Characterizing the Shape of Anatomical Structures with Poisson's Equation

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Abstract. This paper presents a novel approach to analyze the shape of anatomical structures. Our methodology is rooted in classical physics and in particular Poisson's equation, a fundamental partial differential equation [1]. The solution to this equation and more specifically its equipotential surfaces display properties that are useful for shape analysis. We demonstrate the solution of this equation on synthetic and medical images. We present a numerical algorithm to calculate the length of streamlines formed by the gradient field of the solution to this equation for 3D objects. We used the length of streamlines of equipotential surfaces to introduce a new function to characterize the shape of objects. A preliminary study on the shape of the caudate nucleus in Schizotypal Personality Disorder (SPD) illustrates the power of our method.

1 Introduction

Shape analysis methods play a key role in the study of medical images. They enable us to go beyond simple volumetric measures to provide a more intuitive idea of the changes an anatomical structure undergoes. There are mainly three classes of shape analysis methods. The first class relies on a feature vector, such as spherical harmonics or invariant moments [2, 3], as a representation of shape and tries to discriminate between classes of shapes using clustering methods such as principal component analysis. These methods are usually numerically stable and relevant statistics can be computed from them. However, their interpretation is often difficult and they rarely provide an intuitive description of the shape. The second class of methods is based on a surface boundary representation of the object and the study of the mechanical deformations required to transform one object into another [4, 5]. This popular technique is very intuitive, but relies on registration methods which are difficult to implement and not always reliable. Calculating significant statistics from the deformation also poses a challenge. The third class makes use of medial representa

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tions which provide insightful information about the symmetry of the object. Unfortunately, the medial models still need to be registered with each other before any statistics can be derived [6-8]. In clinical studies, different classes of methods are often combined in order to obtain intuitive and statistically significant results, see for example [9].

In this paper, we propose a novel shape analysis method based on Poisson's equation with a Dirichlet boundary condition. This equation, most known in electrostatics, has very interesting properties for the study of shape. Most notably, its solution is always smooth, has one sink point and can be made independent of the scale of the original object. Our approach is to extract one scalar value and use this number to compare different classes of objects.

Section 2 provides details on Poisson's equation and how it can be used for shape analysis. Section 3 illustrates and validates the method on synthetic objects and finally Section 4 presents a preliminary case study of the caudate in Schizotypal Personality Disorder.

2 Methods

2.1 Poisson's Equation

Poisson's equation is fundamental to mathematical physics and has been widely used over a range of phenomena. Examples include electrostatic fields, gravitational fields, thermo-dynamic flows and other applications. Mathematically, Poisson's equation is a second-order elliptic partial differential equation defined as:

$$\Delta u = -1 \tag{1}$$

Poisson's equation is independent of the coordinate system and characterizes the entire domain (volume in 3D) not only its boundary. Functions u satisfying Poisson's equation are called potential functions. These functions have many mathematical properties related to the underlying geometry of the structure. Among the properties are the following:

1. The shape of the potential function is correlated to the geometry of the structure. This correlation gives a mathematical meaning to the medical concept of anatomical sublayers.
2. In the special case of Dirichlet conditions on the outer surface of a closed homogeneous domain, the potential function converges smoothly to a single sink point. Moreover, a unique streamline can be drawn from each point of the boundary to the sink point by following the gradient field of the potential function.
3. The pattern of streamlines is independent of the value of the potential on the boundary and is closely related to the shape of the domain.

The length of each streamline can be calculated by summing the Euclidean distances between neighboring points along the streamline. For example, in electrostatics, this length is called the 'electric displacement'.

In Figure 1, we illustrate the solution of Poisson's equation for simple two dimensional domains, a circle (Figure 1a) and a square (Figure 1b). In both examples, the initial conditions on the boundary were set to u = 100. The solution represents a layered set of equipotential curves making a smooth transition from the outer contour to the center. The equipotential curves for a circle are simply smaller circles. However, the equipotential curves inside a square smoothly change shape as they approach the sink point. This change is illustrated in Figure 1a where two streamlines connect the sink point to two equipotential points inside the circle. Figure 1b shows similar streamlines inside the square. Inside a circle the 'electric displacement' along an equipotential curve is constant. Inside a square, this displacement varies due to the shape of the boundary. Note however, that the variation of displacement decreases from one equipotential level to another while moving towards the sink point. We used this concept of displacement as the basis of our approach to apply Poisson's equation for the analysis of shape of anatomical structures.

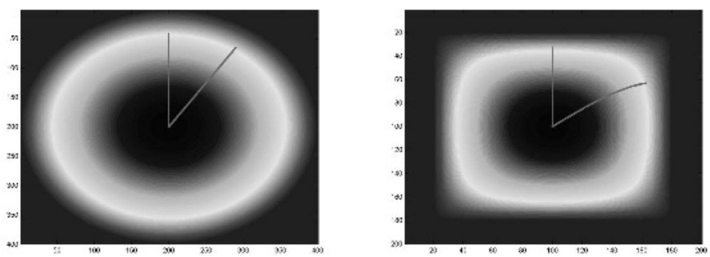

Fig. 1. a) potential function inside a circle with streamlines of two equipotential points. b) potential function inside a square with streamlines of two equipotential points.

2.2 Calculating the Displacement

Since there are many standard numerical methods for solving Poisson's equation (1), we will not discuss in this work the numerical solution of Poisson's equation for 3D MRI. We refer the reader to [10] for details on numerical solutions of standard partial differential equations. We will focus instead on the computation of the displacement for each point in the given domain.

Because all streamlines converge to one unique sink point, P_s, we can design a downwinding algorithm to calculate the displacement at a voxel P_0 by summation of the Euclidean distances between consecutive voxels along the streamline connecting P_0 to P_s.

First, P_s is found by solving the following equation:

$$\|\text{grad}(u)\| = 0 \tag{2}$$

The displacement D at a voxel P_0 is then defined as:

$$D(P_0) = \Sigma L_i \tag{3}$$

L_i is the Euclidean distance between consecutive voxels P_i and P_{i+1} on the streamline connecting P_0 to the sink point. Depending on the direction of the gradient field at voxel P_i, L_i can have one of the following values:

$h_1, h_2, h_3, (h_1^2 + h_2^2)^{1/2}, (h_1^2 + h_3^2)^{1/2}, (h_2^2 + h_3^2)^{1/2}$ *or* $(h_1^2 + h_2^2 + h_3^2)^{1/2}$;
h_1 is the voxel width, h_2 the voxel height and h_3 the slice thickness.

The direction of the gradient is assessed using a simple transformation from a Cartesian to a spherical coordinate system. The azimuth θ and the elevation φ of the gradient are calculated at P_i and used to determine L_i. The coordinates of the following voxel on the streamline, P_{i+1}, can be computed using;

$$\begin{cases} x_{i+1} = x_i + L_i.\sin\varphi_i\cos\theta_i\,; \\ y_{i+1} = y_i + L_i.\sin\varphi_i\sin\theta_i\,; \\ z_{i+1} = z_i + L_i.\sin\theta_i \end{cases} \tag{4}$$

L_i is added to the current $D(P_0)$ using (3), the procedure is then repeated at P_{i+1} until the sink point P_s is reached. Figure 2 presents the displacement maps of a circle and square.

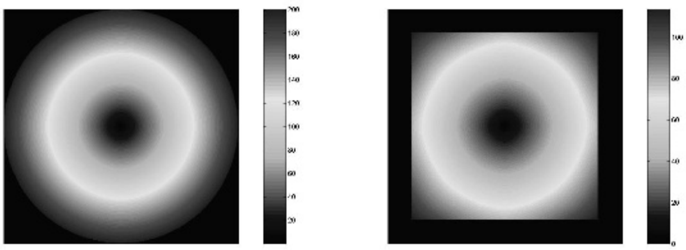

Fig. 2. Displacement map of a circle (left) and a square (right).

2.3 Analysis of Shape Using Poisson's Equation

Earlier, we demonstrated that the dynamics of change of the equipotential surfaces inside the domain while approaching the center is related to the domain geometry. To evaluate this process we first define the normalized drop of potential E at an equipotential surface S_i as:

$$E = (U_0 - U_i)/(U_0 - U_s); \tag{5}$$

U_0, U_s and U_i are the potentials on the boundary, at the sink point and on the current equipotential surface respectively. E characterizes the amount of energy needed to transform the surface boundary into the current equipotential surface.

We then introduce ν, the coefficient of variance of the displacement along the current equipotential surface S_i:

$$\nu(E) = \text{stdev}(D(S_i))/\text{mean}(D(S_i)) \tag{6}$$

Function ν(E), which we call the "shape characteristic", displays some very interesting properties:

1. It is independent of the potential on the outer boundary.
2. It is independent of the overall volume and defined exclusively by the shape.
3. A given value ν corresponds to a known drop of potential E.

4. The slope of $\nu(E)$ dramatically decreases on a well-determined interval for each specific shape. We call the 'critical point' the point $P_c(E_c, \nu_c)$ where the curvature of $\nu(E)$ changes sign. This point characterizes the shape in a unique way.

Thus, the shape of a structure can be represented, through the "shape characteristic", by one unique point $P_c(E_c, \nu_c)$ independent of both the initial conditions on the outer boundary and the volume of the structure. This point defines the amount of energy, needed for the structure to lose its initial shape and to deform into a new one comparable to a sphere.

3 Experiments on Synthetic Data

In addition to the 2D examples given in Section 2, we created two simple phantoms with different shapes to test our algorithm. The first phantom is a single 3D tube, the second is a tubular tree. Poisson's equation was solved, and then the displacement map was calculated. Figure 3 shows the resulting potential functions and displacement maps for a cross section of both objects.

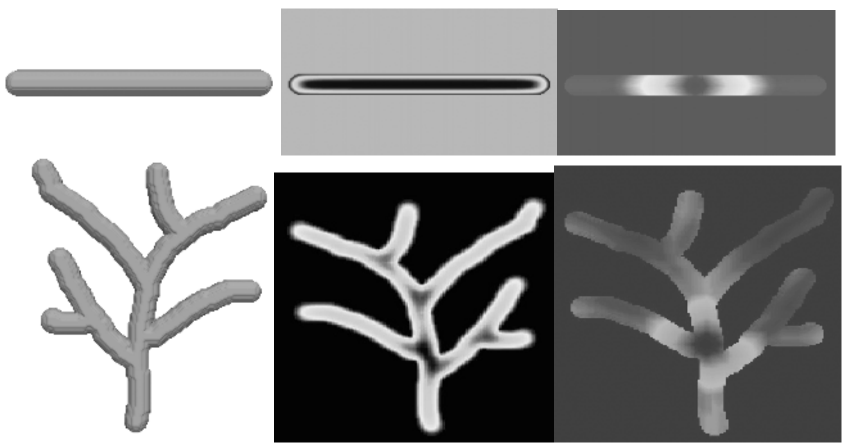

Fig. 3. Potential functions and (middle) and displacement maps (right) for two simple synthetic objects

A graph of the shape characteristic function, $\nu(E)$, is shown in Figure 4. Notice ν is 0 for the circle as expected. For the other phantoms, ν is a monotically decreasing function, i.e., as the equipotential surface approaches the sink point, the shape characteristic approaches zero.

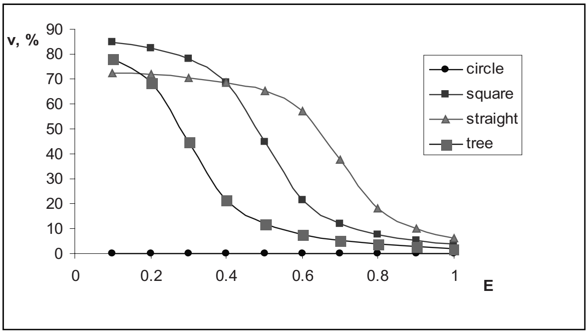

Fig. 4. Shape characteristic ν as a function of E for four phantoms.

4 Shape Analysis of the Caudate Nucleus

The caudate nucleus is an essential part in the "cognitive" circuitry connecting the frontal lobe to subcortical structures of the brain. Pathology in any of the core components of this circuitry may result in neurological disease such as schizophrenia[11, 12] and schizotypal personality disorder (SPD) . Previous studies have shown volumetric and shape differences of the caudate between normal controls and SPD subjects [13, 14]. In this paper we propose to validate our methodology by applying our shape analysis method to the data used in [13, 14] and verify the previously observed shape differences.

4.1 Methods

Fifteen right-handed male subjects with Schizotypal Personality Disorder (SPD) with no previous neuroleptic exposure and fourteen normal comparisons subjects (NC), underwent MRI scanning. Subjects were group matched for parental socioeconomic status, handedness and gender. MRI images were acquired with a 1.5T GE scanner, using a SPoiled Gradient Recalled (SPGR) sequence yielding to a (256x256x124) volume with (0.9375x0.9375x1.5mm) voxel dimensions. The scans were acquired coronally. The caudates were drawn manually and separated by an Anterior/Posterior boundary. Details of the segmentation procedure can be found in [14].

We applied our novel method to the entire caudate to investigate any difference in shape between normal controls and SPD subjects. Poisson's equation was solved for each left and right caudate. Afterwards, the displacement maps were calculated using the algorithm presented in Section 2. An example caudate nucleus and its corresponding potential function and displacement map are shown in Figure 5. The function ν was then computed for each structure. We used E_c = 30% as 'critical' value to calculate the corresponding ν_c. We applied a Mann-Whitney non-parametric test to compare the values of ν_c between the two groups. Figure 6 presents a plot graph of ν_c for our data set.

Fig. 5. 3D rendering of the caudate nucleus (left), sagittal cross section displaying the potential function (middle) and displacement map (right).

4.2 Results and Discussion

Our test revealed a statistically significant difference in the head of the right caudate for the value ν_c ($p< 0.03$) between NC and SPD. No significant group differences ($p< 0.167$) in the head of the left Caudate were found.

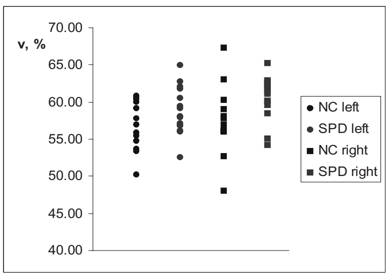

Fig. 6. Plot of the 'critical' value ν_c (E_c=0.3) of the caudate for 14 SPD and 15 controls.

Previous analysis of the same data set revealed a group difference in volume on both sides and a shape difference on the right side using the "shape index" measure[13, 14]. We are glad to report that our method confirms the results previously published.

5 Conclusion

We have developed a novel method for shape analysis of anatomical structures. Our method is based on using the solution of Poisson's equation to assess the dynamics of change of shape of the equipotential surfaces inside the structure. We have developed an algorithm for calculating the displacement maps defined by the length of the streamlines generated by the gradient field of the potential function. We used these maps to introduce a new function, called 'shape characteristic', which characterizes in a unique way the shape of a structure. Our method was validated on synthetic 2D and 3D and on real medical data. Our results on the shape of the caudate nucleus in SPD correlate nicely with the literature. This suggests that our method can be used as a powerful tool to correlate shape of anatomical structures with different factors such as aging or diseases. The "shape characteristic" is one of many tools one can build based on Poisson's equation. In future work, we propose to design more intuitive shape analysis techniques based on this equation and its solutions.

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