Discrete Frontiers

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Abstract. Many applications require to extract the surface of an object from a discrete set of valued points, applications in which the topological soundness of the obtained surface is, in many case, of the utmost importance. In this paper, we introduce the notion of frontier order which provides a discrete framework for defining frontiers of arbitrary objects. A major result we obtained is a theorem which guarantees the topological soundness of such frontiers in any dimension. Furthermore, we show how frontier orders can be used to design topologically coherent "Marching Cubes-like" algorithms.

1 Introduction

The Marching Cubes algorithm[1] provides an efficient way to extract a polygonal surface from an object expressed as a subset of a digital image, or an isosurface from a function. However, the polygonal mesh obtained by this algorithm is not guaranteed to be a topological surface, since artefacts such as holes[2,3,4,5] might appear. While small holes, though a nuisance, might not seem an overly important issue for the visualization of large objects, they can have a dramatic impact on collision detection and most calculations. Consequently, many researches have been directed toward solving this problem[3,4,5,6,7,8].

The approach of J. O. Lachaud [8] is especially interesting: it guarantees the topology of the extracted surface using the topology of the underlying discrete object. Such guarantees are obtained using the framework of digital topology[9] for the underlying object while defining continuous analogs of digital boundaries, and the results hold true for $\mathbb{Z}^n$, $n \in \mathbb{N}^\star$. In a former article[10], we introduced the notion of frontier orders in 2D and 3D partially ordered sets, asserting the possibility to define the frontiers of objects as symmetrical separating surfaces in such spaces. The present article will encompass and extend our previous results: frontier orders will be presented as a purely discrete framework, based on order topology[11,12,13], which provides topological guarantees for a wide variety of spaces of any dimension. The main result of this paper is a theorem establishing that the frontier order of any subset of an n-surface[14] is a union of disjoint $(n-1)$-surfaces. This result allows us to design sound "Marching Cubes-like" algorithms to extract frontiers of objects both in the Khalimsky grid and in $\mathbb{Z}^3$ equipped with the digital topology.

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2 Definitions

Let us first introduce the notations we will use in this article. If X is a set and S a subset of X, $\overline{S}$ denotes the complement of S in X. If λ is a binary relation on X, *i.e.*: a subset of $X \times X$, the *inverse* of λ is the binary relation $\{(x,y) \in X \times X; (y,x) \in \lambda\}$. For any binary relation λ, $\lambda^\square$ is defined by $\lambda^\square = \lambda \setminus \{(x,x); x \in X\}$. For each x of X, $\lambda(x)$ denotes the set $\{y \in X; (x,y) \in \lambda\}$ and for any subset S of X, $\lambda(S)$ denotes the set $\{y \in \lambda(s); s \in S\}$.

2.1 Orders

An *order* is a pair $|X| = (X, \alpha_X)$ where X is a set and α_X is a reflexive, antisymmetric and transitive binary relation on X. The set $\alpha_X(x)$ is called the α_X*-adherence* of x. We denote by β_X the inverse of α_X and by θ_X the union of α_X and β_X. The set $\theta_X(x)$ is called the θ_X*-neighborhood* of x. A *path* from x_0 to x_n in X in $|X|$ is a sequence $x_0, \ldots, x_n$ of elements of X such that $\forall i \in [1 \ldots n]$, $x_{i-1} \in \theta_X(x_i)$. A connected component C of $|X|$ is a maximal subset of X such that for all $x, y \in C$, there exists a path from x to y in C.

The rank of an element x of X is 0 if $\alpha_X^\square(x) = \emptyset$ and is equal to the maximal rank of the elements of $\alpha_X^\square(x)$ plus 1 otherwise; the rank of an order is the maximal rank of its elements. Any element of an order is called a point and it is also called an n-element, n being the rank of this point.

An order $|X|$ is *countable* if X is countable, it is *locally finite* if, for each $x \in X$, $\theta_X(x)$ is a finite set. A *CF-order* is a countable locally finite order.

Let $|X|$ and $|Y|$ be two orders, $|X|$ and $|Y|$ are *order isomorphic* if there exists a bijection $f : X \to Y$ such that, for all $x_1, x_2 \in X$, $x_1 \in \alpha_X(x_2) \Leftrightarrow f(x1) \in \alpha_Y(f(x_2))$.

If (X, α_X) is an order and S is a subset of X, the *sub-order* of $|X|$ relative to S is the order (S, α_S) with $\alpha_S = \alpha_X \cap (S \times S))$. When no confusion may arise, we also write $|S| = (S, \alpha_S)$.

2.2 Discrete Surfaces

We use the general definition for n-dimensional surfaces (or simply n*-surfaces*) proposed by Evako, Kopperman and Mukhin[14]; such surfaces are also known as *Jordan* n*-surfaces*[15]. This definition is both elegant and efficient:
Let $|X| = (X, \alpha_X)$ be a non-empty CF-order.

• The order $|X|$ is a *0-surface* if X is composed of exactly two points x and y such that $y \notin \alpha_X(x)$ and $x \notin \alpha_X(y)$.
• The order $|X|$ is an n*-surface*, $n > 0$, if $|X|$ is connected and if, for each x in X, the order $|\theta_X^\square(x)|$ is an $(n-1)$-surface.

2.3 Simplicial Complexes

Let Λ be a set, any non-empty subset of Λ is called a *simplex*. A subset constituted of $(n+1)$ of elements Λ is also called an n*-simplex*. Now, let C be a family of simplexes of Λ, C is a *simplicial complex* if it is closed by inclusion, which

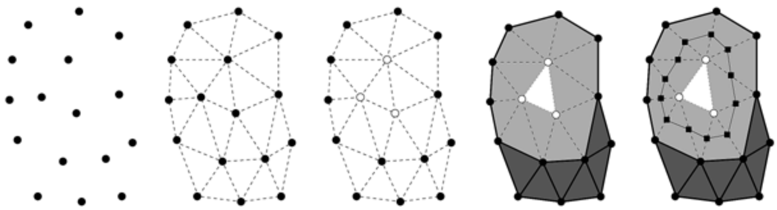

Fig. 1. Schema depicting our methodology. a) Our data is a set of points. b) Upon this set of points is built a simplicial complex. c) Independently from this simplicial complex, some of these points are labeled as object points, the others as background points. d) This bi-partition of the point set induces a tri-partition of the simplicial complex between an object complex [white], a background complex [black] and a frontier complex [grey]. e) The frontier order [depicted by a discrete curve], isomorphic to the frontier complex, is then defined.

means that, if s belongs to C, then any non-empty subset of s also belongs to C. A *(simplicial) n-complex* is a simplicial complex in which maximal elements are n-simplexes. The minimal subset Λ_C of Λ such that any element of C is a subset of Λ_C is called the *support* of C. In this paper, simplicial complexes are also seen as orders: any simplicial complex C will be interpreted as the order $|C| = (C, \subseteq)$. Consequently, C will be said to be an n-surface if $|C|$ is an n-surface.

The simplicial complexes we just defined are often known as *abstract simplicial complexes*, as opposed to other notions of complexes based upon an underlying Euclidean space.

2.4 Chains of an Order

Let $|X|$ be an order, a *chain* of $|X|$ is a fully ordered subset of X. An n-chain is a chain of size $n+1$. We denote by $\mathcal{C}^X$ the set of all the chains of $|X|$, *ie.*: $\mathcal{C}^X = \{S \subseteq X, S \neq \emptyset, \forall s_1, s_2 \in S, s_1 \in \theta_X^\square(s_2)\}$. It should be noted that $(\mathcal{C}^X, \subseteq)$ is an order and that $\mathcal{C}^X$ is a simplicial complex, the support of which is X. Moreover, the topology of $(\mathcal{C}^X, \subseteq)$ is strongly related to the topology of $|X|$, as shown by the following proposition:

Proposition 1 *Let $|X|$ be an order. If $|X|$ is an n-surface then the order $|\mathcal{C}^X| = (\mathcal{C}^X, \subseteq)$ is an n-surface as well.*

The proof of the above proposition is not included in this article due to space restrictions: while not overly long nor difficult by itself, this proof would require several lemmas. This holds true for the other properties introduced in this article.

3 Frontier Orders

If we consider a simplicial complex C (figure 1.b) and its support X (figure 1.a), the partition of X between a set K, *the object*, and its complementary $\overline{K}$, *the background*, (figure 1.c) induces a partition of C into three sets (figure 1.d):

- C_K, the set of all the simplexes which are subsets of K
- $C_{\overline{K}}$, the set of all the simplexes which are subsets of $\overline{K}$
- $C_{K/\overline{K}}$, the set of the simplexes being neither subset of K nor subset of $\overline{K}$

Since a singleton (0-simplex) is either a subset of K or a subset of $\overline{K}$, $C_{K/\overline{K}}$ is not closed for the inclusion and, consequently, is not a simplicial complex. Nevertheless, $|C_{K/\overline{K}}| = (C_{K/\overline{K}}, \subseteq)$ is still the sub-order of $|C|$ relative to $C_{K/\overline{K}}$.

It should be noted that, for any given C and K, $|C_{K/\overline{K}}|$ is order isomorphic to the *frontier order* $|\widetilde{C}_{K/\overline{K}}|$ (figure 1.e) defined as the couple $(\widetilde{C}_{K/\overline{K}}, \alpha_{\widetilde{C}})$ where $\widetilde{C}_{K/\overline{K}} = \{\{A, B\}, A \subseteq K, B \subseteq \overline{K}, A \neq \emptyset, B \neq \emptyset, A \cup B \in C\}$ and $\alpha_{\widetilde{C}}$ is the binary relation such that, considering $M = \{A_1, B_1\}$ and $N = \{A_2, B_2\}$, $M \in \alpha_{\widetilde{C}}(N)$ is equivalent to $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$.

By definition, $C_{K/\overline{K}}$ is both *symmetrical*, since $C_{K/\overline{K}} = C_{\overline{K}/K}$, and *separating*, since any path from $x \in K$ to $y \in \overline{K}$ crosses $C_{K/\overline{K}}$. Consequently, the frontier order, which is symmetrical, can be said to be separating. Furthermore, the following theorem, the main result of this paper, guarantees that a frontier order is a union of discrete surfaces:

Theorem 2 *Let C be a simplicial complex with the property of being an n-surface, $n > 1$, and let X be its support. Now, let K be a non-empty proper subset of X. Then the frontier order $\widetilde{C}_{K/\overline{K}}$ is a union of disjoint $(n-1)$-surfaces.*

As seen previously, to any order can be associated the simplicial complex composed by its chains. So, as a consequence of proposition 1 and theorem 2, we have:

Corollary 3 *Let $|X| = (X, \alpha_X)$ be an order and K a non-empty proper subset of X. If $|X|$ is an n-surface then the frontier order $|\widetilde{\mathcal{C}^X}_{K/\overline{K}}|$ is a union of disjoint $(n-1)$-surfaces.*

4 Marching Cubes and the Khalimsky Grid

The main feature of the Marching Cubes algorithm is a look-up table associating a surface patch to each possible partition of the corners of a unit cube between two sets of points, K and $\overline{K}$. Given a map $f : \mathbb{Z}^3 \to \mathbb{R}$ and a value n, the Marching Cube algorithm sets $K = \{x \in \mathbb{Z}^3, f(x) > n\}$ and $\overline{K} = \mathbb{Z}^3 \setminus K$. Then, for each unit cube of the cubic grid $\mathbb{Z}^3$, the algorithm finds the appropriate surface patch in the look-up table and builds this patch, interpolated according to the values of the eight corners of this unit cube. The union of all those patches constitutes the approximated iso-surface.

This algorithm is often used to extract the surface of an object in a grey-level image, in which case n is interpreted as a threshold. In the case of a binary image, it is sufficient to apply the look-up table without any interpolation.

While the original Marching Cubes algorithm[1] did not consider the topology of the underlying image, and did not guarantee the topology of the extracted surface, we will now explain how to generate a Marching Cube algorithm coherent with the topology of the Khalimsky grid.

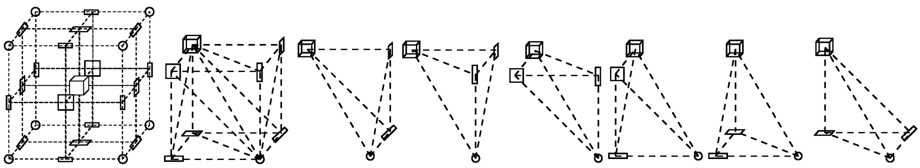

Fig. 2. A unit cube ($\{n, n+1\} \times \{m, m+1\} \times \{l, l+1\}$ and its closure) of H , one of the 8 unit cubes of $\mathbb{Z}$ of which it is made, and the tetrahedra (chains of $|H|$) it contains.

4.1 Khalimsky Grid and Embedded Frontier Order

Let us first introduce now the Khalimsky grids as the family of orders $|H^n| = (H^n, \subseteq)$, defined by:

$$H_0^1 = \{\{a\}, a \in \mathbb{Z}\} \;\; ; \;\; H_1^1 = \{\{a, a+1\}, a \in \mathbb{Z}\}$$
$$H^1 = H_0^1 \cup H_1^1$$
$$H^n = \{h_1 \times \ldots \times h_n, \forall i \in [1, n], h_i \in H^1\}$$

It is important to note that $|H^n|$ is an n-surface for all $n \in \mathbb{N}^*$ as proved by V. A. Evako and al.[14]. This implies, by corollary 3, that the frontier defined for any subset of an order H^n is a union of disjoint $(n-1)$-surfaces.

A natural encoding of the set H^n into the corresponding discrete space $\mathbb{Z}^n$ is defined as follows[11]: to every element $h_1 \times \ldots \times h_n$ of H^n is assigned the vertex of coordinates $(z_1, \ldots, z_n)$ in $\mathbb{Z}^n$, such that $\forall i \in [1 \ldots n]$, $z_i = 2v_i$ if $h_i = \{v_i\}$ and $z_i = 2v_i + 1$ if $h_i = \{v_i, v_i + 1\}$. Figure 2 depicts the cube of H^3 constituted by $\{n, n+1\} \times \{m, m+1\} \times \{l, l+1\}$ and its subsets, which contains 8 unit cubes of $\mathbb{Z}^3$, each of which is itself constituted by 6 tetrahedra, images of the chains of H^3.

This encoding of H^n induces an embedding of the frontier orders based upon it: to each 0-element $\{\{A\}, \{B\}\}$ of the frontier order we assign the vertex of coordinates $(a+b)/2$ where a (resp. b) is the vertex assigned to A (resp. B). Then, to each 1-element we assign the segment joining the vertices associated to the 0-elements of its θ-neighborhood, to each 2-element we assign the corresponding polygon (which is in fact either a triangle or a parallelogram); and so on.

4.2 Marching Cubes-Like Algorithm in Dimension 3

The look-up table obtained for the possible configurations of a unit cube of H^3 is depicted in figure 3. Unlike both the original Marching Cubes algorithm and its correction by Lachaud in the framework of digital topology, our surface generation process is not translation invariant, since the Khalimsky grid itself is not. In practice, it is sufficient to rotate the configuration according to the coordinates of the upper-left-front (or any other) corner of the unit cube. The configurations given figure 3 being based upon chains (tetrahedra) rather than upon cubes, they are more facetized than those of the original Marching-Cubes

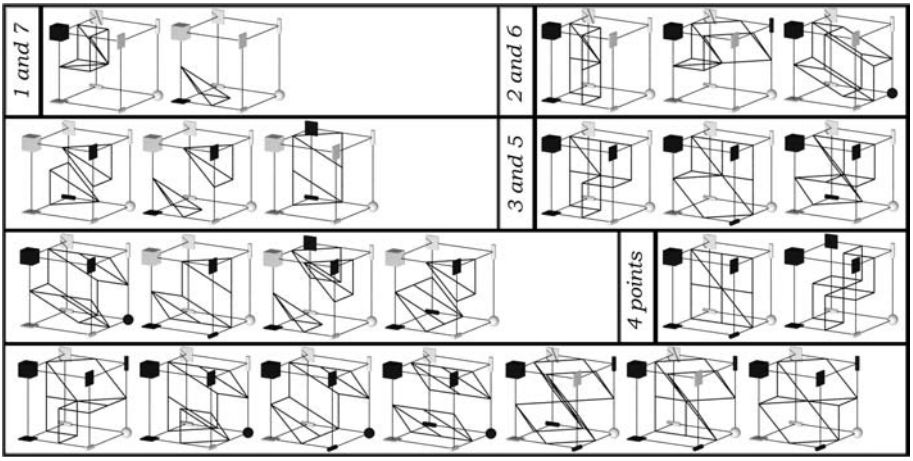

Fig. 3. Configurations obtained for the look-up table of the Marching Cubes-like algorithm in the H^3 case. Whenever several configurations are identical up to rotations and symmetries, only one is presented here. While the original Marching-Cube Algorithm generates from 1 to 4 triangles for each configuration, the count here ranges from 2 to 12 triangles (2 to 6 frontier orders elements, some of them correspond to parallelograms).

algorithm. It is possible to simplify these configurations, with the guarantee to preserve the overall topology, and the guarantee that the surface still separates the object from the background. The simplification process is as follows: the configurations of figure 3 are first triangulated, then anti-stellar and bi-stellar moves[16] are applied to reduce the number of faces. In order to ensure the coherency of the frontier between adjacent unit cubes, we systematicaly replace any point located on a face but not an edge of a cubic cell by the segment connecting its two nearest neighbors in this face as depicted in figure 5. We thereby obtain the configuration table depicted figure 4.

5 Frontier Orders and Digital Topology

In the framework of digital topology[9], a digital image built upon $\mathbb{Z}^3$ can be seen as a quadruple $(\mathbb{Z}^3, m, n, K)$, where $K \subseteq \mathbb{Z}^3$ is the set of the object points (or object), where $\overline{K}$ is the set of the background points (or background) and where $(m, n) \in \{(6, 26), (6, 18), (26, 6), (18, 6)\}$, m being the adjacency of the object and n the adjacency of the background. More precisely, any two points belonging to the object are connected if:

- both belong to a unit edge.
- both belong to a unit face and either $m = 18$ or $m = 26$.
- both belong to a unit cube and $m = 26$.

The same goes for the background, with n instead of m.

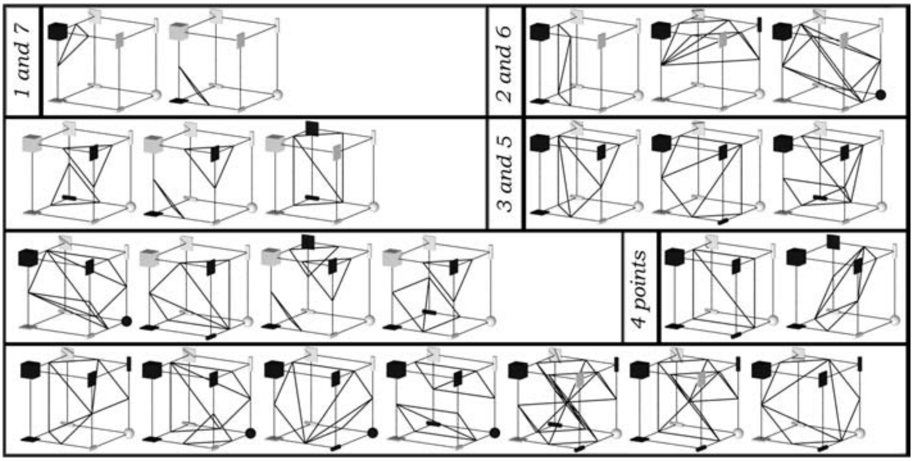

Fig. 4. Simplified configurations obtained for look-up table of the Marching Cubes-like algorithm in the H^3 case, from the configurations presented in figure 3. One should note that some originaly different frontiers have identical simplifications, up to rotations. Most simplified configurations are equivalent to the corresponding configuration of the original Marching-Cubes algorithm; in the sense that they have the same number of triangles, the same intersection with the cube boundary and are stellar equivalent. Nevertheless some new configurations appear whenever two points located on the opposite corner of a face or cube are adjacent according to $|H^3|$ topology; and one of the original algorithm configurations, assuming four non-adjacent corners, has no equivalent here.

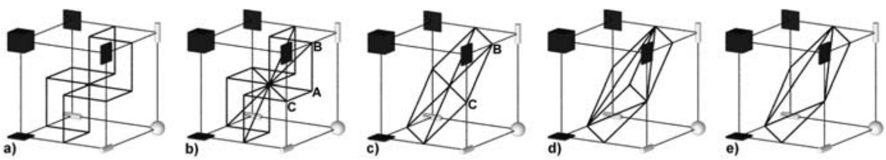

Fig. 5. a) Is an original configuration. b) Is a triangulation of a). c) Is obtained from b) by the anti-stellar move replacing the vertex A by the 1-simplex $\{B, C\}$, this same move being applied to all points located on the centers of the faces (observe that this move has effects not only on this cube but on the neighboring ones as well). d) and e) are then obtained by consecutive bi-stellar moves.

In this framework, Lachaud[7,8] has provided a topologically sound Marching Cubes algorithm using continuous analogs of digital boundaries, we will show how the same result can be reached using purely discrete means: frontier orders.

Since $\mathbb{Z}^3$ equipped with digital topology is not an order, we first need to build a simplicial complex C upon it. However, would C be built using only $\mathbb{Z}^3$ as its support, it would be unable to emulate the various adjacency relations used by digital topology; two points x and y of K located on the opposite corners of a face, for example, would be considered to be adjacent if $\{x, y\} \in C$, whatever the adjacency. In order to take into account the adjacency, we need to introduce two types of intermediary points: face points, which are located in the center of

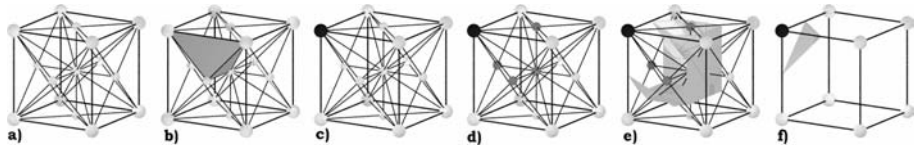

Fig. 6. a) Triangulation of a unit cube, with intermediary (smaller) points. b) One of the 24 identical tetrahedra of this triangulation is outlined in grey. c) Let now assume that one corner point (black) belongs to the object, and all the others (white) to the background. d) Result of the affectation strategy, assuming that the object is 26-connected (which implies that the background is 6-connected). e) Generation of the frontier complex. f) Simplified frontier.

a face, and cube points, which are located in the center of a cube. Then, refering to the previous example, two points of K located on the opposite corners of a face will be considered adjacent if, and only if, the face point associated to this face also belongs to K, which will depend on the adjacency (and, maybe, the other corners of the face). No points are introduced for edges since two points of K located on the same edge are always adjacent. As a result, each cube (figure 6.a) is triangulated into 24 identical tetrahedra defined by 2 points of $\mathbb{Z}^3$, a face point and a cube point (figure 6.b). It should be noted that C is then a 3-surface, which can be easily verified by an exhaustive checking of every existing simplex configuration, thus the hypotheses of theorem 2 are satisfied.

Since the entries of the look-up table are to be entirely determined by the points of $\mathbb{Z}^3$ and the adjacency, the belonging of an intermediary point to either K or $\overline{K}$ is entirely determined by an *affectation strategy* (figure 6.d) defined as follows:

- 6/26-adjacency and 26/6-adjacency (let K be the 26-adjacent set)
  - a face point belongs to K iff at least one corner of this face does
  - a cube point belongs to K iff at least one corner of this cube does
- 6/18-adjacency and 18/6-adjacency (let K be the 18-adjacent set)
  - a face point belongs to K iff at least one corner of this face does
  - a cube point belongs to K iff at least three corners of this cube do

The simplified results, which can be found on figure 7, are obtained from the initial ones by stellar and bi-stellar moves, as in the $|H^3|$ case, and are equivalent to the results obtained by Lachaud for the same configurations.

6 Conclusion

We have introduced frontier orders which allow to define the frontier of a discrete object. We have established that frontier orders are surfaces, which appears as a necessary property for the design of topologically sound Marching Cubes-like algorithms.

An extended version of this paper[17] will provide proofs for the properties stated in this article, as well as other important properties which, due to space limitation, have not been included. In particular we proved that any simplicial

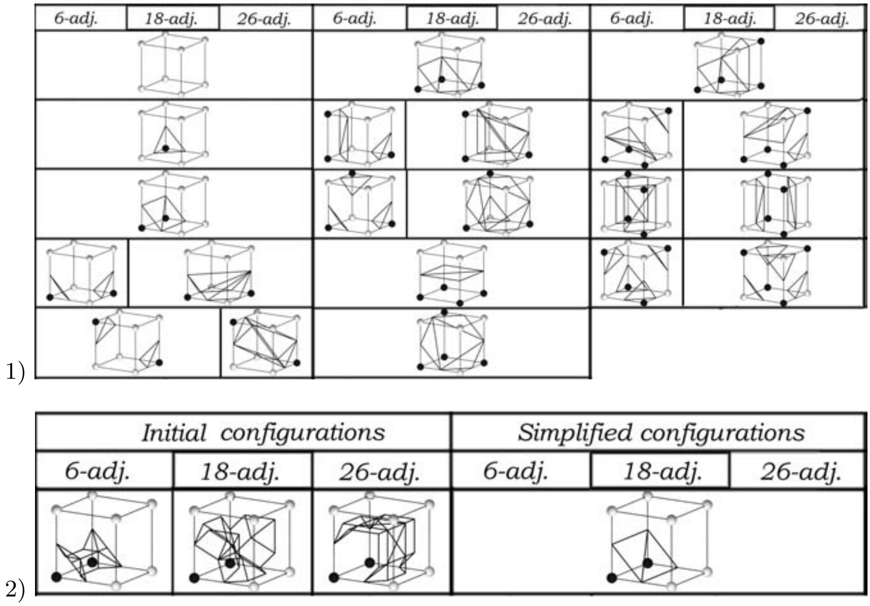

Fig. 7. 1) Simplified configurations, according to the adjacency of the set of black points. As previously (in the H^3 case) the initial configurations are the direct embedding of the frontier order (induced by the subdivision into 24 tetrahedra and the affectation strategy appropriate for the adjacency) into $\mathbb{R}^3$ while the simplified configurations are obtained from the initial ones through stellar and bi-stellar moves. 2) As it can be seen, it may happen that depending on the chosen adjacency, the same vertex configuration produces different frontier order (initial) configurations, which in turn produce the same simplified configuration.

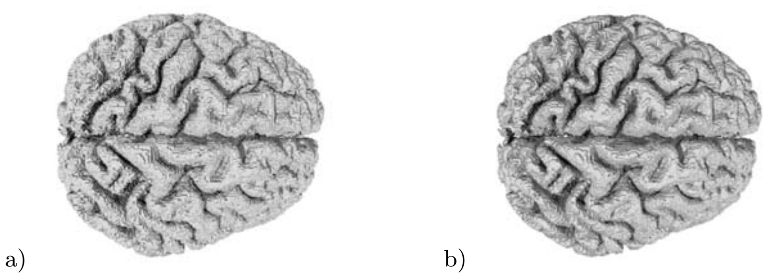

Fig. 8. Results for a segmented cortex (in $|H^3|$), a) using initial configurations b) using simplified configurations.

complex which is an n-surface is an n-pseudomanifold. We will also show the link between frontier orders and regular neighborhoods[18].

The frontier order associated to a cortex segmentation is depicted on figure 8 and those interested in further images may find some at the following address: "`http://www.esiee.fr/~info/xavier/MC03_res.html`".

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