Two Remarks on Reconstructing Binary Vectors from Their Absorbed Projections

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Abstract. We prove two small results on the reconstruction of binary matrices from their absorbed projections: (1) If the absorption constant is the positive root of $x^2 + x - 1 = 0$, then every row is uniquely determined by its left and right projections. (2) If the absorption constant is the root of $x^4 - x^3 - x^2 - x + 1 = 0$ with $0 < x < 1$, then in general a row is not uniquely determined by its left and right projections.

1 Introduction

The reconstruction of binary matrices from their row and column sums is a basic problem in discrete tomography, and work on this problem goes back to a seminal paper of Ryser [5] from 1957. Recently, Kuba and Nivat [4] introduced a new discrete tomography model that they call the *emission discrete tomography* model, EDT for short. In this EDT model, the whole space is filled with some homogeneous partially absorbing material, and the function to be reconstructed represents an object emitting radioactive rays into the surrounding space. Hence, the measurements in EDT are *absorbed* projections that depend on both, the emitting object and the absorption.

Formally, a *picture* is a binary $m \times n$ matrix $A = (a_{i,j})_{m \times n}$. The entries $a_{i,j}$ are from $\{0, 1\}$, and they are sometimes called *pixels*. Measurements of the picture are taken by sending rays through the material; since the material is partially absorbing, these rays become gradually weaker as they proceed through the material. In [2, 3, 4], the absorption behavior is characterized by the *absorption constant*

$$\beta = \frac{1}{2}(-1 + \sqrt{5}), \qquad (1)$$

where $\beta \approx 0.618$ is the positive root of the equation

$$\beta^2 + \beta = 1. \qquad (2)$$

Then the *absorbed left projection* ALP_i of row i is defined as

$$\text{ALP}_i = \sum_{j=1}^{n} a_{i,j} \cdot \beta^j \tag{3}$$

and the *absorbed right projection* ARP_i of row i is defined as

$$\text{ARP}_i = \sum_{j=1}^{n} a_{i,j} \cdot \beta^{n+1-j}. \tag{4}$$

Intuitively speaking, in the left projection the ray enters the row from the left hand side. As the ray passes through the pixels at the beginning of the row, it is still strong, and provides precise information. Then the ray becomes gradually absorbed, and the factors β^j in (3) encode this absorption. In the right projection, the ray enters the row symmetrically from the right hand side, and this leads to the symmetric formula given in (4). Absorbed upward and downward projections of columns are defined analogously.

The papers by Kuba and Nivat [4], Balogh et al. [2], and Kuba et al. [3] discuss a variety of algorithmical and combinatorial questions around reconstruction problems with absorbed projections. Noteworthy, all these papers only study the cases with absorbed *left* projections of rows (but without right projections) and with absorbed *upward* projections of columns (but without downward projections). Barcucci et al. [1] proved that the left and right absorbed projections determine uniquely the binary matrix if the absorbtion coefficient is $\mu = \log((1 + \sqrt{5})/2)$ and they gave an algorithm for the reconstruction.

In this note, we will show that the corresponding problems where the left and right absorbed row projections are known simultaneously are rigid and highly constrained. Our combinatorial proof is different from the proof given by Barcucci et al. [1] in the sense that it does not use the concenpt of switching component.

Theorem 1. *In the EDT model with absorption constant $\beta = \frac{1}{2}(-1 + \sqrt{5})$, the left and the right absorbed projections ALP_i and ARP_i determine the row r_i uniquely.*

Up to now, we have only considered the basic EDT model with absorption constant $\beta = \frac{1}{2}(-1 + \sqrt{5})$ (which arguably is the simplest non-trivial EDT model). Interestingly, the statement in Theorem 1 does *not* generalize to arbitrary values of the absorption constant:

Theorem 2. *Consider the EDT model where the absorption constant*

$$\gamma = \frac{1}{4}\left(\sqrt{13} + 1 - \sqrt{2\sqrt{13} - 2}\right) \approx 0.581 \tag{5}$$

is the unique real root of $\gamma^4 - \gamma^3 - \gamma^2 - \gamma + 1 = 0$ that satisfies $0 < \gamma < 1$. Then the left and the right absorbed projections ALP_i and ARP_i in general do not determine a row r_i uniquely.

2   Proof of the Unique Reconstruction Result

In this section we prove Theorem 1. Throughout, let $\beta = \frac{1}{2}(-1 + \sqrt{5})$ be defined as in (2), and let $\alpha = \frac{1}{2}(1 + \sqrt{5})$ denote the positive root of $\alpha^2 = \alpha + 1$. Note that $\alpha = 1/\beta$. Furthermore, we define the Fibonacci numbers as usual by

$$F_{-3} = -1 \qquad F_{-2} = 1 \qquad F_{-1} = 0 \qquad F_0 = 1 \qquad F_1 = 1 \qquad (6)$$

and by $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.

Proposition 1. *The following statements on the Fibonacci numbers are well-known:*

(i) $\alpha^i = F_{i-1}\alpha + F_{i-2}$ *for all* $i \geq 0$.
(ii) $\beta^i = (-1)^{i-1}F_{i-1}\beta + (-1)^{i-2}F_{i-2}$ *for all* $i \geq 0$.
(iii) $F_{i+2} - 1 = F_i + F_{i-1} + \cdots + F_1 + F_0$ *for all* $i \geq 0$.

Proof. By induction on i. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proposition 2. *Let* $x_i \in \{-1, 0, 1\}$ *for* $i \geq 0$. *Then the following statements hold true:*

(i) *If* $\sum_{i=0}^{\ell} F_{2i} \cdot x_{2i} = 0$, *then* $x_{2i} = 0$ *for* $i = 0, \ldots, \ell$.
(ii) *If* $\sum_{i=0}^{\ell} F_{2i+1} \cdot x_{2i+1} = 0$, *then* $x_{2i+1} = 0$ *for* $i = 0, \ldots, \ell$.

Proof. Proof of statement (i). Suppose for the sake of contradiction that $x_{2i} \neq 0$ for some i, and consider the largest index k with $x_{2k} \neq 0$. Without loss of generality, we may assume that $x_{2k} = -1$ (and otherwise, we multiply all x_{2k} by -1). Then $\sum_{i=1}^{k-1} F_{2i} \cdot x_{2i} - F_{2k} = 0$. This leads to

$$F_{2k} = \sum_{i=0}^{k-1} F_{2i} \cdot x_{2i} \leq \sum_{i=0}^{k-1} F_{2i}$$

$$= \sum_{i=0}^{k-1} F_{2i-2} + F_{2i-1} = \sum_{i=-2}^{2k-3} F_i < 1 + \sum_{i=0}^{2k-2} F_i.$$

This blatantly contradicts statement (iii) in Proposition 1, and proves statement (i). Statement (ii) can be handled similarly: Let k be the largest index k with $x_{2k+1} \neq 0$, and assume that $x_{2k+1} = -1$. Then

$$F_{2k+1} = \sum_{i=0}^{k-1} F_{2i+1} \cdot x_{2i+1} \leq \sum_{i=0}^{k-1} F_{2i+1}$$

$$= \sum_{i=0}^{k-1} F_{2i-1} + F_{2i} = \sum_{i=-1}^{2k-2} F_i = \sum_{i=0}^{2k-2} F_i.$$

This contradicts Proposition 1.(iii) and completes the proof of statement (ii).   □

Now consider two rows $r_1 = \langle a_1, \ldots, a_n \rangle$ and $r_2 = \langle b_1, \ldots, b_n \rangle$ with pixels $a_i, b_i \in \{0, 1\}$ in some binary picture. We assume that the left projections of these two rows are identical, i.e.,

$$\sum_{i=1}^{n} a_i \, \beta^i = \sum_{i=1}^{n} b_i \, \beta^i, \tag{7}$$

and that also their right projections are identical, i.e.,

$$\sum_{i=1}^{n} a_i \, \beta^{n+1-i} = \sum_{i=1}^{n} b_i \, \beta^{n+1-i}. \tag{8}$$

For $i = 1, \ldots, n$ we define $c_i = a_i - b_i$ with $c_i \in \{-1, 0, +1\}$. Furthermore, we introduce the four auxiliary quantities $A = \sum_{i=1}^{n} F_{i-2} \, c_i$, $B = \sum_{i=1}^{n} F_{i-3} \, c_i$, $C = \sum_{i=1}^{n} (-1)^{i-2} F_{i-2} \, c_i$, and $D = \sum_{i=1}^{n} (-1)^{i-3} F_{i-3} \, c_i$.

Now Equation (7) can be written as $\sum_{i=1}^{n} \beta^i c_i = 0$. By using statements (i) and (ii) in Proposition 1, this leads to

$$0 = \sum_{i=1}^{n} \beta^i c_i = \sum_{i=1}^{n} ((-1)^{i-1} F_{i-1} \beta + (-1)^{i-2} F_{i-2}) c_i = \beta(D - C) + C. \tag{9}$$

Similarly, Equation (8) can be written as $\sum_{i=1}^{n} \alpha^i c_i = 0$ and yields

$$0 = \sum_{i=1}^{n} \alpha^i c_i = \sum_{i=1}^{n} (F_{i-1} \alpha + F_{i-2}) c_i = \alpha(A + B) + A. \tag{10}$$

Since α and β are irrational numbers, whereas $D - C$, C, $A + B$, A are integers, the Equations (9) and (10) imply $D - C = 0$, $C = 0$, $A + B = 0$, and $A = 0$. Hence, $A = B = C = D = 0$ holds. From this we derive

$$0 = A + C = \sum_{i=1}^{n} (1 + (-1)^{i-2}) \, F_{i-2} \, c_i = 2 \sum_{i=1}^{\lfloor n/2 \rfloor} F_{2i-2} c_{2i}. \tag{11}$$

Equation (11) together with Proposition 2.(i) now yields that all c_i with an even index are 0. Furthermore, we derive for the odd indices that $A - C = 0$ and therefore

$$0 = \sum_{i=1}^{n} (1 - (-1)^{i-2}) \, F_{i-2} \, c_i = 2 \sum_{i=1}^{\lceil n/2 \rceil} F_{2i-3} c_{2i-1} = 2 \sum_{i=2}^{\lceil n/2 \rceil} F_{2i-3} c_{2i-1}. \tag{12}$$

Here we used in the final step that $F_{-1} = 0$. Equation (12) together with Proposition 2.(ii) now yields that all c_i with an odd index $i \geq 3$ are 0. What about c_1? The value of $B = \sum_{i=1}^{n} F_{i-3} \, c_i$ boils down to $B = F_{-2} c_1 = c_1$, and together with $B = 0$ this implies that also $c_1 = 0$. Summarizing, $c_i = 0$ must hold for all $i = 1, \ldots, n$. This implies $a_i = b_i$ for all $i = 1, \ldots, n$. Hence, the two rows r_1 and r_2 must be identical, and any row is uniquely determined by its left and right absorbed projections. This completes the proof of Theorem 1.

Remark 1. About absorption values for which any binary matrix is uniquely determined by its absorbed row sums see Section 6 in [3].

3   Proof of the Non-unique Reconstruction Result

In this section we prove Theorem 2. Hence, consider an absorption constant γ with $0 < \gamma < 1$ that is a root of $1 + \gamma^4 = \gamma + \gamma^2 + \gamma^3$ as defined in (5). Consider a 2×10 matrix with two rows $r_1 = \langle 1, 0, 0, 0, 1, 1, 0, 0, 0, 1 \rangle$ and $r_2 = \langle 0, 1, 1, 1, 0, 0, 1, 1, 1, 0 \rangle$. Then

$$\begin{aligned} \mathrm{ALP}_1 = \gamma + \gamma^5 + \gamma^6 + \gamma^{10} &= (\gamma + \gamma^6)(1 + \gamma^4) \\ &= (\gamma + \gamma^6)(\gamma + \gamma^2 + \gamma^3) \\ &= \gamma^2 + \gamma^3 + \gamma^4 + \gamma^7 + \gamma^8 + \gamma^9 = \mathrm{ALP}_2. \end{aligned}$$

Since rows r_1 and r_2 both are left-right symmetric, this chain of equations also yields $\mathrm{ARP}_1 = \mathrm{ARP}_2$. Summarizing, these two rows both have the same absorbed left projection and both have the same absorbed right projection. Generally, it is not possible to uniquely reconstruct a row from its two projections. This proves Theorem 2.

Remark 2. Of course, the left and right absorbed projections determine the binary vector uniquely if its size is small enough, that is, if $n \leq 4$. The explanation is that a binary vector is non-uniquely determined by its left and right absorbed projections if and only if the equation $1 + \gamma^4 = \gamma + \gamma^2 + \gamma^3$ can be applied in the description of its value. This equation requires at least 5 binary digits.

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