No-Three-in-Line-in-3D*

Attila Pór[1] and David R. Wood[1,2]

[1] Department of Applied Mathematics, Charles University, Prague, Czech Republic
por@kam.mff.cuni.cz
[2] School of Computer Science, Carleton University, Ottawa, Canada
davidw@scs.carleton.ca

Abstract. The *no-three-in-line* problem, introduced by Dudeney in 1917, asks for the maximum number of points in the $n \times n$ grid with no three points collinear. In 1951, Erdös proved that the answer is $\Theta(n)$. We consider the analogous three-dimensional problem, and prove that the maximum number of points in the $n \times n \times n$ grid with no three collinear is $\Theta(n^2)$. This result is generalised by the notion of a *3D drawing* of a graph. Here each vertex is represented by a distinct gridpoint in $\mathbb{Z}^3$, such that the line-segment representing each edge does not intersect any vertex, except for its own endpoints. Note that edges may cross. A 3D drawing of a complete graph K_n is nothing more than a set of n gridpoints with no three collinear. A slight generalisation of our first result is that the minimum volume for a 3D drawing of K_n is $\Theta(n^{3/2})$. This compares favourably to $\Theta(n^3)$ when edges are not allowed to cross. Generalising the construction for K_n, we prove that every k-colourable graph on n vertices has a 3D drawing with $\mathcal{O}(n\sqrt{k})$ volume. For the k-partite Turán graph, we prove a lower bound of $\Omega((kn)^{3/4})$.

1 Introduction

In 1917, Dudeney [10] asked what is the maximum number of points in the $n \times n$ grid with no three points collinear? This question, dubbed the *no-three-in-line* problem, has since been widely studied [1, 2, 7, 14, 16–19, 21]. A breakthrough came in 1951, when Erdős [14] proved that for any prime p, the set $\{(x, x^2 \bmod p) : 0 \le x \le p-1\}$ contains no three collinear points. If follows that the $n \times n$ grid contains $n/2$ points with no three collinear, and for all $\epsilon > 0$ and $n > n(\epsilon)$, there are $(1-\epsilon)n$ points with no three collinear. The result has been improved to $(3/2-\epsilon)n$ by Hall *et al.* [18] using a different construction. These bounds are optimal if we ignore constant factors, since each gridline contains at most two points, and thus the number of points is at most $2n$. Guy and Kelly [17] conjectured that the maximum number of points in the $n \times n$ grid with no three collinear tends to $(2\pi^2/3)^{\frac{1}{3}}n$ as $n \to \infty$.

In this paper we study the *no-three-in-line-in-3D* problem: what is the maximum number of points in the $n \times n \times n$ grid with no three points collinear? The following is our primary result.

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Theorem 1. *The maximum number of points in the $n \times n \times n$ grid with no three collinear is $\Theta(n^2)$.*

Cohen *et al.* [6] generalised the no-three-in-line problem in a similar direction. They proved that for any prime p, the set $\{(x, x^2 \bmod p, x^3 \bmod p) : 0 \leq x \leq p-1\}$ contains no four coplanar points. It follows that the $n \times n \times n$ grid contains at least $n/2$ and $(1-\epsilon)n$ points with no four coplanar. Each gridplane contains at most three points; thus we have an upper bound of $3n$.

Cohen *et al.* [6] were motivated by three-dimensional graph visualisation. Let G be an (undirected, finite, simple) graph with vertex set $V(G)$ and edge set $E(G)$. A *3D drawing* of G represents each vertex by a distinct point in $\mathbb{Z}^3$ (a *gridpoint*), such that with each edge represented by the line-segment between its endpoints, the only vertices that an edge intersects are its own endpoints. That is, an edge does not 'pass through' a vertex. The *bounding box* of a 3D drawing is the minimum axis-aligned box containing the drawing. If the bounding box has side lengths $X-1$, $Y-1$ and $Z-1$, then we speak of an $X \times Y \times Z$ drawing with *volume* $X \cdot Y \cdot Z$. That is, the volume of a 3D drawing is the number of gridpoints in the bounding box. This definition is formulated so that 2D drawings have positive volume.

Distinct edges in a 3D drawing *cross* if they intersect at a point other than a common endpoint. Based on the observation that the endpoints of a pair of crossing edges are coplanar, Cohen *et al.* [6] proved that the minimum volume for a crossing-free 3D drawing of K_n is $\Theta(n^3)$. The lower bound here is based on the observation that no axis-perpendicular gridplane can contain five vertices, as otherwise there is a planar K_5. Note that it is possible for four vertices to be in a single gridplane, provided that they are not in convex position. Subsequent to the work of Cohen *et al.* [6], crossing-free 3D drawings have been widely studied [4–6, 8, 9, 11, 12, 15, 20, 23]. This paper initiates the study of volume bounds for 3D drawings of graphs, in which crossings are allowed. The following simple observation is immediate.

Observation 1. *A set V of n gridpoints in $\mathbb{Z}^3$ determines a 3D drawing of K_n if and only if no three points in V are collinear.* □

Thus, the following result is a slight strengthening of Theorem 1.

Theorem 2. *The minimum volume for a 3D drawing of K_n is $\Theta(n^{3/2})$.*

A *k-colouring* of a graph G is an assignment of one of k colours to each vertex, so that adjacent vertices receive distinct colours. We say G is *k-colourable*. The *chromatic number* $\chi(G)$ is the minimum k such that G is k-colourable. The Turán graph $T(n, k)$ is the n-vertex complete k-partite graph with $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$ vertices in each colour class. Theorem 2 generalises as follows.

Theorem 3. *Every k-colourable graph on n vertices has a 3D drawing with $\mathcal{O}(n\sqrt{k})$ volume. Moreover, every 3D drawing of the Turán graph $T(n, k)$ has $\Omega((kn)^{3/4})$ volume.*

Note that 2D drawings of k-colourable graphs were studied by Wood [25], who proved an $\mathcal{O}(kn)$ area bound, which is best possible for the Turán graph.

The remainder of this paper is organised as follows. In Section 2 we prove the lower bounds in Theorems 1 and 2, which imply the upper bound in Theorem 1. In Section 3 we prove the upper bounds in Theorems 1 and 2, which imply the lower bound in Theorem 1.

2 Lower Bounds

An axis-parallel line through a gridpoint is called a *gridline*. A gridline that is parallel to the X-axis (respectively, Y-axis and Z-axis) is called an *X-line* (*Y-line* and *Z-line*). An axis-perpendicular plane through a gridpoint is called a *gridplane*.

Lemma 1. *There are at most $2n^2$ points in the $n \times n \times n$ grid with no three collinear.*

Proof. Every X-line contains at most two points, and there are n^2 X-lines. □

The idea in Lemma 1 can be generalised to give a universal lower bound on the volume of a 3D drawing of a graph.

Lemma 2. *Every 3D drawing of a graph G has at least $\chi(G)^{3/2}/\sqrt{8}$ volume.*

Proof. Say G has an $A \times B \times C$ drawing. The vertices on a single Z-line induce a set of paths, as otherwise an edge passes through a vertex. The set of paths is 2-colourable. Using a distinct pair of colours for each Z-line, we obtain a $2AB$-colouring of G. Thus $\chi(G) \leq 2AB$. Similarly, $\chi(G) \leq 2AC$ and $\chi(G) \leq 2BC$. Thus $8(ABC)^2 \geq \chi(G)^3$, and the volume $ABC \geq \sqrt{\chi(G)^3/8}$. □

The bound in Lemma 2 is only of interest if $\chi(G) \geq 2n^{2/3}$, since n is a trivial lower bound on the volume of a 3D drawing.

The following lemma proves the lower bound in Theorem 3.

Lemma 3. *For all $n \equiv 0 \pmod{k}$, every 3D drawing of $T(n,k)$ has at least $(kn)^{3/4}/\sqrt{8}$ volume.*

Proof. Consider an $A \times B \times C$ drawing of $T(n,k)$. Let a_i (respectively, b_i and c_i) be the number of X-lines (Y-lines and Z-lines) that contain a vertex in the i-th colour class. Considering the arithmetic and harmonic means of $\{a_i : 1 \leq i \leq k\}$ we have,

$$k^2 \;\leq\; \left(\sum_i a_i\right)\left(\sum_i \frac{1}{a_i}\right) \;.$$

The X- and Y-lines that contain a vertex coloured i intersect in at most $a_i b_i$ gridpoints. There are n/k vertices coloured i. Thus $a_i b_i \geq n/k$, implying $1/a_i \leq kb_i/n$.

Hence,

$$k^2 \;\leq\; \left(\sum_i a_i\right)\left(\sum_i \frac{kb_i}{n}\right) \;.$$

That is,

$$kn \ \leq \ \left(\sum_i a_i\right)\left(\sum_i b_i\right) \ .$$

There are at most two distinct colours represented in each gridline, as otherwise an edge passes through a vertex. There are BC distinct X-lines. Thus $\sum_i a_i \leq 2BC$. Similarly, $\sum_i b_i \leq 2AC$. Thus $kn \leq (2BC)\,(2AC)$. That is, $ABC^2 \geq kn/4$. By symmetry, $ACB^2 \geq kn/4$ and $BCA^2 \geq kn/4$. Thus $(ABC)^4 \geq (kn/4)^3$, implying that the volume $ABC \geq (kn/4)^{3/4}$. □

Since $\chi(K_n) = n$ and $K_n = T(n, n)$, Lemmata 2 and 3 both prove the lower bound in Theorem 2.

Corollary 1. *Every 3D drawing of K_n has volume at least $n^{3/2}/\sqrt{8}$.* □

3 Upper Bounds

The next lemma is the main component in the proof of our upper bounds. For all primes p, define

$$V_p \ = \ \left\{\left(x,\, y,\, (x^2+y^2) \bmod p\right) \ : \ 0 \leq x, y \leq p-1\right\} \ .$$

Lemma 4. *For all primes p, the set V_p contains three collinear points if and only if $p \equiv 1 \pmod 4$.*

Proof. The result is trivial for $p = 2$. Now assume that p is odd. Suppose V_p contains three collinear points a, b, and c. Then there exists a vector $\boldsymbol{v} = (v_x, v_y, v_z)$ such that $b = k\boldsymbol{v} + a$ and $c = \ell\boldsymbol{v} + a$, for distinct nonzero integers k and ℓ. (Precisely, $v_x = \gcd(b_x - a_x, c_x - a_x)$, $v_y = \gcd(b_y - a_y, c_y - a_y)$, and $v_z = \gcd(b_z - a_z, c_z - a_z)$.) Since $b \in V_p$,

$$(kv_x + a_x)^2 + (kv_y + a_y)^2 \ \equiv \ kv_z + a_z \pmod p \ .$$

That is,

$$k^2(v_x^2 + v_y^2) + a_x^2 + a_y^2 \ \equiv \ kv_z + a_z - 2k(v_x a_x + v_y a_y) \pmod p \ .$$

Since $a \in V_p$, we have $a_x^2 + a_y^2 \equiv a_z \pmod p$. Since p is a prime and $k \neq 0$,

$$k(v_x^2 + v_y^2) \ \equiv \ v_z - 2(v_x a_x + v_y a_y) \pmod p \ .$$

By the same argument applied to c,

$$\ell(v_x^2 + v_y^2) \ \equiv \ v_z - 2(v_x a_x + v_y a_y) \pmod p \ .$$

Thus,

$$k(v_x^2 + v_y^2) \ \equiv \ \ell(v_x^2 + v_y^2) \pmod p \ .$$

That is,

$$(k-\ell)(v_x^2+v_y^2) \equiv 0 \pmod{p} .$$

Since $k \neq \ell$ and p is a prime,

$$v_x^2+v_y^2 \equiv 0 \pmod{p} .$$

Now v_x and v_y are both not zero, as otherwise a, b and c would be in a single Z-line. Without loss of generality, $v_x \neq 0$. Thus v_x has a multiplicative inverse modulo p, and

$$(v_y v_x^{-1})^2 \equiv -1 \pmod{p} .$$

That is, -1 is a quadratic residue. A classical result found in any number theory textbook states that -1 is a quadratic residue modulo an odd prime p if and only if $p \equiv 1 \pmod{4}$.

Now we prove the converse. Suppose that $p \equiv 1 \pmod{4}$. By the above-mentioned result there is an integer t such that $1+t^2 \equiv 0 \pmod{p}$. We can assume that $0 \leq t \leq (p-1)/2$ as otherwise $p-t$ would do. Thus $(1,t,0) \in V_p$ and $(2,2t,0) \in V_p$, and the three points $\{(0,0,0),(1,t,0),(2,2t,0)\}$ are collinear. □

To apply Lemma 4 we need primes $p \not\equiv 1 \pmod{4}$.

Lemma 5 ([3, 13]).

(a) *For all* $t \in \mathbb{N}$, *there is a prime* $p \not\equiv 1 \pmod{4}$ *with* $t \leq p \leq 2t$.

(b) *For all* $\epsilon > 0$ *and* $t > t(\epsilon)$, *there is a prime* $p \equiv 3 \pmod{4}$ *with* $t \leq p \leq (1+\epsilon)t$.

Proof. Part (a) is a strengthening of Bertrand's Postulate due to Erdős [13]. Baker *et al.* [3] proved that for all sufficiently large t, the interval $[t, t+t^{0.525}]$ contains a prime. The proof can be modified to give primes $\equiv 3 \pmod{4}$ in the same interval [Glyn Harman, personal communication, 2004]. Clearly this implies (b). □

We can now prove the upper bound in Theorem 2.

Lemma 6. *Every complete graph* K_n *has a 3D drawing with* $(2+o(1))n^{3/2}$ *volume, and for all* $\epsilon > 0$ *and* $n > n(\epsilon)$, K_n *has a 3D drawing with* $(1+\epsilon)n^{3/2}$ *volume.*

Proof. By Lemma 5 with $t = \lceil\sqrt{n}\rceil$, there is a prime $p \not\equiv 1 \pmod{4}$ with $\lceil\sqrt{n}\rceil \leq p \leq 2\lceil\sqrt{n}\rceil$ and $p \leq (1+\epsilon)\lceil\sqrt{n}\rceil$. By Observation 1 and Lemma 4, the set V_p defines a $p \times p \times p$ drawing of K_{p^2}. By choosing the appropriate vertices, we obtain a $\lceil n/p\rceil \times p \times p$ drawing of K_n. The volume is $(2+o(1))n^{3/2}$ and $(1+\epsilon)n^{3/2}$. □

The same proof gives the lower bound in Theorem 1.

Lemma 7. *There are at least* $n^2/4$ *points in the* $n \times n \times n$ *grid with no three collinear. For all* $\epsilon > 0$ *and* $n > n(\epsilon)$, *there are at least* $(1-\epsilon)n^2$ *points in the* $n \times n \times n$ *grid with no three collinear.* □

Lemma 6 generalises to give the following construction of a 3D drawing of $T(n,k)$.

Lemma 8. *Every Turán graph $T(n,k)$ has a 3D drawing with $(2+o(1))n\sqrt{k}$ volume. For all $\epsilon > 0$ and $k > k(\epsilon)$, $T(n,k)$ has a 3D drawing with $(1+\epsilon)n\sqrt{k}$ volume.*

Proof. Index the colour classes $\{(x,y) : 0 \leq x, y \leq \lceil\sqrt{k}\rceil - 1\}$. By Lemma 5, there is a prime $p \not\equiv 1 \pmod 4$ with $\lceil\sqrt{k}\rceil \leq p \leq 2\lceil\sqrt{k}\rceil$ and $p \leq (1+\epsilon)\lceil\sqrt{k}\rceil$. For each $1 \leq i \leq \lceil n/k\rceil$, put the i-th vertex in colour class (x,y) at $\big(x,\, y,\, ip + (x^2 + y^2) \bmod p\big)$. Each colour class occupies its own Z-line. Thus, if an edge passes through a vertex, then three vertices from distinct colour classes are collinear. Observe that for every vertex at (a_x, a_y, a_z), we have $a_x^2 + a_y^2 \equiv a_z \pmod p$. Thus the same argument from Lemma 4 applies here, and no three vertices from distinct colour classes are collinear. Thus no edge passes through a vertex, and we obtain a 3D drawing of $T(n,k)$. The bounding box is $\lceil\sqrt{k}\rceil \times \lceil\sqrt{k}\rceil \times p\lceil n/k\rceil$. The volume is $(1+o(1))np$, which is $(2+o(1))n\sqrt{k}$ and $(1+\epsilon)n\sqrt{k}$. □

Pach *et al.* [23] proved that every k-colourable graph on n vertices is a subgraph of $T(2n+2k, 2k-1)$. Thus Lemma 8 implies the upper bound in Theorem 3.

Lemma 9. *Every k-colourable graph on n vertices has a 3D drawing with $(4\sqrt{2}+o(1))n\sqrt{k}$ volume. For all $\epsilon > 0$ and $k > k(\epsilon)$, every k-colourable graph on n vertices has a 3D drawing with $(2\sqrt{2}+\epsilon)n\sqrt{k}$ volume.* □

4 Open Problems

Open Problem 1. Does every k-colourable graph have a crossing-free 3D drawing with $\mathcal{O}(kn^2)$ volume? The best known upper bound is $\mathcal{O}(k^2n^2)$ due to Pach *et al.* [23]. A $\mathcal{O}(kn^2)$ bound would match the $\Theta(n^3)$ bound for the minimum volume of a crossing-free 3D drawing of K_n.

For $1 \leq \ell \leq d-1$, let $\mathrm{vol}(n,d,\ell)$ be the minimum bounding box volume for n vertices in $\mathbb{Z}^d$, such that no $\ell+2$ vertices are in any ℓ-dimensional subspace. We have the following lower bound.

Lemma 10. *For $1 \leq \ell \leq d-1$, $\mathrm{vol}(n,d,\ell) \geq \left(\dfrac{n}{\ell+1}\right)^{d/(d-\ell)}$.*

Proof. Consider n vertices in a d-dimensional box of volume $\mathrm{vol}(n,d,\ell)$, such that no $\ell+2$ vertices are in any ℓ-dimensional subspace. The box can be partitioned into $\mathrm{vol}(n,d,\ell)^{(d-\ell)/d}$ subspaces of dimension ℓ, each of which have at most $\ell+1$ vertices by assumption. Thus $n \leq (\ell+1)\,\mathrm{vol}(n,d,\ell)^{(d-\ell)/d}$, and $\mathrm{vol}(n,d,\ell)$ is as claimed. □

Open Problem 2. What is $\mathrm{vol}(n,d,\ell)$?

Consider the case of $\mathrm{vol}(n,d,d-1)$. Erdős [14] and Cohen *et al.* [6] proved that $\mathrm{vol}(n,2,1) \in \Theta(n^2)$ and $\mathrm{vol}(n,3,2) \in \Theta(n^3)$, respectively. Let $V = \{(x, x^2 \bmod p, \ldots, x^d \bmod p) : 0 \leq x \leq n-1\}$, where p is a prime with $n-1 \leq p \leq 2n$. The proofs of Erdős [14] and Cohen *et al.* [6] generalise to show that V contains no $d+1$ points in any $(d-1)$-dimensional subspace. Thus $\mathrm{vol}(n,d,d-1) \leq 2^{d-1}n^d$. By Lemma 10, $\mathrm{vol}(n,d,d-1) \in \Theta(n^d)$ for any constant d.

Open Problem 3. What is vol$(n, d, 1)$? Erdős [14] proved that vol$(n, 2, 1) \in \Theta(n^2)$. Theorem 2 proves that vol$(n, 3, 1) \in \Theta(n^{3/2})$. This problem is unsolved for all constant $d \geq 4$. Note that for $d \geq \log_2 n$ the problem becomes trivial. Just place the vertices at $\{(x_1, \dots, x_d) : x_i \in \{0, 1\}\}$, and vol$(n, d, 1) \in \Theta(n)$.

Open Problem 4. What is vol$(n, d, 2)$? This case is interesting as it relates to crossing-free drawings. Cohen *et al.* [6] proved vol$(n, 3, 2) \in \Theta(n^3)$. Wood [24] proved that for $d = 2\log n + \mathcal{O}(1)$, we have vol$(n, d, 2) \in \mathcal{O}(n^2)$. In particular, K_n has a $2 \times 2 \times \cdots \times 2$ crossing-free d-dimensional drawing with $\mathcal{O}(n^2)$ volume. What is the minimum volume for a crossing-free drawing of K_n, irrespective of dimension, is of some interest.

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