Once we use "ordinary" transitions labelled by pairs (q, a) to represent the probabilistic transitions we have no problem to distinguish two "different" transitions $p \xrightarrow{a}_{q'} p'$, $p \xrightarrow{a}_{q''} p''$, if $p' \neq p''$. However, in such a case it would not be adequate to treat the case $p' = p''$ in a different way. This is why we use $\mathcal{M}_1$ instead of $\mathcal{P}_1$ to define our probabilistic multi-transition systems (abbreviated as pmts).

We can easily translate the classical definition of probabilistic bisimulation between probabilistic transition systems in [9], to our own pmts's as follows.

[2] Although Larsen and Skou defined their systems following the reactive aproach [4], and therefore the sum of their probabilities is 1 for each action a, we prefer to follow in this paper the generative aproach, so that the total addition of all the probabilities is 1. This is done to simplify the notation, since all the results in this paper are equally valid for the reactive model.

Definition 1. *A probabilistic bisimulation on a coalgebra* $p : X \to \mathcal{M}_1([0,1] \times A \times X)$ *is an equivalence relation* $\equiv_p$ *on* X *such that, whenever* $x_1 \equiv_p x_2$*, taking* $p(x_i) = \sum t^j_j \cdot (p^i_j, a^i_j, x^i_j)$*, we also have* $\sum\{t^1_j \cdot p^1_j \mid a^1_j = a,\, x^1_j \in E\} = \sum\{t^2_j \cdot p^2_j \mid a^2_j = a,\, x^2_j \in E\}$*, for all* $a \in A$ *and every equivalence class* E *in* $X/{\equiv_p}$*.*

In [17] it is proved that probabilistic bisimilarity defined by probabilistic bisimulations coincides with categorical $\mathcal{D}$-bisimilarity. By applying the functor $\mathcal{D}_M$ we can transform our pmts's into their presentation as Larsen and Skou's pts's. Then it is trivial to check that the corresponding notions of probabilistc bisimulation coincide, and therefore they also coincide with categorical $\mathcal{D}$-bisimilarity.

However, that is clearly not the case for plain categorical $\mathcal{M}_1([0,1] \times A \times \cdot)$-bisimulations. This is so because when we consider the functor $\mathcal{M}_1([0,1] \times A \times \cdot)$, probabilistic transitions are considered as plain transitions labelled with pairs over $[0,1] \times A$, whose first component has no special meaning. As a result, we have, for instance, no bisimulation relating x and y if we consider $X = \{x\}$, $Y = \{y\}$, $p_a : X \to \mathcal{M}_1([0,1] \times A \times X)$ with $p_a(x) = 1 \cdot (1, a, x)$ and $p_b : Y \to \mathcal{M}_1([0,1] \times A \times Y)$ with $p_b(y) = 2 \cdot (\frac{1}{2}, a, y)$.

All these facts prove that our probabilistic multi-transition systems are too concrete a representation of probabilistic distributions, which is formally captured by the fact that the components of the natural transformation $\mathcal{D}_M$ are not injective. As a consequence, by using them we do not have a pure coalgebraic characterization of probabilistic bisimulations. By contrast, the original definition of pts's stands apart from the operational way, mixing different transitions into a single distribution. Besides it has to consider the quotient set $X/{\equiv_p}$ when defining probabilistic bisimulations. Our goal will be to obtain a characterization of the notion of probabilistic bisimilarity in terms of our pmts's, and this will be done using the notion of categorical simulation, as we will see in Section 4. Next, we present a collection of general interesting results. First we will see that bisimulations are preserved by natural transformations.

Theorem 1 ([12]). *If* $R \subseteq X \times Y$ *is a bisimulation relating* $a : X \longrightarrow FX$ *and* $b : Y \longrightarrow FY$*, then* R *is also a bisimulation relating* $a' : X \longrightarrow GX$*, given by* $a' = \alpha_X \circ a$*, and* $b' : Y \longrightarrow GY$*, given by* $b' = \alpha_Y \circ b$*.*

Corollary 1. *For* a *and* $a' = \alpha_X \circ a$*, bisimulation equivalence in* a *is included in bisimulation equivalence in* a'*, that is,* $x_1 \equiv_a x_2$ *implies* $x_1 \equiv_{a'} x_2$*.*

A general converse result cannot be expected because in general there is no canonical way to transform G into F. Since the main objective in this paper is to relate $\mathcal{M}$-bisimulations with $\mathcal{P}$ and $\mathcal{D}$-bisimulations, we searched for particular properties of the natural transformations relating these functors which could help us to get the desired general results covering in particular these two cases. This is how we have obtained the concept of quotient functors that we develop in the following.

Definition 2. *Let F be an endofunctor on* **Sets** *and $\equiv$ a functorial equivalence relation $\equiv_X \subseteq FX \times FX$. We define the* quotient functor *$F/\!\equiv$ by $(F/\!\equiv)(X) = FX/\!\equiv_X$, and for any $f : X \longrightarrow Y$, $u \in FX$, and $\overline{u}$ its equivalence class, $(F/\!\equiv)(f)(\overline{u}) = \overline{F(f)(u)}$, that is well defined since $\equiv$ is functorial.*

Definition 3. *1. We say that a functor G is the quotient of F under a functorial equivalence relation $\equiv$ whenever $F/\!\equiv$ and G are isomorphic, which means that there is a pair of natural transformations $\alpha : F/\!\equiv \Rightarrow G$ and $\beta : G \Rightarrow F/\!\equiv$ such that $\beta \circ \alpha = Id_{F/\equiv}$ and $\alpha \circ \beta = Id_G$.*
2. Given a natural transformation $\alpha : F \Rightarrow G$, we write $\equiv^\alpha$ for the family of equivalence relations $\equiv^\alpha_X \subseteq FX \times FX$ defined by the kernel of α: $u_1 \equiv^\alpha_X u_2 \iff \alpha_X(u_1) = \alpha_X(u_2)$.

Proposition 3. *For every natural transformation $\alpha : F \Rightarrow G$, $\equiv^\alpha$ is functorial.*

Proof. We need to show that, for any $f : X \longrightarrow Y$, whenever $u_1 \equiv^\alpha_X u_2$, that is, $\alpha_X(u_1) = \alpha_X(u_2)$, we also have $Ff(u_1) \equiv^\alpha_Y Ff(u_2)$, that is $\alpha_Y(F(f)(u_1)) = \alpha_Y(F(f)(u_2))$; this follows because $\alpha_Y \circ F(f) = G(f) \circ \alpha_X$. □

If every component α_X of a natural transformation is surjective, α is said to be epi.

Proposition 4. *Whenever α is epi, G is the quotient of F under $\equiv^\alpha$, just considering the inverse natural transformation $\alpha^{-1} : G \Rightarrow F/\!\equiv$ given by $\alpha^{-1}_X : G(X) \longrightarrow (F/\!\equiv^\alpha)(X)$ with $\alpha^{-1}_X(v) = \overline{u}$ where $\alpha_X(u) = v$.*

Corollary 2. *$\mathcal{P}$ is the quotient of $\mathcal{M}$ under the kernel of the natural transformation $\{\cdot\} : \mathcal{M} \Rightarrow \mathcal{P}$.*

Corollary 3. *$\mathcal{D}$ is the quotient of $\mathcal{M}_1([0,1] \times \cdot)$ under the kernel of the natural transformation $\mathcal{D}_\mathcal{M} : \mathcal{M}_1([0,1] \times \cdot) \Rightarrow \mathcal{D}$.*

4 $\equiv^\alpha$-simulations Through Quotients of Bisimulations

Let us start by studying the relationships between coalgebras corresponding to functors related by an epi natural transformation.

Definition 4. *Let $\alpha : F \Rightarrow G$ be a natural transformation and $a : X \longrightarrow FX$ an F-coalgebra. We define the α-image of a as the coalgebra $a_\alpha : X \longrightarrow GX$ given by $a_\alpha = \alpha_X \circ a$.*

Definition 5. *Given a natural transformation $\alpha : F \Rightarrow G$ and a G-coalgebra $b : X \longrightarrow GX$, we say that $a : X \longrightarrow FX$ is a concrete F-representation of b iff $b = \alpha_X \circ a$.*

The following result follows immediately from the previous definitions.

Proposition 5. *If α is epi then every G-coalgebra has an F-representation.*

Next we relate G-bisimulations with $\equiv^\alpha$-simulations:

Theorem 2. *Let $\alpha : F \Rightarrow G$ be an epi natural transformation and $b_1 : X_1 \longrightarrow GX_1$, $b_2 : X_2 \longrightarrow GX_2$ two G-coalgebras, with concrete F-representations $a_1 : X_1 \longrightarrow FX_1$ and $a_2 : X_2 \longrightarrow FX_2$. Then, the G-bisimulations relating b_1 and b_2 are precisely the $\equiv^\alpha$-simulations relating a_1 and a_2.*

Proof. Let us show[3] that, for every relation $R \subseteq X_1 \times X_2$,

$$\mathrm{Rel}(F)_{\equiv^\alpha}(R) = \{(u,v) \in FX_1 \times FX_2 \mid (\alpha_{X_1}(u), \alpha_{X_2}(v)) \in \mathrm{Rel}(G)(R)\}\,.$$

We have, unfolding the definition of $\mathrm{Rel}(F)_{\equiv^\alpha}(R)$ and using the fact that α is a natural transformation:

$$\begin{aligned}
\mathrm{Rel}(F)_{\equiv^\alpha}(R) &= \{(u,v) \in FX_1 \times FX_2 \mid \exists w \in FR.\, u \equiv^\alpha Fr_1(w) \wedge Fr_2(w) \equiv^\alpha v\} \\
&= \{(u,v) \in FX_1 \times FX_2 \mid \exists w \in FR.\, \alpha_{X_1}(u) = \alpha_{X_1}(Fr_1(w)) \wedge \\
&\qquad\qquad\qquad\qquad\qquad\qquad \alpha_{X_2}(v) = \alpha_{X_2}(Fr_2(w))\} \\
&= \{(u,v) \in FX_1 \times FX_2 \mid \exists w \in FR.\, \alpha_{X_1}(u) = Gr_1(\alpha_R(w)) \wedge \\
&\qquad\qquad\qquad\qquad\qquad\qquad \alpha_{X_2}(v) = Gr_2(\alpha_R(w))\}\,.
\end{aligned}$$

On the other hand,

$$\mathrm{Rel}(G)(R) = \{(x,y) \in GX_1 \times GX_2 \mid \exists z \in GR.\, Gr_1(z) = x \wedge Gr_2(z) = y\}\,.$$

Now, if $(u,v) \in \mathrm{Rel}(F)_{\equiv^\alpha}(R)$, by taking $\alpha_R(w)$ as the value of $z \in GR$ we have that $(\alpha_{X_1}(u), \alpha_{X_2}(v)) \in \mathrm{Rel}(G)(R)$. Conversely, if $(\alpha_{X_1}(u), \alpha_{X_2}(v)) \in \mathrm{Rel}(G)(R)$ is witnessed by z, let $w \in FR$ be such that $\alpha_R(w) = z$, which must exists because α is epi; it follows that $(u,v) \in \mathrm{Rel}(F)_{\equiv^\alpha}(R)$.

Then, $(b_1(x), b_2(y)) \in \mathrm{Rel}(G)(R)$ if and only if $(a_1(x), a_2(x)) \in \mathrm{Rel}(F)_{\equiv^\alpha}(R)$, from where it follows that R is a G-bisimulation if and only if it is a $\equiv^\alpha$-simulation. □

Corollary 4. *(i) Bisimulations between labelled transition systems are $\equiv^{\{\cdot\}}$-simulations between multi-transition systems. (ii) Bisimulations between probabilistic systems are just $\equiv^{\mathcal{D}_M}$-simulations between (an appropriate class of) multi-transition systems.*

Example 1. Let us illustrate this result by means of some simple examples using the natural transformation $\{\cdot\} : \mathcal{M} \to \mathcal{P}$.

1. If we consider the ordinary transition systems $s_X : \{x, x'\} \longrightarrow \mathcal{P}(\{x, x'\})$, with $s_X(x) = \{x'\}$, $s_X(x') = \emptyset$, and $s_Y : \{y, y'_1, y'_2\} \longrightarrow \mathcal{P}(\{y, y'_1, y'_2\})$ with $s_Y(y) = \{y'_1, y'_2\}$, $s_Y(y'_1) = \emptyset$, and $s_Y(y'_2) = \emptyset$, we have a simple $\mathcal{P}$-bisimulation relating the initial states x and y, given by $R = \{(x,y), (x', y'_1), (x', y'_2)\}$.

[3] It is not difficult to present this proof as a commutative diagram. Then one has to check that all the "small squares" in the diagram are indeed commutative, in order to be able to conclude commutativity of the full diagram. This is what we have carefully done in our proof above.

Denoting by s^1_X and s^1_Y the canonical $\mathcal{M}$-representations of s_X and s_Y, obtained by the embedding of sets into multisets, it is obvious that there is no $\mathcal{M}$-bisimulation relating x and y. But if we consider $s^2_X(x) = \{2 \cdot x'\}$, $s^2_X(x') = \emptyset$, we have now an $\mathcal{M}$-bisimulation between the multi-transition systems s^2_X and s^1_Y relating x and y. And, by Theorem 2, we have that s^1_X is also $\equiv^{\{\cdot\}}$-simulated by s^1_Y, since $\{s^1_X\}_{\mathcal{M}} = \{s^2_X\}_{\mathcal{M}} = s_X$ and s_X and s_Y are $\mathcal{P}$-bisimilar. Obviously, the same happens for any $\{\cdot\}$-representation of s_X, s^k_X with $s^k_X = \{k \cdot x'\}$ and $s^k_X(x') = \emptyset$.

2. In the example above we got the $\equiv^{\{\cdot\}}$-simulation by proving that there are $\mathcal{M}$-representations of the considered coalgebras for which the given relation is also an $\mathcal{M}$-bisimulation. However, this is not necessary as the following shows. Let us consider $t_X : \{x\} \longrightarrow \mathcal{P}(\{x\})$ with $t_X(x) = \{x\}$ and $Y = \{\beta \mid \beta \in \mathbb{N}^*, \beta_i \leq i\}$ with $t_Y(\beta) = \{\beta \circ \langle j\rangle \mid \beta \circ \langle j\rangle \in Y\}$. It is clear that $R = \{(x,\beta) \mid \beta \in Y\}$ is the (only) $\mathcal{P}$-bisimulation relating x and ϵ, the initial states of t_X and t_Y. However, in this case there exists no $\mathcal{M}$-bisimulation relating two $\mathcal{M}$-representations of t_X and t_Y, because $|t_Y(\beta)| = |\beta| + 1$ and therefore we would need a representation t^k_X with $t^k_X(x) = \{k \cdot x\}$ such that $k \geq l$ for all $l \in \mathbb{N}$, which is not possible because the definition of multiset does not allow the infinite repetition of any of its members. Instead, Theorem 2 shows that any two $\mathcal{M}$-representations of t_X and t_Y are $\equiv^{\{\cdot\}}$-similar.

The reason why we had an $\mathcal{M}$-bisimulation relating the appropriate $\mathcal{M}$-representations of the compared $\mathcal{P}$-coalgebras in our first example was because we were under the hypothesis of the following proposition.

Proposition 6. *Let $\alpha : F \Rightarrow G$ be an epi natural transformation. Whenever a G-bisimulation R relating $b_1 : X \longrightarrow GX$ and $b_2 : Y \longrightarrow GY$ is* near injective, *which means that $|\{b_2(y) \mid (x,y) \in R\}| \leq 1$ for all $x \in X$ and $|\{b_1(x) \mid (x,y) \in R\}| \leq 1$ for all $y \in Y$, there exist some F-representations of b_1 and b_2, $a_1 : X \longrightarrow FX$ and $a_2 : Y \longrightarrow FY$, respectively, such that R is also a bisimulation relating a_1 and a_2.*

Proof. By Theorem 2, R is also a $\equiv^\alpha$-simulation for any pair of F-representations of b_1 and b_2; let a_1, a_2 be any such pair. Then, for all $(x,y) \in R$ we have $(a_1(x), a_2(y)) \in (\equiv^\alpha \circ \mathrm{Rel}(F) \circ \equiv^\alpha)(R)$, and hence there exist $a'_1(x,y) \in FX$, $a'_2(x,y) \in FY$ such that

$$a_1(x) \equiv^\alpha a'_1(x,y),\ a'_2(x,y) \equiv^\alpha a_2(y) \ \text{ and } \ (a'_1(x,y), a'_2(x,y)) \in \mathrm{Rel}(F)(R)\,.$$

We now define an equivalence relation $\equiv$ on R by considering the transitive closure of:

- $(x, y_1) \equiv (x, y_2)$ for all (x, y_1), $(x, y_2) \in R$.
- $(x_1, y) \equiv (x_2, y)$ for all (x_1, y), $(x_2, y) \in R$.

Since R is near injective, it follows that if $(x_1, y_1) \equiv (x_2, y_2)$ then $b_1(x_1) = b_1(x_2)$ and $b_2(y_1) = b_2(y_2)$, and thus $a'_1(x_1, y_1) \equiv^\alpha a'_1(x_2, y_2)$ and $a'_2(x_1, y_1) \equiv^\alpha a'_2(x_2, y_2)$.

We consider $R/\equiv$ and for each equivalence class of the quotient set we choose a canonical representative $\overline{(x,y)}$. Obviously we have that $(x,y_1),(x,y_2) \in R$ implies $\overline{(x,y_1)} = \overline{(x,y_2)}$ and that $(x_1,y),(x_2,y) \in R$ implies $\overline{(x_1,y)} = \overline{(x_2,y)}$.

Let us now define two coalgebras $a_1' : X \longrightarrow FX$ and $a_2' : Y \longrightarrow FY$ as follows:

– If there exists some y such that $(x,y) \in R$ we take $a_1'(x) = a_1'\overline{(x,y)}$ for any such y; otherwise, we define $a_1'(x)$ as $a_1(x)$.
– If there exists some x such that $(x,y) \in R$ we take $a_2'(y) = a_2'\overline{(x,y)}$ for any such x; otherwise, $a_2'(y)$ is $a_2(y)$.

With the above definitions,

$$a_1'(x) = a_1'\overline{(x,y)} \equiv^\alpha a_1'(x,y) \equiv^\alpha a_1(x)\,,$$

and similarly $a_2'(y) \equiv^\alpha a_2(y)$, so that a_1', a_2' are F-representations of b_1 and b_2. Besides,

$$\text{if } (x,y) \in R \text{ then } (a_1'(x), a_2'(y)) \in \mathrm{Rel}(F)(R)$$

and R is an F-bisimulation relating them. □

Let us conclude this illustration of our main theorem by explaining why we needed an infinite coalgebra to get a counterexample of the result between bisimulations relating G-coalgebras and those relating their F-representations. As a matter of fact, in the case of the multiset and the powerset functors we could prove the result in Proposition 6 not only for near injective bisimulations but for any relation where no element is related with infinitely many others. However, we will not prove this fact here since it does not seem to generalize to arbitrary natural transformations relating two functors.

Example 2. Next we present an example for the natural transformation $\mathcal{D}_M : \mathcal{M}_1([0,1] \times X) \Rightarrow \mathcal{D}(X)$. If we consider the two probabilistic transition systems s_X and s_Y given by their multisets of probabilistic transitions: $s_X = \{(\frac{1}{2}, x, x_1'), (\frac{1}{2}, x, x_2')\}$, $s_Y = \{(\frac{1}{3}, y, y_1'), (\frac{1}{3}, y, y_2'), (\frac{1}{3}, y, y_3')\}$, where each triple (p, x, x') represents the probabilistic transition $x \xrightarrow{p} x'$, we have the following $\mathcal{D}$-bisimulation relating the initial states x and y: $R = \{(x,y)\} \cup \{(x_i', y_j') \mid i = 1,2, j = 1,2,3\}$. It is easy to see that for the two $\mathcal{M}_1$-representations $s_X^3 = \{3\cdot(\frac{1}{6}, x, x_1'), 3\cdot(\frac{1}{6}, x, x_2')\}$ and $s_Y^2 = \{2\cdot(\frac{1}{6}, y, y_1'), 2\cdot(\frac{1}{6}, y, y_2'), 2\cdot(\frac{1}{6}, y, y_3')\}$, R is also an $\mathcal{M}_1$-bisimulation between them, using the facts that $(x_1', y_1') \in R$, $(x_2', y_2') \in R$ and $(x_1', y_3') \in R$, $(x_2', y_3') \in R$. From this result we immediately conclude that any two $\mathcal{M}_1$-representations of s_X and s_Y are $\equiv^{\mathcal{D}_\mathcal{M}}$-similar.

5 Natural Transformations and Simulations

In this section we will see that all our results about bisimulations in the previous sections can be extended to categorical simulations defined by means of an order on the corresponding functors. Therefore, our first result concerns the preservation of functorial orders by means of natural transformations.

Definition 6. *Given a natural transformation* $\alpha : F \Rightarrow G$ *and* $\sqsubseteq_G$ *an order on* G*, we define the induced order* $\sqsubseteq_G^{\alpha-}$ *on* F *by:* $x \sqsubseteq_G^{\alpha-} x' \iff \alpha_X(x) \sqsubseteq_G \alpha_X(x')$.

It is immediate that $\sqsubseteq_G^{\alpha-}$ is indeed an order on F; given $f : X \longrightarrow Y$ and $x, x' \in X$:

$$\begin{aligned} x \sqsubseteq_G^{\alpha-} x' &\iff \alpha_X(x) \sqsubseteq_G \alpha_X(x') \\ &\implies Gf(\alpha_X(x)) \sqsubseteq_G Gf(\alpha_X(x')) \\ &\iff \alpha_Y(Ff(x)) \sqsubseteq_G \alpha_Y(Ff(x')) \\ &\iff Ff(x) \sqsubseteq_G^{\alpha-} Ff(x'), \end{aligned}$$

where the implication follows because $\sqsubseteq_G$ is functorial.

Example 3. Taking $\{\cdot\} : \mathcal{M} \Rightarrow \mathcal{P}$ and $\sqsubseteq_\mathcal{P} = \subseteq$, then the induced order $\sqsubseteq_\mathcal{P}^{\{\cdot\}-}$ on $\mathcal{M}$ is defined as $u \sqsubseteq_\mathcal{P}^{\{\cdot\}-} v$ iff $\{u\} \subseteq \{v\}$: that is, it coincides with multiset inclusion.

Another example corresponds to the equality relation on G.

Proposition 7. *The induced order* $=_G^{\alpha-}$ *on* F *is just the relation* $\equiv^\alpha$.

Proof. The definition of $\equiv^\alpha$ is just the particular case of our definition of $\sqsubseteq_G^{\alpha-}$ for the equality relation on G as an order on it. □

Orders on F can be also translated to G through a natural transformation $\alpha : F \Rightarrow G$.

Definition 7. *Given a natural transformation* $\alpha : F \Rightarrow G$ *and* $\sqsubseteq_F$ *an order on* F*, we define the projected order* $\sqsubseteq_F^\alpha$ *on* G *as the transitive closure of the relation* $x \sqsubseteq_F^\alpha x'$*, which holds if:*

there exist x_1*,* x_1' *such that* $x = \alpha_X(x_1)$*,* $x' = \alpha_X(x_1')$ *and* $x_1 \sqsubseteq_F x_1'$*, or* $x = x'$.

We need to add the last condition in the definition above in order to cover the case in which α is not an epi. Obviously, we can remove it whenever α is indeed an epi, and in the following we will see that we only need that condition in order to guarantee reflexivity of $\sqsubseteq_F^\alpha$ in the whole of GX, because all of our results concerning this order will be based on its restriction to the images of the components of the natural transformation α_X.

Again, it is easy to prove that $\sqsubseteq_F^\alpha$ is indeed an order on G. By definition, it is reflexive and transitive. It is also functorial: given $f : X \longrightarrow Y$ and $x \sqsubseteq_F^\alpha x'$, with $x = \alpha_X(x_1)$ and $x' = \alpha(x_1')$ such that $x_1 \sqsubseteq_F x_1'$, we need to show $Gf(x) \sqsubseteq_F^\alpha Gf(x')$. Since $Gf(x) = Gf(\alpha(x_1)) = \alpha(Ff(x_1))$, $Gf(x') = Gf(\alpha(x_1')) = \alpha(Ff(x_1'))$, and $Ff(x_1) \sqsubseteq_F Ff(x_1')$, the result follows by the definition of $\sqsubseteq_F^\alpha$.

Theorem 3 (Simulations are preserved by natural transformations). *If* $R \subseteq X \times Y$ *is a* $\sqsubseteq_F$*-simulation relating* $a : X \longrightarrow FX$ *and* $b : Y \longrightarrow FY$*, and* $\alpha : F \Rightarrow G$ *is a natural transformation, then* R *is also a* $\sqsubseteq_F^\alpha$*-simulation relating* $a' = \alpha_X \circ a$ *and* $b' = \alpha_Y \circ b$.

Proof. Let $(x, y) \in R$: we need to show that $(a'(x), b'(y)) \in \mathrm{Rel}(G)_{\sqsubseteq_F^\alpha}(R)$. Since R is a $\sqsubseteq_F$-simulation, $(a(x), b(x)) \in \mathrm{Rel}(F)_{\sqsubseteq_F}(R)$. This means that there exists $w \in FR$ such that $a(x) \sqsubseteq_F Fr_1(w)$ and $Fr_2(w) \sqsubseteq_F b(x)$, and hence that there exists $z = \alpha_R(w) \in GR$ such that $a'(x) \sqsubseteq_F^\alpha \alpha_X(Fr_1(w)) = Gr_1(z)$ and $Gr_2(z) = \alpha_Y(Fr_2(w)) \sqsubseteq_F^\alpha b'(x)$; therefore, $(a'(x), b'(x)) \in \mathrm{Rel}(G)_{\sqsubseteq_F^\alpha}(R)$. □

As said before, bisimulations are just the particular case of simulations corresponding to the equality relation. Obviously we have that $=_F^\alpha$ is $=_G$ and therefore Theorem 1 about the preservation of bisimulations by natural transformations is a particular case of our new preservation theorem covering arbitrary $\sqsubseteq_F$-simulations.

Analogously, we now generalized Theorem 2 to arbitrary $\sqsubseteq_G$-simulations.

Theorem 4. *Let $\alpha : F \Rightarrow G$ be an epi natural transformation, $\sqsubseteq_G$ an order on G and $b_1 : X_1 \longrightarrow GX_1$, $b_2 : X_2 \longrightarrow GX_2$ two coalgebras, with $a_1 : X_1 \longrightarrow FX_1$, $a_2 : X_2 \longrightarrow FX_2$ arbitrary concrete F-representations. Then, the $\sqsubseteq_G$-simulations relating b_1 and b_2 are precisely the $\sqsubseteq_G^{\alpha-}$-simulations relating a_1 and a_2.*

Proof. Just like Theorem 2, the result follows from showing that, for every relation $R \subseteq X_1 \times X_2$,

$$\mathrm{Rel}(F)_{\sqsubseteq_G^{\alpha-}}(R) = \{(u, v) \in FX_1 \times FX_2 \mid (\alpha_{X_1}(u), \alpha_{X_2}(v)) \in \mathrm{Rel}(G)_{\sqsubseteq_G^\alpha}(R)\}\,.$$

Unfolding the definition of $\mathrm{Rel}(F)_{\sqsubseteq_G^{\alpha-}}(R)$ and using the fact that α is a natural transformation:

$$\begin{aligned}\mathrm{Rel}(F)_{\sqsubseteq_G^{\alpha-}}(R) &= \{(u, v) \in FX_1 \times FX_2 \mid \exists w \in FR.\, u \sqsubseteq_G^{\alpha-} Fr_1(w) \wedge \\ &\qquad\qquad Fr_2(w) \sqsubseteq_G^{\alpha-} v\} \\ &= \{(u, v) \in FX_1 \times FX_2 \mid \exists w \in FR.\, \alpha_{X_1}(u) \sqsubseteq_G \alpha_{X_1}(Fr_1(w)) \wedge \\ &\qquad\qquad \alpha_{X_2}(Fr_2(w)) \sqsubseteq_G \alpha_{X_2}(v)\} \\ &= \{(u, v) \in FX_1 \times FX_2 \mid \exists w \in FR.\, \alpha_{X_1}(u) \sqsubseteq_G Gr_1(\alpha_R(w)) \wedge \\ &\qquad\qquad Gr_2(\alpha_R(w)) \sqsubseteq_G \alpha_{X_2}(v)\}\,.\end{aligned}$$

On the other hand,

$$\mathrm{Rel}(G)_{\sqsubseteq_G}(R) = \{(x, y) \in GX_1 \times GX_2 \mid \exists z \in GR.\, x \sqsubseteq_G Gr_1(z) \wedge Gr_2(z) \sqsubseteq_G y\}\,.$$

Now, if $(u, v) \in \mathrm{Rel}(F)_{\sqsubseteq_G^{\alpha-}}(R)$, by taking $\alpha_R(w)$ as the value of $z \in GR$ we have that $(\alpha_{X_1}(u), \alpha_{X_2}(v)) \in \mathrm{Rel}(G)_{\sqsubseteq_G}(R)$. Conversely, if $(\alpha_{X_1}(u), \alpha_{X_2}(v)) \in \mathrm{Rel}(G)_{\sqsubseteq_G}(R)$ is witnessed by z, let $w \in FR$ be such that $\alpha_R(w) = z$, which must exist because α is epi; it follows that $(u, v) \in \mathrm{Rel}(F)_{\sqsubseteq_G^{\alpha-}}(R)$. □

6 Combining Non-determinism and Probabilistic Choices

Probabilistic choice appears as a quantitative counterpart of non-deterministic choice. However, it has been also argued that the motivations supporting the use of these two constructions are different, so that it is also interesting to be able to manage both together. The literature on the subject is full of proposals in this direction [13,10,14], but it has been proved in [16] that there is no distributive law of the probabilistic monad V over the powerset monad P. As a consequence, if we want to combine the two categorical theories to obtain a common framework, we have to sacrifice some of the properties of one of those monads. Varacca and Winskel have followed this idea by relaxing the definition of the monad V, removing the axiom $A \oplus_p A = A$, so that they are aware of the probabilistic choices taken along a computation even if they are superfluous.

We have not yet studied that general case, whose solution in [16] is technically correct, but could be considered intuitively not too satisfactory since one would like to maintain the idempotent law $A \oplus_p A = A$, even if this means that only some practical cases can be considered.

As a first step in this direction we will present here the simple case of alternating probabilistic systems, which in our multi-transition system framework can be defined as follows:

Definition 8. *Alternating multi-transition systems are defined as* $(\mathcal{M}(A \times \cdot) \cup \mathcal{M}_1([0,1] \times A \times \cdot))$*-coalgebras: any state of a system represents either a nondeterministic choice or a probabilistic choice; however, probabilistic and nondeterministic choices cannot be mixed together.*

By combining the two natural transformations $\{\cdot\}$ and $\mathcal{D}_M$ we obtain the natural transformation $\mathcal{D}_M^a$, that captures the behaviour of alternating transition systems.

Definition 9. *We use the term* alternating probabilistic systems *to refer to the* $(\mathcal{P}(A \times \cdot) \cup \mathcal{D}(A \times \cdot))$*-coalgebras. By combining the classical definition of bisimulation and that of probabilistic bisimulations we obtain the natural definition of probabilistic bisimulation for alternating probabilistic systems.*

We define $\mathcal{D}_{M_X}^a : \mathcal{M}(A \times \cdot) \cup \mathcal{M}_1([0,1] \times A \times \cdot) \Rightarrow \mathcal{P}(A \times \cdot) \cup \mathcal{D}(A \times \cdot)$ *as* $\mathcal{D}_{M_X}^a(M) = \{\cdot\}(M)$, $\mathcal{D}_{M_X}^a(M_1) = \mathcal{D}_M(M_1)$, *where* $M \in \mathcal{M}(A \times X)$, $M_1 \in \mathcal{M}_1([0,1] \times A \times X)$.

Then we can consider the induced functorial equivalence $\equiv^{\mathcal{D}_M^a}$ which roughly corresponds to the application of $\equiv^{\{\cdot\}}$ in the non-deterministic states, and the application of $\equiv^{\mathcal{D}_M}$ in the probabilistic states. As a consequence of Theorem 2 we obtain the following corollary.

Corollary 5. *Bisimulations between alternating probabilistic systems are just* $\equiv^{\mathcal{D}_M^a}$*-simulations between alternating multi-transition systems.*

Example 4. Let $X = \{x, x_1', x_2', x_3', x_4'\}$, $Y = \{y, y_1', y_2', y_3', y_4'\}$ and let us define (disregarding actions) the alternating multi-transition systems $a_X : X \longrightarrow \mathcal{M}(X) \cup \mathcal{M}_1([0,1] \times X)$ and $a_Y : Y \longrightarrow \mathcal{M}(Y) \cup \mathcal{M}_1([0,1] \times Y)$ as $a_X(x) = \{1 \cdot (\frac{1}{2}, x_1'), 1 \cdot (\frac{1}{2}, x_2')\}$, $a_X(x_1') = \{1 \cdot x_3'\}$, $a_X(x_2') = \{1 \cdot x_4'\}$, $a_X(x_3') = a_X(x_4') = \emptyset$, $a_Y(y) = \{1 \cdot (\frac{1}{3}, y_1'), 1 \cdot (\frac{1}{3}, y_2'), 1 \cdot (\frac{1}{3}, y_3')\}$, $a_Y(y_1') = a_Y(y_2') = a_Y(y_3') = \{1 \cdot y_4'\}$,

$a_Y(y_4') = \emptyset$. a_X and a_Y induce the canonical alternating probabilistic systems $b_X : X \longrightarrow \mathcal{P}(X) \cup \mathcal{D}(X)$ and $b_Y : Y \longrightarrow \mathcal{P}(Y) \cup \mathcal{D}(Y)$ (for example, $b_X(x) = \frac{1}{2}x_1' + \frac{1}{2}x_2'$ and $b_Y(y_3') = \{y_4'\}$).

Now, if we want to know if there is a bisimulation between b_X and b_Y we can use the fact that $R = \{(x,y)\} \cup \{(x_i', y_j') \mid i = 1,2, j = 1,2,3\} \cup \{(x_i', y_4') \mid i = 3,4\}$ is a $\equiv^{\mathcal{D}^a_M}$-bisimulation between a_X and a_Y (using a similar argument to that in Example 2), and apply Corollary 5 to conclude that there is a $(\mathcal{P} \cup \mathcal{D})$-bisimulation between b_X and b_Y.

7 Conclusion

In this paper we have shown that multitransition systems are a common framework wherein bisimulation of ordinary and probabilistic transition systems almost collapse into the same concept of multiset (bi)simulation. Indeed, the definition of bisimulation for the multiset functor is extremely simple, which supports the idea that multisets are the natural framework in which to justify the use of bisimulation as the canonical notion of equivalence between (states of) systems.

These results have been obtained by exploiting the fact that natural transformations between two functors relate in a nice way bisimulations over their corresponding coalgebras. We have illustrated these general results by means of the natural transformations that connect the powerset and the probabilistic distributions functors with the multiset functor.

The categorical notion of simulation proposed by Hughes and Jacobs has played a very important role in our work; this fact, in our opinion, is far from being casual. In particular, categorical simulations based on equivalence relations always define equivalence relations weaker than bisimulation equivalence. Besides, as illustrated by their use in this paper, they can be used to relate the bisimulation equivalence corresponding to functors connected by a natural transformation.

Related to our work is [2], where probabilistic bisimulations are studied in connection with natural transformations and other categorical notions. Even though some connections can be found, there are very important differences; in particular they do not consider categorical simulations nor use the multiset functor as a general framework in which to study both ordinary and probabilistic bisimulations. We can also mention [15], where the functor $\mathcal{D}$ is replaced with a functor of indexed valuations so that it can be combined with the powerset functor.

A direction for further study that we intend to explore concerns other classes of bisimulations, like the forward-backward ones estudied in [5]. Besides we will study more general combinations of non-deterministic and probabilistic choices, comparing in detail our approach with the use of indexed valuations in [15,16] to combine the monads defining the corresponding functors.

We are confident we will be able to study them in a common setting by generalizing and adapting all the appropriate notions on categorical simulations.

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