Hamiltonian-with-Handles Graphs and the k-Spine Drawability Problem*

Emilio Di Giacomo[1], Walter Didimo[1],
Giuseppe Liotta[1], and Matthew Suderman[2]

[1] Università degli Studi di Perugia, Perugia, Italy
{digiacomo,didimo,liotta}@diei.unipg.it
[2] School of Computer Science, McGill University, Montreal, Canada
msuder@cs.mcgill.ca

Abstract. A planar graph G is k-spine drawable, $k \geq 0$, if there exists a planar drawing of G in which each vertex of G lies on one of k horizontal lines, and each edge of G is drawn as a polyline consisting of at most two line segments. In this paper we: (i) Introduce the notion of hamiltonian-with-handles graphs and show that a planar graph is 2-spine drawable if and only if it is hamiltonian-with-handles. (ii) Give examples of planar graphs that are/are not 2-spine drawable and present linear-time drawing techniques for those that are 2-spine drawable. (iii) Prove that deciding whether or not a planar graph is 2-spine drawable is $\mathcal{NP}$-Complete. (iv) Extend the study to k-spine drawings for $k > 2$, provide examples of non-drawable planar graphs, and show that the k-drawability problem remains $\mathcal{NP}$-Complete for each fixed $k > 2$.

1 Introduction

Many graph drawing applications require that the vertices of the graph be placed on some set of horizontal lines. Such drawings have applications in visualization, DNA mapping, and VLSI layout [10, 8]. A common aesthetic requirement is that it be easy to locate the end-vertices of each edge. One way to achieve this is by representing edges as polylines composed of a small number of line segments, and by placing the vertices so that polylines from different edges cross a minimum number of times, if at all. Hence, we have the k-spine drawability problem: Given a planar graph G and an integer $k \geq 0$, is there a planar drawing of G such that the vertices of G lie on k horizontal lines called *spines* and each edge is drawn as a polyline consisting of at most two line segments? For $k \geq 0$, we say that a graph is k-spine drawable, or has a k-spine planar drawing, if it is a yes-instance to the k-spine drawability problem.

The k-spine drawability problem for $k = 1$ is a classic topic in the graph drawing and computational geometry literature, where 1-spine drawings are commonly called 2-page book embeddings or 2-stack layouts. Bernhart and

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Kainen [1] show that a planar graph has a 2-page book embedding if and only if it is sub-hamiltonian, which implies that the 1-spine drawability testing problem is in general NP-hard. Meaningful subclasses of planar graphs that admit 2-page book embeddings (i.e. they are 1-spine drawable) are described in the literature (see, e.g. [1, 3]).

The k-spine drawability problem for $k \geq 1$ has also been widely investigated in the case that the edges cannot bend, i.e they are straight-line segments. There are several papers devoted to this problem, both under the assumption that no two vertices on the same spine can be adjacent (see, e.g. [5, 7]) and under the assumption that there can be intra-spine edges (see, e.g. [4, 6, 9]). In particular, Cornelsen, Shank, and Wagner [4] characterize the family of graphs that admit a straight-line 2-spine drawing with intra-spine edges. They show that the graphs in this family are a proper subset of outerplanar graphs and describe a linear time test algorithm.

The present paper studies k-spine drawings for $k \geq 2$. It is assumed that edges can bend at most once and that two edges on the same spine can be adjacent. We are interested in testing whether or not a graph G admits a k-spine drawing, and, if so, computing such a drawing. The main results in this paper are as follows:

- We introduce and study the notion of hamiltonian-with-handles planar graphs. We show that a planar graph admits a 2-spine drawing if and only if it is sub-hamiltonian-with-handles.
- We study the relationship between hamiltonian-with-handles graphs and planar graphs. Namely, we show that there exist planar graphs that are not sub-hamiltonian-with-handles; consequently, they do not admit a 2-spine drawing. We also prove that every 2-outerplanar graph G is sub-hamiltonian-with-handles and that an embedding-preserving 2-spine drawing of G can be computed from a 2-outerplanar embedding in linear time.
- Motivated by these results, we study the problem of deciding whether or not a planar graph admits a 2-spine drawing. We show that this problem is $\mathcal{NP}$-Complete.
- We extend the investigation to $k > 2$ spines and prove that in this case not all planar graphs are k-spine drawable. We show that the problem of testing k-spine drawability remains $\mathcal{NP}$-Complete for any fixed integer $k > 2$.

For reason of space, some proofs are sketched or omitted.

2 Preliminaries

A *k-spine planar drawing* of G ($k \geq 1$) is a planar drawing of G in which the vertices of G are drawn as points on one of k horizontal straight lines (called *spines*), and the edges of G are drawn as polylines consisting of at most two segments (i.e. each edge is drawn with at most one bend). If G admits a k-spine planar drawing, then G is said to be *k-spine drawable*.

Let Γ be a k-spine planar drawing of G. A *jumping segment* to vertex v is a straight-line segment $\overline{pv}$ contained in an edge incident on v in Γ such that p and

v lie on different spines. We say that p is its first endpoint and v is its second endpoint. A *jumping sequence* J from a vertex v to a vertex w is a sequence $f_0, f_1, \ldots, f_h$ of jumping segments in Γ such that:

1. The first endpoint of f_0 is on the same spine as v, coinciding with v or to the right of v;
2. The second endpoint of f_h is on the same spine as w coinciding with w or to the left of w;
3. If f_i and f_{i+1} are consecutive segments in J, and p is the second endpoint of f_i and q is the first endpoint of f_{i+1}, then p and q lie on the same spine and p is to the left of q.

The *landing segments* of J are the horizontal line segments between the second endpoint of each f_i and the first endpoint of its successor in J, along with the horizontal segment between v and the first endpoint of f_0 and the horizontal segment between the second endpoint of f_h and w. Thus, the *landing sequence* $L_{v,w}(J)$ from v to w of the jumping sequence J is the sequence of landing segments of J whose order corresponds to the order of the segments in J. The *jumping vertex sequence* $V_{v,w}(J)$ of jumping sequence J from vertex v to vertex w is the sequence of vertices that lie on the landing segments of $L_{v,w}(J)$. The order of the vertices corresponds to the order that their segments appear in $L_{v,w}(J)$, and then to their left-to-right order in Γ. Whenever the jumping vertex sequence $V_{v,w}(J)$ is a simple path with $prev(w) = \emptyset$ and $next(w) = \emptyset$, we call it a *cutting path* of G in Γ. Similarly, if $V_{v,w}(J)$ can be augmented by edge addition while maintaining planarity to be a simple path with $prev(w) = \emptyset$ and $next(w) = \emptyset$, then we call it an *augmenting cutting path* of G in Γ.

Cutting paths will be essential to our characterization of 2-spine drawable graphs later. Very roughly, a cutting path splits the graph into two subgraphs that are each 1-spine drawable. The following lemma can be proved.

Lemma 1. *For each 2-spine planar drawing Γ of a planar graph G, there exists an augmenting cutting path of G in Γ.*

3 Hamiltonian-with-Handles Graphs

In this section we characterize the class of 2-spine drawable graphs. First, we require a few additional definitions.

Let G be an embedded planar graph. A *base path* of G is a simple path Π of G such that the first and the last end-vertices of Π are on the external face of G. Let Π be a base path and let η be a simple path of G such that no vertex of η is a vertex of Π. Path η is a *handle* of Π if for each end-vertex of η there exists an edge e, called a *bridge*, connecting the end-vertex to Π. The end-vertex of e in Π is called an *anchor vertex* of η. Its other end-vertex is called an *extreme vertex* of η. The subpath of Π between the anchor vertices of η is called the *co-handle* of η and is denoted $\widehat{\eta}$. The subgraph of G composed of the cycle C_η formed by η, its bridge edges and $\widehat{\eta}$, along with any edges and vertices inside C_η is called the *handle graph* of η and is denoted G_η.

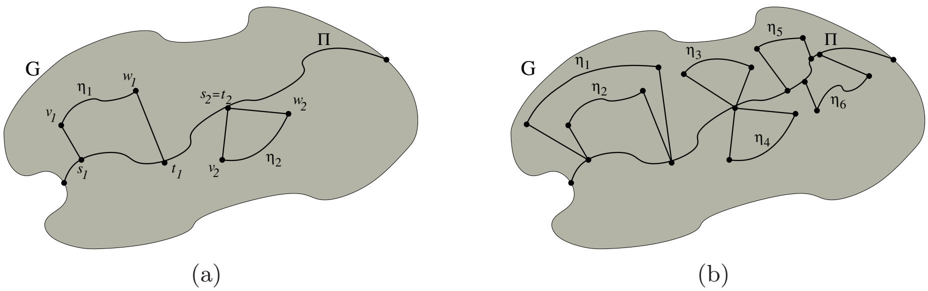

Fig. 1. (a) Illustration of handles along a path Π. η_1 is a non-dangling left handle and η_2 is a dangling right handle. Edges (s_1, v_1) and (t_1, w_1) are the bridges of η_1, and edges (s_2, v_2) and (t_2, w_2) are the bridges of η_2. Vertices s_1 and t_1 are the anchor vertices of η_2, and vertex $s_2 = t_2$ is the anchor vertex of η_2. Vertices v_1 and w_1 are the extreme vertices of η_1, and vertices v_2 and w_2 are the extreme vertices of η_2. (b) Some examples of interleaving handles.

If the two anchor vertices of a handle coincide, then the handle is called a *dangling handle*. If we walk along path Π from one end to the other, then every edge of G that is not in Π is either on the left-hand side of Π or on the right-hand side. Handles on the left-hand side are called *left handles*, and handles on the right-hand side are called *right handles*. Figure 1(a) illustrates these definitions.

Let η_1 and η_2 be two handles, and let s_1 and t_1 be the anchor vertices of η_1 such that s_1 is encountered before t_1 when walking along Π. Similarly, let s_2 and t_2 be the anchor vertices of η_2 such that s_2 is encountered before t_2 when walking along Π. Handles η_1 and η_2 are said to be *interleaving* if one of the following two cases holds:

- G_{η_1} and G_{η_2} share more than one vertex or share a vertex that is not an anchor for η_1 or η_2 (see, for example, handles η_1 and η_2 or handles η_5 and η_6 in Figure 1(b)); or
- η_1 is a left dangling handle, η_2 is a right dangling handle, and $s_1 = s_2 = t_1 = t_2$ (see, for example, handles η_3 and η_4 in Figure 1(b)).

A planar graph G is *hamiltonian-with-handles* if either G has at most two vertices or, for some planar embedding of G, the vertices of G can be covered by a cycle C and a set of paths $\eta_1, \eta_2, \ldots \eta_p$ such that: (i) C is a simple cycle, (ii) C is the union of a base path Π and an edge, and (iii) $\eta_1, \eta_2, \ldots \eta_p$ are non-interleaving handles of Π. G is *sub-hamiltonian-with-handles* if it can be augmented by adding edges in such a way that the resulting augmented graph is still planar and hamiltonian-with-handles.

4 Characterizing 2-Spine Drawable Graphs

In this section we prove the following characterization:

Theorem 1. *A planar graph G is 2-spine drawable if and only if it is sub-hamiltonian-with-handles.*

4.1 Proof of Necessity

We first prove that if a planar graph G is 2-spine drawable, then G is sub-hamiltonian-with-handles. Let Γ be a 2-spine planar drawing of a planar graph G. By Lemma 1, there exists an augmenting cutting path $\Pi = V_{v,w}(J)$ of G in Γ. We will use Π as our base path. It remains then to prove that the vertices of G outside Π can be covered with a set of non-interleaving handles.

Let $J = f_0,\ f_1,\ \ldots,\ f_h$, and use λ_i to denote the landing segment before each jumping segment f_i in the landing sequence $L_{v,w}(J)$. In addition, let λ_{h+1} denote the landing segment after f_h. We call the first and last vertices, denoted v_i and w_i, of each λ_i its *corner vertices*. We use π_i $(i = 0, \ldots, h+1)$ to denote the subpath of Π consisting of:

- the vertex immediately preceding v_i, if it exists;
- all the vertices in λ_i; and,
- the vertex immediately following w_i, if it exists.

We call each π_i a *pocket*. Each pocket has an associated portion of a spine called its *pocket lead*:

- Pocket lead $\widehat{\pi_0}$ is the portion of spine that is before λ_1;
- Pocket lead $\widehat{\pi_i}$ $(i = 1, \ldots, h)$ is the portion of spine that is between λ_{i-1} and λ_{i+1}; and,
- Pocket lead $\widehat{\pi_{h+1}}$ is the portion of spine that is after λ_h.

A maximal sequence of consecutive vertices in a pocket lead is called *candidate handle*.

Lemma 2. *Let Γ be a 2-spine planar drawing of a planar graph G, and let Π be a cutting path of G in Γ. Let π_i be a pocket of Π and let $\widehat{\pi_i}$ be the pocket lead of π_i $(0 \leq i \leq h+1)$. Let η be a candidate handle in $\widehat{\pi_i}$, and let v_η and w_η be the first vertex and the last vertex of η, respectively. Then, there exist two vertices $s_\eta, t_\eta \in \pi_i$ such that either there exist edges (v_η, s_η) and (w_η, t_η) in Γ or these edges can be added to Γ while maintaining the planarity of Γ. Furthermore, vertex s_η is on the spine that does not contain the vertices of η.*

Lemma 2 shows that G can be augmented by edge addition so that the resulting augmented graph can be covered by the cutting path Π plus a set of handles of Π. In order to prove that G is sub-hamiltonian-with-handles we need to prove that these handles are pairwise non-interleaving.

Lemma 3. *Let Γ be a 2-spine planar drawing of a planar graph G, let Π be the cutting path of G in Γ, and let $\eta_1, \eta_2, \ldots, \eta_p$ be a set of candidate handles of G in Γ. Then, Γ can be augmented so that $\eta_1, \eta_2, \ldots, \eta_p$ are pairwise non-interleaving handles.*

Proof. By Lemma 2, Γ can be augmented so that each η_j is a handle, and, if η_j is in pocket lead $\widehat{\pi_i}$, then its anchors s_j and t_j belong to π_i. We now prove that each pair of handles is non-interleaving. Without loss of generality, we consider the pair η_1 and η_2. By way of contradiction, assume that η_1 and η_2 are interleaving. According to the definition there are two cases.

- G_{η_1} and G_{η_2} share more than one vertex or share a vertex that is not an anchor for η_1 or η_2.
  By definition, η_1 and η_2 are disjoint so, by Lemma 2, the vertices that G_{η_1} and G_{η_2} share are also shared by the pockets corresponding to η_1 and η_2.
  We first consider the case where η_1 and η_2 belong to different pockets. Two pockets share vertices only if they are consecutive so we assume, without loss of generality, that η_1 belongs to pocket π_i and η_2 belongs to the next pocket π_{i+1}. In that case, the pockets share two vertices, w_i and v_{i+1}, which are consecutive on path Π. Thus, η_1 belongs to the same spine as v_{i+1} and is left of v_{i+1}. On the opposite spine, η_2 is to the right of w_i. By Lemma 2, s_2 does not belong to the spine of η_2 so s_2 appears after v_{i+1} in path Π, or coincides with v_{i+1}. Vertex t_2 appears after s_2 in Π or coincide with s_2. Hence G_{η_1} and G_{η_2} can share at most an anchor vertex. Therefore, η_1 and η_2 must belong to the same pocket.
  Since η_1 and η_2 belong to the same pocket, we assume, without loss of generality, that η_1 is to the left of η_2 on some spine. Let w_{η_1} be the last vertex of η_1 and let v_{η_2} be the first vertex of η_2. The two handles are interleaving only if the subpaths s_1 to t_1 of Π and s_2 to t_2 of Π share an edge. This implies that t_1 is to the right of s_2. By definition, $next(w_{\eta_1})$ is a crossing c_1 and $prev(v_{\eta_2})$ is also a crossing c_2 to the right of c_1. In addition, an edge incident on t_1 contains the segment $\overline{c_1 t_1}$ and another edge incident on s_2 contains the segment $\overline{c_2 s_2}$. Since c_1 is left of c_2 and s_2 is left of t_1, we have an edge crossing so η_1 and η_2 do not interleave.
- η_1 is a left dangling handle, η_2 is a right dangling handle, and $s_1 = s_2 = t_1 = t_2$. Since η_1 is a left dangling handle and η_2 is a right dangling handle then they are on different spines. By Lemma 2 also s_1 and s_2 are on different spines, but this is impossible since they coincide. □

Together, Lemmas 2 and 3 prove the necessary condition of our characterization:

Lemma 4 (Necessary Condition). *If a graph G is 2-spine drawable, then G is sub-hamiltonian-with-handles.*

4.2 Proof of Sufficiency

To prove the sufficiency of the characterization of Theorem 1, we describe an algorithm that constructs a 2-spine planar drawing of any graph that is sub-hamiltonian-with-handles. For reasons of space only an outline of the algorithm is given.

Suppose that G is sub-hamiltonian-with-handles for some planar embedding and base path Π. Thus, Π divides G into two subgraphs, one to the left of Π and the other to the right of Π. Very roughly, the algorithm first draws the base path on the two spines so that it is possible to draw the subgraph that is to the left of Π, above the drawing of Π, and the subgraph that is to the right of Π, below the drawing of Π (see also Figure 2). The algorithm performs the following steps:

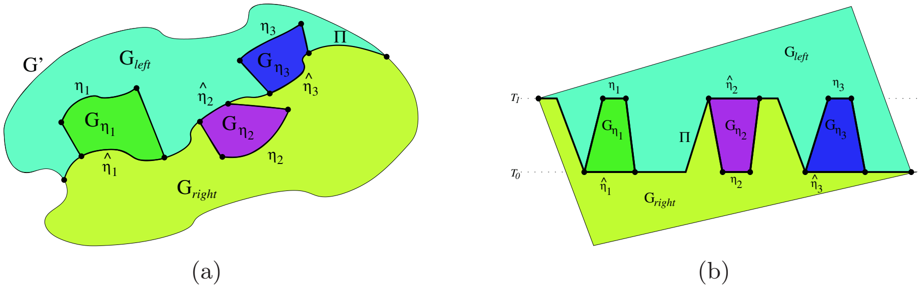

Fig. 2. Illustration of the drawing algorithm. (a) The graph G', obtained after the removal of the dangling handles, can be decomposed into two graphs G_{left} and G_{right} plus one handle graph for each handle. (b) The drawing technique assigns vertices to each spine so that the left handles can be drawn on spine T_1 and the right handles can be drawn on spine T_0. G_{left} is drawn completely above Π, and G_{right} is drawn completely below Π. The drawings of G_{left} and G_{right} share only vertices of Π.

Drawing the Vertices of Π: The algorithm starts by drawing the vertices of Π in G on the two spines. Each vertex is assigned to one of the two spines so that each co-handle of a left handle is on the lower spine and each co-handle of a right handle is on the higher spine. A position on the spine, i.e. an x-coordinate, is also assigned to each vertex of Π.

Removing the Dangling Handles: In order to simplify the algorithm, the dangling handles are removed and replaced with a set of new edges. The resulting graph G' then has only non-dangling handles but may have multiple edges. The removed handles are re-inserted back into the graph in the last step of the algorithm.

Drawing the Vertices of the Non-dangling Handles: The vertices of G' that are not in Π (i.e. the vertices of the non-dangling handles of G) are assigned an x-coordinate and a spine.

Drawing the Edges of G_{left} and of G_{right}: Recall that Π divides G' into two subgraphs, one to the left and the other to the right. We roughly define G_{left} to be the subgraph induced by the edges to the left of Π minus any handle graph edges. We similarly roughly define G_{right} to the be the subgraph induced by the edges to the right of Π minus any handle graph edges. Thus, the algorithm draws the edges of G_{left} and G_{right} separately, using the same technique for each, and then merges the two drawings together.

Drawing the Edges of the Handle Graph: After the edges of G_{left} and G_{right} are drawn, the edges of each handle graph are added to the drawing.

Re-inserting the Dangling Handles: Finally, the dangling handles are re-inserted into the drawing after removing the edges that were inserted earlier to replace the handle.

Lemma 5 (Sufficient Condition). *If a planar graph G is sub-hamiltonian-with-handles, then G is 2-spine drawable.*

Together, Lemmas 4 and 5 prove Theorem 1.

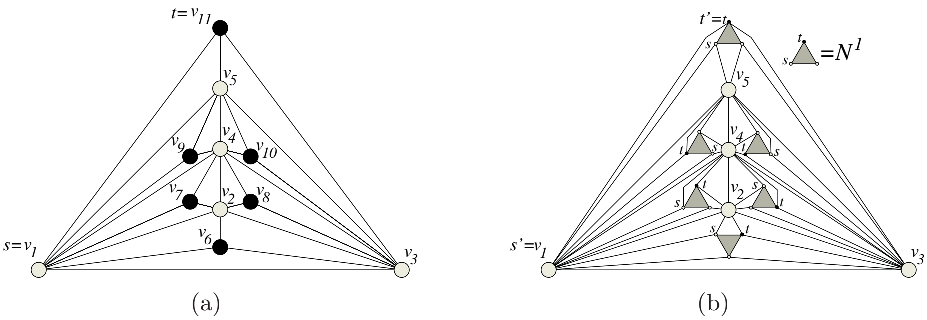

Fig. 3. (a) Maximal planar graph N; (b) The graph N^2 for the proofs of Theorem 2.

5 2-Spine Drawability Testing

The characterization result of Theorem 1 naturally raises two related questions: (i) Is every planar graph 2-spine drawable? (ii) How hard is it to decide whether or not a planar graph is 2-spine drawable? In this section we address both questions.

Theorem 2. *There exists a planar graph that is not* 2*-spine drawable.*

Sketch of Proof: Let N^1 be the maximal planar graph of Figure 3(a). Graph N^1 is non-hamiltonian [2] and therefore not 1-spine drawable [1]. Let H_5 be the subgraph of N^1 obtained by removing the vertices of degree three (the black vertices) from N^1. Given the embedding of H_5 in Figure 3(a), let N^2 be the maximal planar graph obtained by inserting a copy of N^1 into each face of H_5 and then triangulating the result (see Figure 3(b)).

We prove that graph N^2 is not 2-spine drawable. To this aim we consider a weaker version of the necessary condition in Theorem 1: if maximal planar graph G is 2-spine drawable, then G contains a simple cycle C such that $G \setminus C$ is 1-spine drawable. If G is 2-spine drawable, then, by Lemma 1, there exists an augmenting cutting path Π for a 2-spine planar drawing Γ of G. The end-vertices of Π are on the external face of G, so, since G is maximal, they are adjacent. Therefore, Π plus the edge connecting its end-vertices form a simple cycle C. Since no edge of G crosses Π in Γ, if we remove C from the drawing of G, we are left with a set of subgraphs of G that are drawn on one spine and are therefore 1-spine drawable.

We now prove that N^2 is not 2-spine drawable. Suppose, by way of contradiction, that N^2 is 2-spine drawable. By the above necessary condition, there exists a simple cycle in N^2 such that $N^2 \setminus C$ is 1-spine drawable. Since N^1 is not 1-spine drawable, then C must contain at least one vertex from each copy of N^1. In the embedding of N^2 in Figure 3(b), each copy of N_1 is inside a different face of H_5. Thus, given any two vertices v_1 and v_2 from different copies of N_1, there must be a vertex of H_5 between v_1 and v_2 in C. Since there are six copies of N^1 and five vertices in H_5, then all the vertices of H_5 are in C. Thus, C contains at least one vertex from each copy of N^1 and all the vertices of H_5; however, this implies that there exists a hamiltonian circuit in N^1, a contradiction. □

While Theorem 2 gives a negative result, the following theorem describes a meaningful class of 2-spine drawable graphs.

Theorem 3. *Every embedded* 2*-outerplanar graph is* 2*-spine drawable and a* 2*-spine planar drawing of* G *can be computed in linear time.*

Sketch of Proof: By Theorem 1 it is sufficient to prove that G is sub-hamiltonian-with-handles. We assume that G is biconnected. If it is not biconnected, then we can easily make it biconnected by edge addition, while maintaining a 2-outerplanar embedding. Since G is biconnected, the external face of G is a simple cycle C. Let G_0 be the subgraph of G induced by the vertices of C. We choose our base path Π to be C minus an edge. Each internal vertex, that is, each vertex that is not on the external face, is either adjacent to a vertex of the external face or can be made adjacent to a vertex of the external face by adding an edge. Each internal vertex v is a handle of length one and the edge connecting v to a vertex of the external face is its bridge. As for the time complexity, we remark that finding C and the handles takes linear time, and that the drawing procedure described in Section 4.2 requires linear time if C and the handles are given. □

Based on the above theorem, one can ask whether embedded 2-outerplanar graphs can be drawn on less than two spines. We observe that the graph of Figure 3(b) is 2-outerplanar and that, as observed in the proof of Theorem 2, it is not 1-spine drawable.

Motivated by the results in Theorems 2 and 3, we investigate the complexity of deciding whether a planar graph is 2-spine drawable. The next theorem states that this problem is $\mathcal{NP}$-complete. In fact, we prove that the problem is $\mathcal{NP}$-complete when restricted to embedded maximal planar graphs and embedding-preserving 2-spine planar drawings. The original problem and this restricted version are polynomially equivalent because maximal planar graphs have a linear number of planar embeddings that can be efficiently computed.

The reduction is from HC-EMP: given an embedded maximal planar graph G, determine whether or not G is external hamiltonian, i.e. G has a hamiltonian circuit with an edge on the external face. Wigderson [11] has proved that HC-MP (the hamiltonian circuit problem for maximal planar graphs) is $\mathcal{NP}$-Complete. These two problems are polynomially equivalent, once again because each maximal planar graph has a linear number of embeddings. The proof of the next theorem is omitted for reasons of space.

Theorem 4. *The problem of determining whether or not a planar graph is* 2*-spine drawable is* $\mathcal{NP}$*-complete.*

6 *k*-Spine Drawability Testing

We extend the study of the 2-spine drawability to the case of the k-spine drawability. The following results can be proved by inductively generalizing the the proofs for the 2-spine drawing results.

Theorem 5. *For each fixed integer $k > 2$, there exists a planar graph that is not k-spine drawable.*

Sketch of Proof: The proof of this theorem is an extension of the proof of Theorem 2 and is based on a necessary condition for a planar graph to be k-spine drawable: if planar graph G is k-spine drawable, then G contains a simple cycle C such that $G \setminus C$ is $(k-1)$-spine drawable. We inductively describe a sequence of maximal planar graphs N^k that are not k-spine drawable for $k \geq 1$: (i) N^1 is the graph of Figure 3(a); (ii) N^k, for $k \geq 2$, is obtained from H_5 by inserting a copy of N^{k-1} into each face of H_5 (assuming the embedding of H_5 in Figure 3(a)) and then triangulating. We prove that N^k is not k-spine drawable by induction on k. N^1 is not 1-spine drawable since it is not hamiltonian. Assume that N^{k-1} is not $(k-1)$-spine drawable and, suppose, by way of contradiction, that N^k is k-spine drawable. By the necessary condition above, there exists a simple cycle C of N^k such that $N^k \setminus C$ is $(k-1)$-spine drawable. Since N^{k-1} is not $(k-1)$-spine drawable, then C must contain at least one vertex from each copy of N^{k-1}. In the planar embedding of N^k, each copy of N^{k-1} is inside a different face of H_5. Thus, given any two vertices v_1 and v_2 from different copies of N^{k-1}, there must be a vertex of H_5 between v_1 and v_2 in C. Since there are six copies of N^{k-1} and five vertices in H_5, then all the vertices of H_5 are in C. Thus, C contains at least one vertex from each copy of N^{k-1} and all the vertices of H_5. This implies that there exists a hamiltonian circuit in N^1 which is impossible. □

The proof of $\mathcal{NP}$-Completeness for 2-spine drawability testing can be extended to k-spine drawability for $k > 2$.

Theorem 6. *For each fixed integer $k > 2$, the problem of determining whether or not a planar graph is k-spine drawable is $\mathcal{NP}$-Complete.*

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