Combinatorial Topologies for Discrete Planes

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Abstract. A discrete analytical plane **DAP** is defined as a set of lattice points which satisfy two inequalities. In this paper, we define a discrete combinatorial plane **DCP** and show relations between **DAP**s and **DCP**s such that a **DCP** is a combinatorial surface of a **DAP**. From the relations, we derive new combinatorial topological properties of **DAP**s.

1 Introduction

A plane **P** in the 3-dimensional Euclidean space $\mathcal{R}^3$ is given by an analytical form such that

$$\mathbf{P} = \{(x, y, z) \in \mathcal{R}^3 \; : \; ax + by + cz + d = 0\} \tag{1}$$

where a, b, c, d are real numbers. Let $\mathcal{Z}$ be the set of integers; $\mathcal{Z}^3$ denotes the set of lattice points whose coordinates are all integers. The discrete analytical form of discrete planes in $\mathcal{Z}^3$, called discrete analytical planes, was introduced by Reveillès [9] and defined such that

$$\mathbf{DAP} = \{(x, y, z) \in \mathcal{Z}^3 \; : \; 0 \leq ax + by + cz + d < w\} \tag{2}$$

where a, b, c, d are all integers. We call w a width of a **DAP**. If $w = |a|+|b|+|c|$, a **DAP** is called a standard plane **SP** [2,5], and if $w = max\{|a|, |b|, |c|\}$, a **DAP** is called a naive plane **NP** [9].

In this paper, we define discrete planes which have combinatorial topological structures, called discrete combinatorial planes **DCP**s. We construct a **DCP** by applying our algorithm of combinatorial boundary tracking [8] to one of digitized half spaces separated by **P**. Any **DCP** is defined as the combinatorial boundary of a polyhedral complex which is considered to be a polygonal decomposition of border points of such a separated region. Thus, a **DCP** is a topological space and not a subset of $\mathcal{Z}^3$ as a **DAP**.

Our main aim is to show relations between **DAP**s and **DCP**s such that a **DCP** is a combinatorial surface of a **DAP**. In [7], we have already shown such

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relations between **NP**s and **DCP**s. Discrete combinatorial planes defined in [7] are based on simplicial complexes [1,10], but in this paper, they are based on polyhedral complexes [11]. This is because we would like to make use of our recent results from the polyhedral approach [8] to obtain relations between **DAP**s and **DCP**s some of which could not be derived from the simplicial approach [7], for example, relations between **SP** and **DCP**. From the relations, we obtain combinatorial topological properties of **DAP**s. First we consider configurations of points in a **DAP** at local regions which project on the coordinate plane $z = 0$ as rectangles whose sizes are 2×2 are called bicubes. If the sizes of rectangles are 3×3, they are called tricubes. It has been shown in [4,6] that there exist five different bicubes and forty different tricubes in **NP**s where $0 \leq a \leq b \leq c$, $c > 0$. Note that they have been obtained only for **NP**s, but it is possible to extend the results to **SP**s if we use local differences of combinatorial topological structures between an **NP** and an **SP**. We therefore obtain combinatorial topological structures in bicubes and tricubes, called combinatorial bicubes and tricubes, for both **SP**s and **NP**s. By observing combinatorial bicubes and tricubes, we show that a **DCP** is a 2-dimensional combinatorial manifold of a **DAP**. Similar properties have been given for **SP**s in [5] and for **NP**s in [6], respectively, but no proof and no detail are given in [6]. We also study connectivities of points in a **DAP** and the complement $\overline{\mathbf{DAP}}$ and derive the same results as the previous work [3,5] whose proofs are different from ours.

2 Discrete Combinatorial Planes

In $\mathcal{R}^n$, a convex polyhedron σ is the convex hull of a finite set of points in some $\mathcal{R}^d$ where $d \leq n$. The dimension of σ is the dimension of its affine hull. An n-dimensional convex polyhedron σ is abbreviated to an n-polyhedron. A linear inequality $\boldsymbol{a} \cdot \boldsymbol{x} \leq z$ is valid for σ if it is satisfied for all points $\boldsymbol{x} \in \sigma$. A face of σ is defined by any set $\delta = \sigma \cap \{\boldsymbol{x} \in \mathcal{R}^d : \boldsymbol{a} \cdot \boldsymbol{x} \leq z\}$ where $\boldsymbol{a} \cdot \boldsymbol{x} \leq z$ is valid for σ.

Definition 1. *A polyhedral complex* **K** *is a set of convex polyhedra such that*

1. *the empty polyhedron is in* **K***,*
2. *if $\sigma \in$* **K***, then all faces of σ are also in* **K***,*
3. *the intersection $\sigma \cap \tau$ of two convex polyhedra $\sigma, \tau \in$* **K** *is a face both of σ and of τ.*

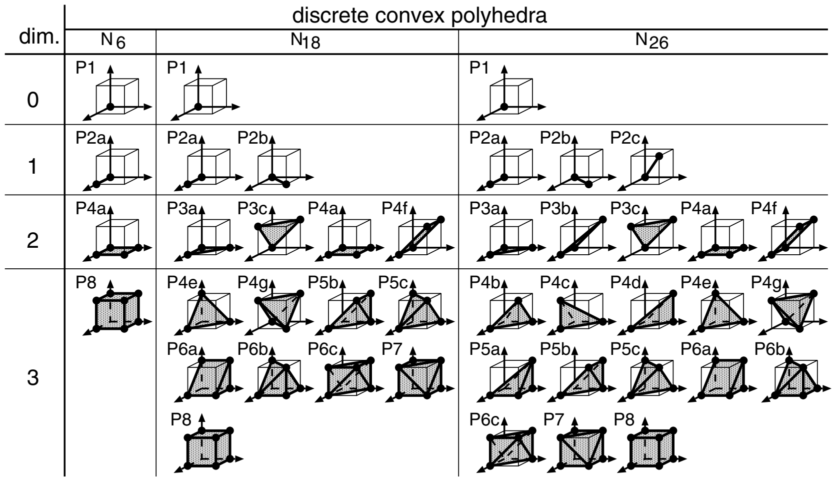
The dimension of **K** is the largest dimension of a convex polyhedron in **K**.

In $\mathcal{Z}^3$ m-neighborhoods are defined by

$$\mathbf{N}_m(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathcal{Z}^n : \|\boldsymbol{x} - \boldsymbol{y}\|^2 \leq t\} \tag{3}$$

setting $t = 1, 2, 3$ for each $m = 6, 18, 26$. We consider all convex polyhedra in $\mathcal{Z}^3$ such that the vertices are all lattice points and any adjacent vertices are m-neighboring each other for either $m = 6, 18, 26$. We call such convex polyhedra discrete convex polyhedra. We illustrate all discrete convex polyhedra with the

Table 1. All discrete n-polyhedra for $n = 0, 1, 2, 3$.



dimension of $n = 0, 1, 2, 3$ and with the m-neighborhood relations between the adjacent vertices for $m = 6, 18, 26$ in Table 1. We then construct a discrete polyhedral complex which is a collection of discrete convex polyhedra satisfying the three conditions in Definition 1 for each m-neighborhood system. Hereafter, n-dimensional discrete convex polyhedra and n-dimensional discrete polyhedral complexes are called discrete n-polyhedra and discrete n-complexes.

We give some topological notions for discrete polyhedral complexes [1]. A discrete n-complex $\mathbf{K}$ is said to be pure if each of discrete n'-polyhedra of $\mathbf{K}$ is a face of a discrete n-polyhedron of $\mathbf{K}$ where $n' < n$. If $\mathbf{K}_0$ is any subcomplex of $\mathbf{K}$, the complex consisting of all the elements of $\mathbf{K}_0$ and of all the elements of $\mathbf{K}$ each of which is a face of at least one element of $\mathbf{K}_0$ is called the combinatorial closure $Cl(\mathbf{K}_0)$ of $\mathbf{K}_0$ in $\mathbf{K}$. We consider a discrete polyhedral complex $\mathbf{C}$ as a topological representation of any subset $\mathbf{V} \subset \mathcal{Z}^3$, i.e. a topological space by topologizing $\mathbf{V}$. We first obtain a pure discrete 3-subcomplex $\mathbf{O} \subseteq \mathbf{C}$ and then define the combinatorial boundary $\partial\mathbf{O}$ of $\mathbf{O}$ in the following [10].

Definition 2. *Let* $\mathbf{O}$ *be a pure discrete* 3*-complex and* $\mathbf{Q}$ *be the set of all discrete* 2*-polyhedra in* $\mathbf{O}$ *each of which is a face of exactly one discrete* 3*-polyhedron in* $\mathbf{O}$. *The combinatorial boundary of* $\mathbf{O}$ *is defined such that* $\partial\mathbf{O} = Cl(\mathbf{Q})$.

From Definition 2, we see that the boundary $\partial\mathbf{O}$ of a pure discrete 3-complex $\mathbf{O}$ is a pure discrete 2-subcomplex of $\mathbf{O}$.

Because discrete convex polyhedra are defined for each m-neighborhood system where $m = 6, 18, 26$, a discrete polyhedral complex $\mathbf{C}$, a discrete pure 3-polyhedron $\mathbf{O}$ and the combinatorial boundary $\partial\mathbf{O}$ are also defined for each m-neighborhood system. When we insist an m-neighborhood system considering for them, they are denoted by $\mathbf{C}_m$, $\mathbf{O}_m$ and $\partial\mathbf{O}_m$ instead.

Given a finite lattice-point set $\mathbf{V} \subset \mathcal{Z}^3$, we show in [8] how to construct a pure discrete 2-complex $\partial\mathbf{O}_m$ which is the combinatorial boundary of $\mathbf{V}$. The idea is simple: we first obtain a discrete polyhedral complex $\mathbf{C}_m$ putting as many discrete convex polyhedra as possible into $\mathbf{V}$ so that all the vertices of discrete convex polyhedra are points in $\mathbf{V}$ and the dimensions of discrete convex polyhedra are maximum. Then we cut away less than 3-dimensional parts of $\mathbf{C}_m$ to obtain a pure discrete 3-complex $\mathbf{O}_m$. Finally, we extract the combinatorial boundary $\partial\mathbf{O}_m$ from $\mathbf{O}_m$ by following Definition 2. The full details and the more effective algorithm for obtaining $\partial\mathbf{O}_m$ from $\mathbf{V}$ are found in [8].

In the sense of general topology, we define the set of border points of $\mathbf{V}$ such that

$$Br_m(\mathbf{V}) = \{\boldsymbol{x} \in \mathbf{V} : \mathbf{N}_m(\boldsymbol{x}) \cap \overline{\mathbf{V}} \neq \emptyset\}. \tag{4}$$

Let $Sk(\mathbf{K})$ be the set of all vertices of discrete convex polyhedra in a discrete complex $\mathbf{K}$. We then have the following important relations which will be used later in this paper. The proof is omitted in this paper because of the page limitation; it will be seen in our prepared paper[1].

Theorem 1. *For any subset* $\mathbf{V} \subset \mathcal{Z}^3$*, we have relations such as*

$$\begin{aligned} Br_6(\mathbf{V}) &= Sk(\partial\mathbf{O}_{26}) \cup (Sk(\mathbf{C}_{26}) \setminus Sk(\mathbf{O}_{26})), \\ Br_{26}(\mathbf{V}) &= Sk(\partial\mathbf{O}_6) \cup (Sk(\mathbf{C}_6) \setminus Sk(\mathbf{O}_6)). \end{aligned}$$

A plane $\mathbf{P}$ of (1) defines two digitized half spaces such as

$$\begin{aligned} \mathbf{I}^- &= \{(x,y,z) \in \mathcal{Z}^3 : ax + by + cz + d \leq 0\}, \\ \mathbf{I}^+ &= \{(x,y,z) \in \mathcal{Z}^3 : ax + by + cz + d \geq 0\}. \end{aligned} \tag{5}$$

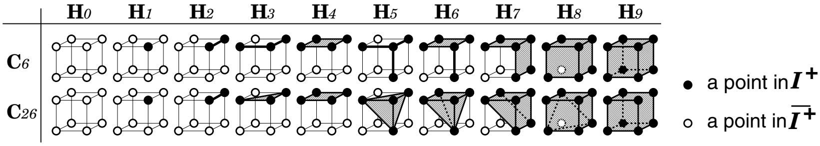
We apply the algorithm of combinatorial boundary tracking shown in the previous subsection to $\mathbf{I}^+$, instead of $\mathbf{V}$, for obtaining a discrete combinatorial plane $\mathbf{DCP}_m$ which is a pure discrete 2-complex $\partial\mathbf{O}_m$. Table 2 illustrates how to obtain a $\mathbf{C}_m$ for each $m = 6, 26$ from $\mathbf{I}^+$; depending on a point configuration $\mathbf{H}_i$, $i = 0, \ldots, 9$, of $\mathbf{I}^+$ at each unit cubic region, we have a discrete polyhedral complex and we set $\mathbf{C}_m$ to be the union of these discrete polyhedral complexes for all unit cubic regions in $\mathcal{Z}^3$. Obviously $\mathbf{I}^+$ is a infinite set. Therefore, from a computational viewpoint, the algorithm will not end if it is applied to $\mathbf{I}^+$. However, from a mathematical viewpoint, we see that $\mathbf{DCP}_m$ is uniquely obtained from $\mathbf{I}^+$. We then have the following inclusion relations. The proof which is omitted here can be similar to that of Lemma 1 in [7] [2].

Property 1. For any plane $\mathbf{P}$, we have the relations such that $Sk(\mathbf{DCP}_6) \supseteq Sk(\mathbf{DCP}_{18}) = Sk(\mathbf{DCP}_{26})$.

[1] Similar relations are shown in [8] with some illustrations; $Br_6(\mathbf{V}) = Sk(\partial\mathbf{O}_{26}) \cup Sk(\mathbf{C}_{26} \setminus \mathbf{O}_{26})$, $Br_{26}(\mathbf{V}) = Sk(\partial\mathbf{O}_6) \cup Sk(\mathbf{C}_6 \setminus \mathbf{O}_6)$.

[2] In [7] a discrete combinatorial plane is a simplicial complex, but it is easy to show that it has the same set of vertices as that of our **DCP** which is a polyhedral complex.

Table 2. A discrete polyhedral complex $\mathbf{C}_m$ for each configuration $\mathbf{H}_i$, $i = 0, \ldots, 9$, of points of $\mathbf{I}^+$ for $m = 6, 26$. We consider cases of $0 \leq a \leq b \leq c$, $c > 0$ in the table.



Similarly to $Br_m(\mathbf{V})$ of (4), we can define a set of border points of $\mathbf{I}^+$, called a discrete morphological plane such that

$$\mathbf{DMP}_m = \{\boldsymbol{x} \in \mathbf{I}^+ : \mathbf{N}_m(\boldsymbol{x}) \cap \overline{\mathbf{I}^+} \neq \emptyset\} \tag{6}$$

for each $m = 6, 18, 26$. We then derive the next relations between **DMP** and **DCP** corresponding to Theorem 1 about the relations between $Br(\mathbf{V})$ and $\partial\mathbf{O}$.

Lemma 1. *For any plane* $\mathbf{P}$*, we have relations such as*

$$\begin{aligned} \mathbf{DMP}_6 &= Sk(\mathbf{DCP}_{18}) = Sk(\mathbf{DCP}_{26}), \\ \mathbf{DMP}_{26} &= Sk(\mathbf{DCP}_6). \end{aligned} \tag{7}$$

Proof. From Theorem 1, replacing $\mathbf{V}$ with $\mathbf{I}^+$, we have $\mathbf{DMP}_6 = Sk(\mathbf{DCP}_{26}) \cup (Sk(\mathbf{C}_{26}) \setminus Sk(\mathbf{O}_{26}))$, $\mathbf{DMP}_{26} = Sk(\mathbf{DCP}_6) \cup (Sk(\mathbf{C}_6) \setminus Sk(\mathbf{O}_6))$. From Property 1, it is easily seen that we only need to show that the second terms are all empty, namely, $\mathbf{C}_m = \mathbf{O}_m$ for $m = 6, 26$. For the proof, we show that $\mathbf{C}_m$ is pure so that each of discrete n-polyhedron of $\mathbf{C}_m$ where $n < 3$ is a face of a discrete 3-polyhedron of $\mathbf{C}_m$.

We first consider the case of $m = 6$. Let us consider a discrete 2-polyhedron σ_2 in Table 2, for example, H4. Setting $\mathbf{H}(i, j, k)$ to be a configuration of points of $\mathbf{I}^+$ at a unit cube whose vertices are eight lattice points such as $(i+\epsilon_1, j+\epsilon_2, k+\epsilon_3)$ for $\epsilon_i = 0$ or 1 for $i = 1, 2, 3$. From the configuration $\mathbf{H}(i, j, k)$ of H4, we see that $\mathbf{H}(i, j, k+1)$ can be only H9 which has a 3-polyhedron σ_3. Thus σ_2 is a face of σ_3; the faces of σ_2 are also faces of σ_3. Similarly, we can show that other discrete 2-polyhedra of H6, H7 and H8 are also faces of some discrete 3-polyhedra of H9 if we consider the possible point configurations of the adjacent cubes. Let us consider discrete 1-polyhedra which are not faces of discrete 2-polyhedra in Table 2, for example, a discrete 1-polyhedron σ_1 of H2. From the configuration $\mathbf{H}(i, j, k)$ of H2, we see that $\mathbf{H}(i, j, k+1)$ can be only H7 and σ_1 is a face of the right-side discrete 2-polyhedron σ_2. Such σ_2 is a face of a discrete 3-polyhedron of H9 as we have already shown in the above. Similarly, we can show that other discrete 1-polyhedra of H3, H5 and H6 are also faces of some discrete 2-polyhedra which are faces of some discrete 3-polyhedra of H9. Finally, let us consider discrete 0-polyhedra which are not faces of any discrete 1-polyhedra in Table 2, such as a discrete 0-polyhedron σ_0 of H1. From the 1-point configuration $\mathbf{H}(i, j, k)$ of H1, we see that $\mathbf{H}(i, j, k+1)$ can be H5 or H6 which has a discrete 1-polyhedron σ_1 such that σ_0 is a face of σ_1 and σ_1 is a face of a discrete 3-polyhedron of H9.

Let us consider the cases of $m = 26$. In this case, we need to check only discrete 0-, 1- and 2-polyhedra of H1, H2, H3 and H4. Similarly to the case of $m = 6$, we find possible configurations $\mathbf{H}(i,j,k+1)$ adjacent to $\mathbf{H}(i,j,k)$ of H1, H2, H3 and H4: $\mathbf{H}(i,j,k+1)$ can be only H5 or H6 for $\mathbf{H}(i,j,k)$ of H1, H7 for H2, H8 for H3, and H6 or H9 for H4. Therefore, all discrete 0-, 1- and 2-polyhedra are faces of some discrete 3-polyhedra. *(Q.E.D.)*

3 Relations between DAPs and DCPs

Given a plane $\mathbf{P}$ in $\mathcal{R}^3$, obtaining $\mathbf{SP}$, $\mathbf{NP}$, $\mathbf{DCP}_m$ for $m = 6, 18, 26$ respectively, we derive the next theorem.

Theorem 2. *For any* $\mathbf{P}$*, we have relations such as*

$$\mathbf{SP} = Sk(\mathbf{DCP}_6), \tag{8}$$

$$\mathbf{NP} = Sk(\mathbf{DCP}_{18}) \quad = \quad Sk(\mathbf{DCP}_{26}). \tag{9}$$

The relations of (9) have been already proved in [7] (Theorem 2 in [7]).[2] In this paper, we give a proof for (8). Our approach in the following is completely different from that given in [7] for (9). For a proof of (8), due to (7) in Lemma 1, we need to show only the following lemma. Note that it is easy to modify the following lemma for (9) such as $\mathbf{NP} = \mathbf{DMP}_6$.

Lemma 2. *For any plane* $\mathbf{P}$*, we have* $\mathbf{SP} = \mathbf{DMP}_{26}$.

In order to prove this lemma, we need the following lemma.

Lemma 3. *For any plane* $\mathbf{P}$ *such that* $0 \le a \le b \le c$*,* $c > 0$*, if a point* $(u-1, v-1, w-1) \in \mathbf{I}^+$*, then* $\mathbf{N}_{26}(u,v,w) \subset \mathbf{I}^+$.

Proof. Because $(u-1, v-1, w-1) \in \mathbf{I}^+$, we obtain that $a(u-1) + b(v-1) + c(w-1) + d \ge 0$ from (5). Setting $(u', v', w') \in \mathbf{N}_{26}(u,v,w)$, we have $u-1 \le u'$, $v-1 \le v'$, $w-1 \le w'$, thus $au' + bv' + cw' + d \ge a(u-1) + b(v-1) + c(w-1) + d \ge 0$ because a, b, c are not negative. *(Q.E.D.)*

Proof of Lemma 2. For simplification. we set $w = a + b + c$ for $\mathbf{SP}$ of (2) such that $0 \le a \le b \le c$, $c > 0$. Similar proofs are easily derived for other $\mathbf{P}$s. Let us consider two Euclidean planes, $\mathbf{P}$ of (1) and $\mathbf{P}' = \{(x,y,z) \in \mathcal{R}^3 : ax + by + cz + d = a + b + c\}$. We see that $\mathbf{SP}$ is a set of lattice points between $\mathbf{P}$ and $\mathbf{P}'$. Obviously, a point $(p,q,r) \in \mathcal{R}^3$ is on $\mathbf{P}'$ if $(p-1, q-1, r-1) \in \mathcal{R}^3$ is on $\mathbf{P}$. Geometrically, this means that there is a unit cube between $\mathbf{P}$ and $\mathbf{P}'$ such that the two vertices (p,q,r) and $(p-1, q-1, r-1)$ of the unit cube are on $\mathbf{P}$ and $\mathbf{P}'$, respectively.

(i) For any point $(u, v, w) \in \mathbf{SP}$, i.e. a point (u, v, w) between $\mathbf{P}$ and $\mathbf{P}'$ (can be on $\mathbf{P}$ but not be on $\mathbf{P}'$), we have $0 \leq au+bv+cw+d < a+b+c$ from (2). Thus, $-(a+b+c) \leq a(u-1)+b(v-1)+c(w-1)+d < 0$, so that $(u-1, v-1, w-1) \in \overline{\mathbf{I}^+}$. Because $(u-1, v-1, w-1) \in \mathbf{N}_{26}(u, v, w)$, we have $\mathbf{N}_{26}(u, v, w) \cap \overline{\mathbf{I}^+} \neq \emptyset$.
(ii) For any point $(u, v, w) \in \mathbf{I}^+ \setminus \mathbf{SP}$, we have $au+bw+cw+d \geq a+b+c$, thus $a(u-1)+b(v-1)+c(w-1)+d \geq 0$. Therefore we say $(u-1, v-1, w-1) \in \mathbf{I}^+$ and obtain $\mathbf{N}_{26}(u, v, w) \cap \overline{\mathbf{I}^+} = \emptyset$ from Lemma 3.
(iii) From **(i)** and **(ii)**, we have $\mathbf{SP} = \mathbf{DMP}_{26}$. *(Q.E.D.)*

From Theorem 2, we see that $\mathbf{DCP}_{18}$ and $\mathbf{DCP}_{26}$ are topological spaces on $\mathbf{NP}$ and $\mathbf{DCP}_6$ is a topological space on $\mathbf{SP}$.

4 Combinatorial Topological Properties of DAPs

4.1 Combinatorial Bicubes and Tricubes

It has been shown in [4,6] that there exist five bicubes and forty tricubes in $\mathbf{NP}$s where $0 \leq a \leq b \leq c$, $c > 0$. Considering each of five bicubes [6] at a region $\mathbf{B}(i, j, k) = \{(p, q, r) \in \mathcal{Z}^3 \ : \ p = i, i+1 \ ; \ q = j, j+1 \ ; \ r = k, k+1\}$ for $(i, j, k) \in \mathcal{Z}^3$, we obtain a combinatorial bicube such that $\mathbf{CB}_m(i, j, k) = \{\sigma \in \mathbf{DCP}_m : Sk(\{\sigma\}) \subseteq \mathbf{B}(i, j, k)\}$ for $m = 6, 18, 26$. Similarly, considering each of forty tricubes [4] to be a union of eight bicubes, we obtain combinatorial tricubes such that $\mathbf{CT}_m(i, j, k) = \cup_{(p,q,r) \in \mathbf{B}(i,j,k)} \mathbf{CB}_m(p, q, r)$ for $m = 6, 18, 26$. We illustrate all $\mathbf{CT}_m(i, j, k)$ for $m = 6, 26$ in Fig. 1 and 2. We can easily obtain $\mathbf{CT}_{18}(i, j, k)$ which is similar to $\mathbf{CT}_{26}(i, j, k)$ by replacing $\mathbf{CB}_{26}(p, q, r)$ with $\mathbf{CB}_{18}(p, q, r)$ for $(p, q, r) \in \mathbf{B}(i, j, k)$; there are not many differences between $\mathbf{CB}_{18}(p, q, r)$ and $\mathbf{CB}_{26}(p, q, r)$.

4.2 Combinatorial Topological Properties

Let $\mathbf{K}$ be a polyhedral complex. For each vertex $\boldsymbol{v} \in Sk(\mathbf{K})$, the subcomplex consisting of all convex polyhedra σ of $\mathbf{K}$ which contain $\boldsymbol{v}$ such that $\boldsymbol{v} \in Sk(\{\sigma\})$ is called the star $St(\boldsymbol{v}, \mathbf{K})$ of $\boldsymbol{v}$ in $\mathbf{K}$ [1,11]. The link of $\boldsymbol{v}$ is then defined such as $Lk(\boldsymbol{v}, \mathbf{K}) = Cl(St(\boldsymbol{v}, \mathbf{K})) \setminus St(\boldsymbol{v}, \mathbf{K})$ in [1,11] [3]. A star $St(\boldsymbol{v}, \mathbf{K})$ is said to be cyclic if $Lk(\boldsymbol{v}, \mathbf{K})$ is a simple closed broken line (i.e., if its elements are disposed in cyclic order, like the elements of a circle split up into sectors) [1]. If a star is cyclic, it is combinatorial equivalent of a disc and called an umbrella [10].

For each combinatorial tricube in Fig. 1 and 2, if we consider a star of each white vertex, we obtain 8 and 34 different configurations of stars for $m = 6, 26$, respectively. Note that we also obtain 34 configurations for $m = 18$. We see stars as polygons with diagonal lines in Fig. 1 and 2 and it is obvious that they are cyclic, i.e. umbrellas. The number of umbrellas is less than 40 because there are umbrellas of the same shape for different tricubes. Therefore, we obtain the following property. Note that similar properties are presented in [5,6]; a different proof is seen in [5] and no proof is presented in [6].

[3] The link is called the outer boundary in [1].

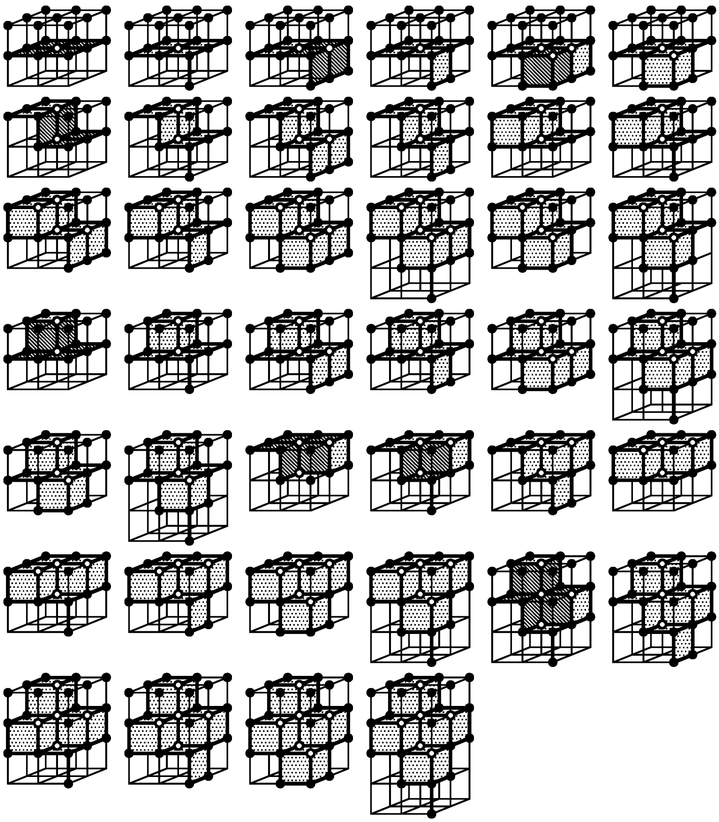


Fig. 1. Combinatorial tricubes $\mathbf{CT}_6(i, j, k)$s for **SP**s with combinatorial structures obtained from $\mathbf{DCP}_6$s. Eight umbrellas of white vertices which are different from others are also shown as polygons with diagonal lines.

Property 2. Let us consider **SP**, **NP** and $\mathbf{DCP}_m$, $m = 6, 18, 26$ for a given **P**. We then see that $\mathbf{DCP}_6$ is a 2-dimensional combinatorial manifold of **SP** and that $\mathbf{DCP}_{18}$ and $\mathbf{DCP}_{26}$ are those of **NP**.

4.3 Connectivity Properties

A subset $\mathbf{A} \subset \mathcal{Z}^3$ is said to be m-connected if any pair of elements $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{A}$ has a path $\boldsymbol{a}_1 = \boldsymbol{a}, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots, \boldsymbol{a}_p = \boldsymbol{b}$ such that $\boldsymbol{a}_{i+1} \in \mathbf{N}_m(\boldsymbol{a}_i)$ and $\boldsymbol{a}_i \in \mathbf{A}$ for every $i = 1, 2, \ldots, p-1$. Andrès derived connectivity properties of **DAP** with the definitions of k-tunnel and k-separating [3] [4]. If the complement $\overline{\mathbf{DAP}}$ of **DAP** in $\mathbf{Z}^3$ is not k-connected, **DAP** is said to be k-separating for $k = 6, 18, 26$. Considering the two regions such as $\mathbf{A} = \{(x, y, z) \in \mathcal{Z}^3 : ax + by + cz + d < 0\}$,

[4] In [3], k is set to be $0, 1, 2$. In this paper we set $k = 26, 18, 6$ instead to avoid the confusion.

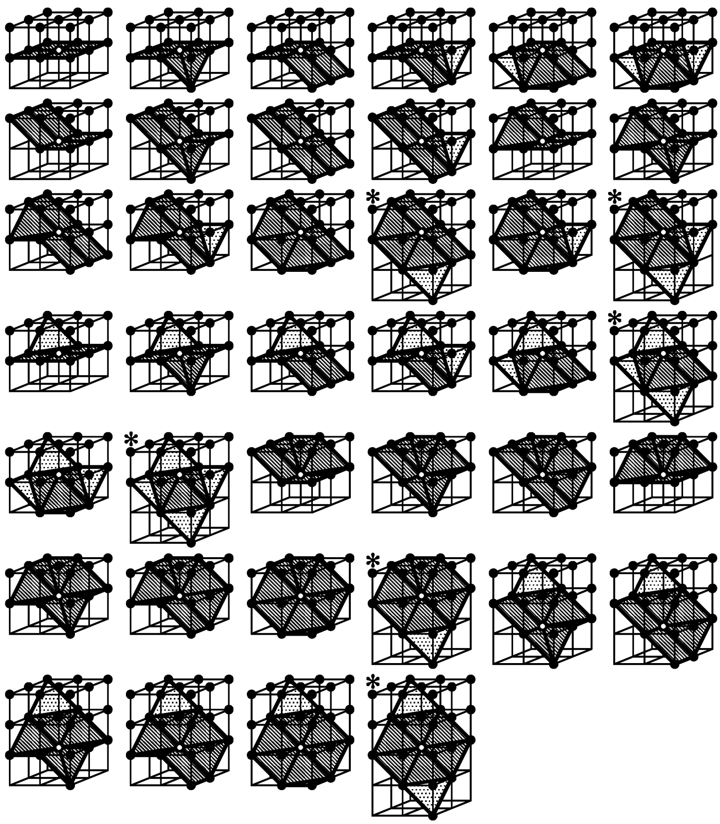


Fig. 2. Combinatorial tricubes $\mathbf{CT}_{26}(i, j, k)$s for **NP**s with combinatorial structures obtained from $\mathbf{DCP}_{26}$s. There are thirty-four different umbrellas of white vertices which are shown as polygons with diagonal lines. Six combinatorial tricubes with asterisks have the umbrellas which are the same as the others.

$\mathbf{B} = \{(x, y, z) \in \mathcal{Z}^3 : ax + by + cz + d \geq w\}$, if there are two k-neighboring points $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a} \in \mathbf{A}$, $\boldsymbol{b} \in \mathbf{B}$, **DAP** is said to have a k-tunnel for $k = 6, 18, 26$. The following properties are already presented in [3,5] and we derive them differently and more simply making use of Theorem 2; from Lemmas 4 and 5 and Theorem 2, we derive Properties 3 and 4

Property 3. A standard plane **SP** is tunnel free, and 6-connected.

Property 4. A naive plane **NP** may have 18-tunnel but no 6-tunnel, and is 6-separating, i.e. **NP** is 18-connected but not 6-connected.

Lemma 4. *Any* $Sk(\mathbf{DCP}_m)$ *is* m*-connected for each* $m = 6, 18, 26$.

Proof. From the definition of discrete polyhedra, a set $Sk(\{\sigma\})$ for a discrete 2-polyhedron σ is m-connected. From Property 2, we see that any vertex $\boldsymbol{v}$ in

$\mathbf{DCP}_m$ has the star $St(\boldsymbol{v}, \mathbf{DCP}_m)$ which is an umbrella, and that $Sk(St(\boldsymbol{v}, \mathbf{DCP}_m))$ is also m-connected. Because $\mathbf{DCP}_m$ is a union of connected $St(\boldsymbol{v}, \mathbf{DCP}_m)$s, $Sk(\mathbf{DCP}_m)$ is m-connected. *(Q.E.D.)*

Lemma 5. *Any* $\mathbf{DMP}_m$ *for* $m = 6, 18, 26$ *is* m*-separating.*

Proof. From (6), any two points $\boldsymbol{a}$, $\boldsymbol{b}$ such that $\boldsymbol{a} \in \mathbf{I}^+ \setminus \mathbf{DMP}_m$ and $\boldsymbol{b} \in \mathbf{I}^-$ are not m-neighboring. Thus, a $\mathbf{DMP}_m$ is m-separating. *(Q.E.D.)*

5 Conclusions

We defined discrete combinatorial planes $\mathbf{DCP}_m$ for $m = 6, 18, 26$ and showed the relations between **DAP**s and $\mathbf{DCP}_m$s as given in Theorem 2, such that a $\mathbf{DCP}_6$ is a combinatorial topology on an **SP** and a $\mathbf{DCP}_{18}$ or $\mathbf{DCP}_{26}$ is a combinatorial topology on an **NP**. From the relations, we obtained combinatorial topological properties of **DAP**s, called combinatorial bicubes and tricubes. By using them, we proved that any **DAP** is considered to be a 2-dimensional combinatorial manifold and also derived their connectivities properties.

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References

1. P. S. Alexandrov, *Combinatorial Topology*, Vol. 1, Graylock Press, Rochester, New York, 1956.
2. E. Andrès, "Le plan discret," in Actes du 3e Colloque Géométrie discrète en imagerie: fondements et applications, Strasbourg, september 1993.
3. E. Andrès, R. Acharya, C. Sibata, "Discrete analytical hyperplanes," Graphical Models and Image Processing, Vol. 59, No. 5, pp. 302–309, 1997.
4. I. Debled-Renesson, "Etude et reconnaissance des droites et plans discrets," Thèse de doctorat de l'université Louis Pasteur, 1995.
5. J. Françon, "Sur la topologie d'un plan arithmétique," Theoretical Computer Science Vol. 156, pp. 159-176, 1996.
6. J. Françon, J. M. Schramm, M. Tajine, "Recognizing arithmetic straight lines and planes," In LNCS 1176; Discrete Geometry for Computer Imagery, Proceedings of 9th International Workshop, DGCI'96, pp. 141–150, Springer-Verlag, Berlin, Heidelberg, 1996.
7. Y. Kenmochi, A. Imiya, "Naive planes as discrete combinatorial surfaces," in LNCS 1953; Discrete Geometry for Computer Imagery, Proceedings of 9th International Conference, DGCI2000, pp. 249–261, Springer, 2000.
8. Y. Kenmochi, A. Imiya, "Discrete polyhedrization of a lattice point set," in LNCS 2243; Digital and image geometry, pp. 148–160, Springer, 2001.
9. J-P. Reveillès, "Géométrie discrète, calcul en nombres entiers et algorithmique," Thèse d'état soutenue à l'université Louis Pasteur, 1991.
10. J. Stillwell, *Classical topology and combinatorial group theory*, Springer-Verlag: New York, 1993.
11. G. M. Ziegler, *Lectures on polytopes*, Springer-Verlag: New York, 1994.