Pseudospectral Iterated Method for Differential Equations with Delay Terms

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Abstract. New efficient numerical methods for hyperbolic and parabolic partial differential equations with delay terms are investigated. These equations model a development of cancer cells in human bodies. Our goal is to study numerical methods which can be applied in a parallel computing environment. We apply our new numerical method to the delay partial differential equations and analyse the error of the method. Numerical experiments confirm our theoretical results.

Keywords: Linear delay equations, numerical approximations, parallel computing, error estimations, numerical experiments.

1 Introduction

1.1 Delay Partial Differential Problems

In this paper investigate numerical solutions to the linear initial boundary value problem with a delay term

$$\begin{aligned} &\frac{\partial}{\partial t}u(x,t)=\epsilon\frac{\partial^2}{\partial x^2}u(x,t)+c\frac{\partial}{\partial x}u(x,t)-\nu u(x,t-\tau_0)+g(x,t),\ 0<t\leq T,\\ &\quad u(x,t)=f_0(x,t),\quad -\tau_0\leq t\leq 0,\ -L\leq x\leq L. \end{aligned} \tag{1.1}$$

Here, $\epsilon>0$, $c\in\mathbb{R}$, $\tau_0\geq 0$, $L>0$ and $T>0$ are given constants. The choice of $\epsilon=0$ gives the hyperbolic equation, $c=0$ gives the parabolic equation and the choice of $\epsilon\neq 0$ and $c\neq 0$ gives the parabolic advection-diffusion equation.

Different types of boundary conditions are required for the two cases $\epsilon\neq 0$ and $\epsilon=0$. For the parabolic case ($\epsilon\neq 0$) there are two boundary conditions

$$u(\pm L,T)=f_{\pm}(t),$$

while for the hyperbolic case ($\epsilon=0$, $c\neq 0$) there is one boundary condition, either

$$u(L,t)=f_+(t)\ (\text{if } c>0)\quad \text{or}\quad u(-L,t)=f_-(t)\ (\text{if } c<0).$$

Here, f_0, $f_{\pm}$ and g are given continuous functions.

Delay problems like (1.1) are used to model cancer cells in human tumors, see [1]. For other applications in population dynamics see [4].

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1.2 Purpose of the Paper

We study the Chebyshev pseudospectral (ChPS) spatial discretization of (1.1) (see e.g. [3], [5]) with the Kosloff and Tal-Ezer (KTE) transformation [6], together with Jacobi waveform relaxation methods for time integration (see e.g. [2], [8]). The ChPS method is known from its high accuracy. Another advantage of ChPS is that convergence of waveform relaxation (WR) is faster when it is applied to ChPS semi-discrete systems than it is applied to finite difference semi-discrete systems, see [8]. The advantage of WR methods is that they are efficient in parallel computing environments and for linear equations like (1.1) they allow direct application of implicit methods for integration in time (they allow the use of much larger stepsizes for time integration which fulfill stability restrictions, as compared with the explicit methods).

The goal of this paper is to show that since the KTE transformation makes most of the entries of the differentiation matrices significantly smaller, the convergence of WR is faster with the KTE transformation than without. Morever, for every type of equation the rate of WR convergence increases with increasing parameter $\alpha \in [0,1]$. We show this by using error bounds and actual errors which occur in the process of computations. Using extensive numerical data we also compare WR errors with their error bounds. Our results confirm the conclusions derived in [7].

2 Numerical Solution to Delay Problem (1.1)

We apply the process of pseudospectral spatial discretization ([3], [5]) with KTE transformation ([6]) and replace (1.1) by the following numerical scheme

$$\begin{aligned} \frac{d}{dt}U(t) &= Q_\alpha U(t) - \nu U(t-\tau_0) + f_\alpha(t), \quad 0 < t \le T, \\ U(t) &= \tilde{f}_0^\alpha(t), \quad -\tau_0 \le t \le 0. \end{aligned} \tag{2.2}$$

Here, Q_α is a matrix which depends on the parameter $\alpha \in [0,1]$ of the KTE transformation and the constants ϵ and c (see [3], [5] and [6]). The components of the vector function $U(t)$ provide us with approximations to the values $u(x,t)$ of the exact solution to problem (1.1), that is,

$$U_i(t) \approx u(x_i^\alpha, t),$$

where x_i^α, $i = 0, \ldots, N$, are the transformed Chebyshev grids (see [6]). Further, the vector function $f_\alpha(t)$ is related to $g(x,t)$ and $f_\pm(t)$; the vector function $\tilde{f}_0^\alpha(t)$ is related to the initial function $f_0(x,t)$.

We simplify the system (2.2) by splitting the matrix Q_α into two matrices:

$$\tilde{A}_\alpha = diag(Q_\alpha), \qquad \tilde{B}_\alpha = Q_\alpha - diag(Q_\alpha).$$

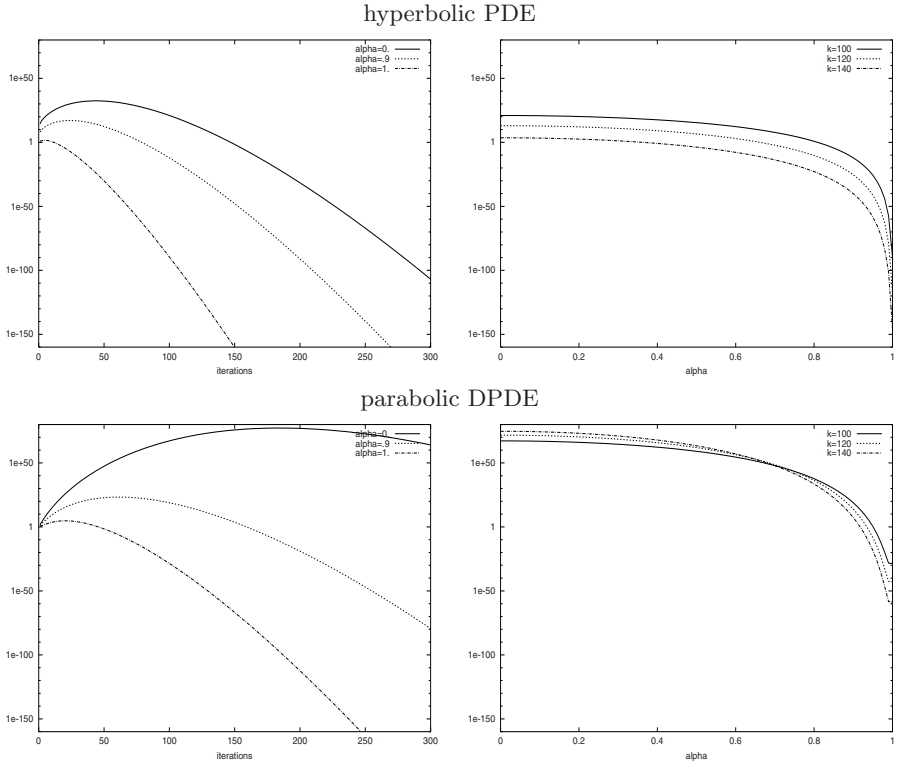

Fig. 1. Error bounds (2.5) as functions of the iteration index k and as functions of the parameter α.

Then the solution $U(t)$ to (2.2) can be approximated by successive iterates $U^k(t)$ computed according to the following Jacobi WR scheme

$$\begin{aligned} \frac{d}{dt}U^{k+1}(t) &= \tilde{A}_\alpha U^{k+1}(t) + \tilde{B}_\alpha U^k(t) - \nu U^k(t-\tau_0) + f(t), \quad 0 < t \le T, \\ U^{k+1}(t) &= \tilde{f}_0^\alpha(t), \quad -\tau_0 \le t \le 0, \end{aligned} \tag{2.3}$$

(see [2], [8] and [7]). Here, $k=0,1,\ldots$ is an iteration index and U^0 is an arbitrary starting function. Since the matrix $A_\alpha^{(l)}$ is diagonal, each equation of the system (2.3) can be solved independently by a different processor.

To study convergence of the waveform relaxation process (2.3) we consider the error defined by

$$e_\alpha^k(t) = U^k(t) - U(t). \tag{2.4}$$

To investigate an error estimation for (2.4) we denote by $\|\cdot\|$ an arbitrary vector norm or the induced matrix norm. It is shown in [8] that error estimations for (2.4) are more delicate if the following logarithmic norm

$$\mu(M) = \lim_{\delta \to 0^+} \frac{\|I + \delta M\| - 1}{\delta},$$

defined for an arbitrary matrix M, is used. Here, I is an identity matrix.

An error estimation for nonlinear delay differential equations (more general than the linear equation in (1.1)) is derived in [7, Theorem 4.3]. The estimation presented in [7, Theorem 4.3] is derived under the assumption that the right-hand sides of the nonlinear delay differential equations are Lipschitz continuous with respect to the delay terms. Since the delay equation (1.1) is linear, the coefficient ν is the Lipschtiz constant, which we use to apply [7, Theorem 4.3]. This leads to the following error estimation

$$\|e_\alpha^{k+1}(t)\| \le \frac{(\|\tilde{B}_\alpha\| + \nu)^{k+1}}{k!} \int_0^t s^k \exp\Big(s\mu(\tilde{A}_\alpha)\Big)\, ds \max_{0 \le \tau \le t} \|e_\alpha^0(\tau)\|, \tag{2.5}$$

for $k = 0, 1, \ldots$, $t \in [0, T]$.

The estimation (2.5) has an advantage over the traditional estimation which, when applied to (1.1), after some computations (different than these included in the proof of [7, Theorem 4.3]) gives

$$\|e_\alpha^{k+1}(t)\| \le \frac{\Big(t(\|\tilde{B}_\alpha\| + \nu)\Big)^{k+1}}{(k+1)!} \exp\Big(t\|\tilde{A}_\alpha\|\Big) \max_{0 \le \tau \le t} \|e_\alpha^0(\tau)\|. \tag{2.6}$$

It is easily seen that estimation (2.5) is sharper than estimation (2.6). We use the sharper estimation (2.5) and confirm the conclusion derived in [7] that the error bound (2.5) decreases for increasing α.

Figure 1 presents the error bounds (2.5) with $N = 32$ as functions of k for $\alpha = 0, 0.9, 1$ and as functions of $\alpha \in [0, 1]$ for $k = 100, 120, 140$. The error bounds are plotted for problems posed for $x \in [-10, 10]$ and $t \in [0, 1]$.

The error bounds are shown for delay partial differential equations (DPDEs) and for partial differential equations (PDEs) without delay terms. They are plotted for the hyperbolic PDE with $\epsilon = 0$, $c = 1$, $\nu = 0$ and the parabolic DPDE with $\epsilon = 1$, $c = 0$, $\nu = 5$. The pictures for the mixed problems with $\epsilon = 1$, $c = 1$ are similar to the pictures for the parabolic DPDE. We refer the reader to [7] for the pictures with parabolic PDEs, hyperbolic DPDEs and mixed PDEs and DPDEs.

We see significant improvement in convergence of WR when $\alpha = 0.9$ and $\alpha = 1$. In the next section we present the errors $\|e_\alpha^k(t)\|$ by means of extensive numerical data. It is shown in Section 3 that the errors $\|e_\alpha^k(t)\|$ behave like their error bounds (i.e. they decrease as α increases).

3 Numerical Experiments

In this section we present results of numerical experiments for the test problem (1.1). We choose $L = 10$ and $T = 1$ and consider six problems: the advective-diffusive problem with $\epsilon = c = 1$, the diffusive problem with $\epsilon = 1$, $c = 0$ and

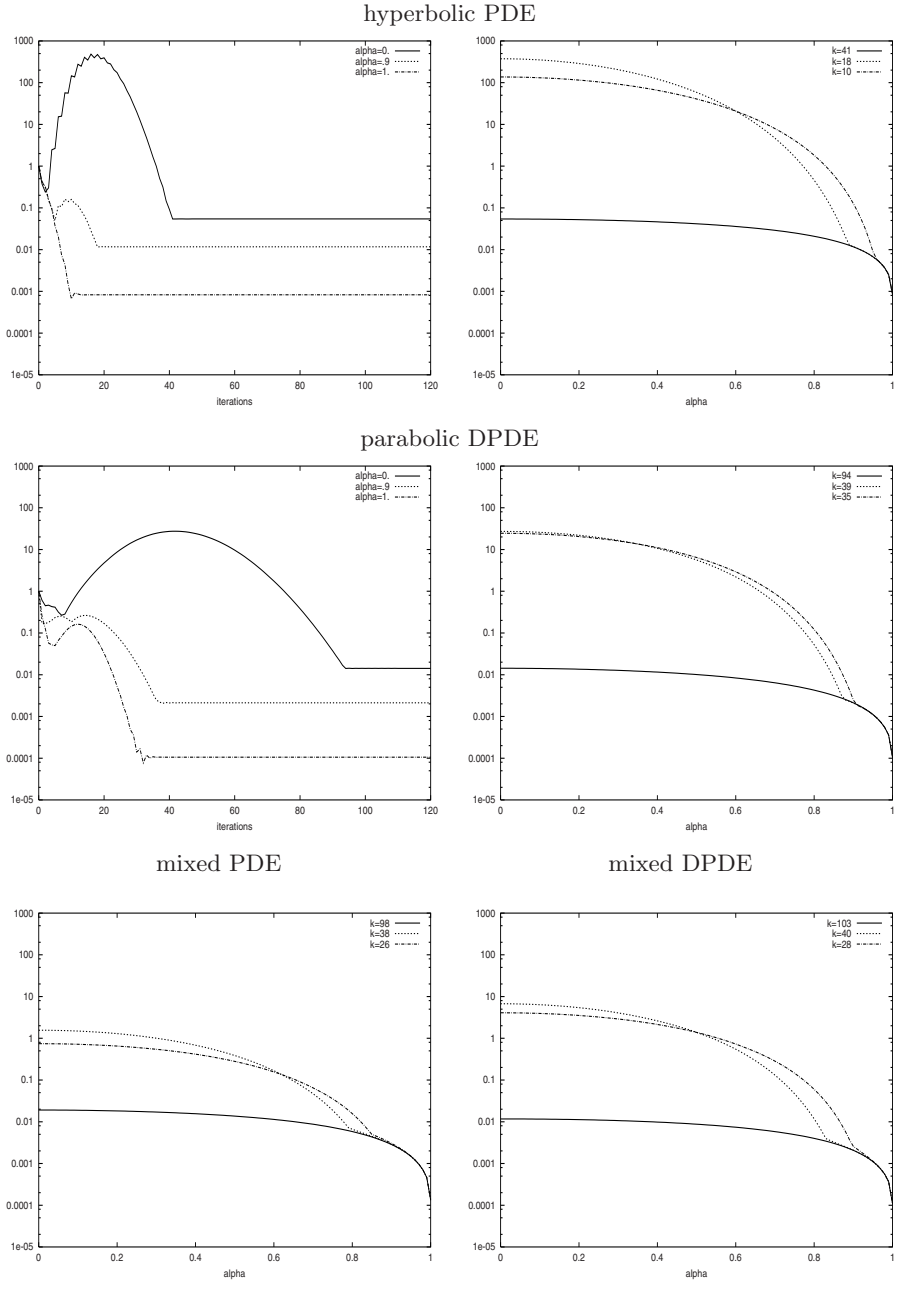

Fig. 2. Errors (3.7) with $N = 32$ as functions of k and as functions of α.

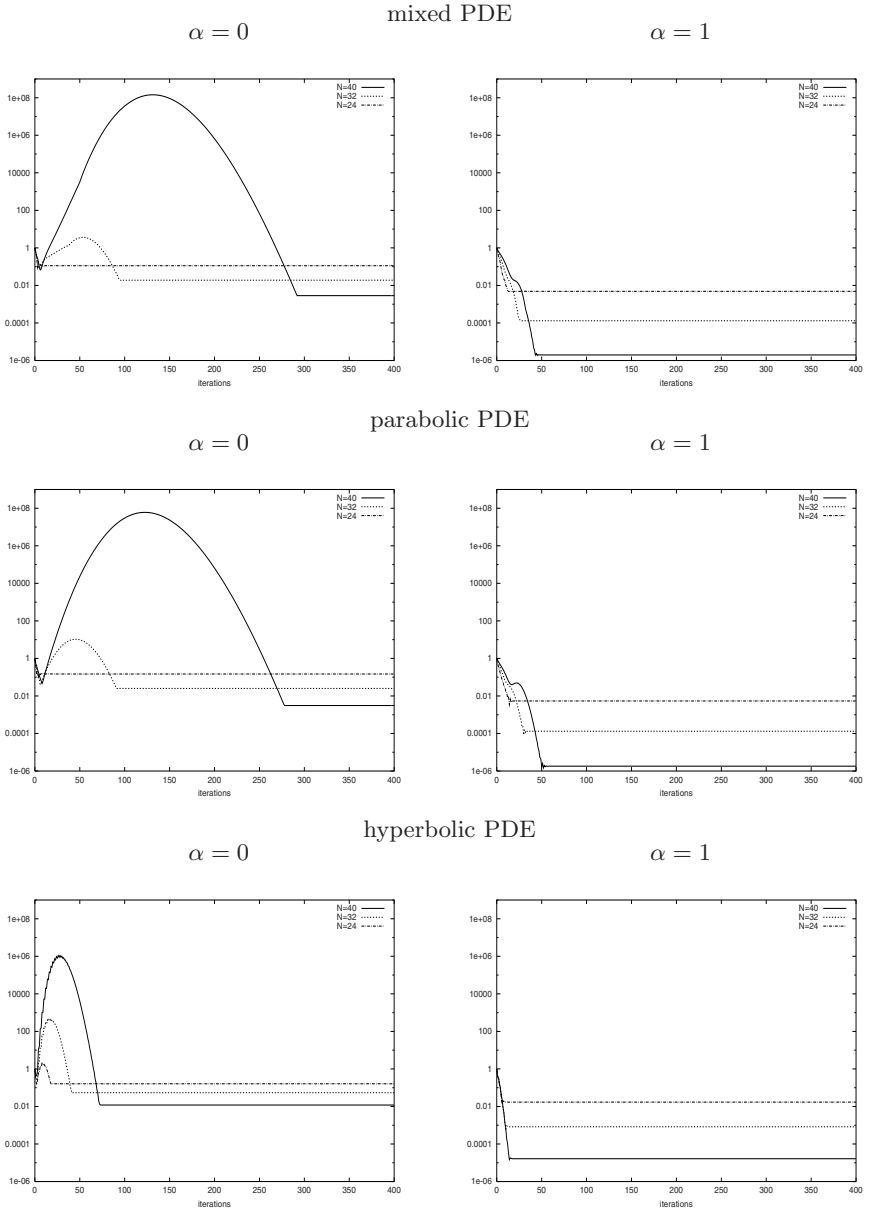

Fig. 3. Errors (3.7) for $N = 40, 32, 24$ as functions of k with $\alpha = 0$ (left pictures) and $\alpha = 1$ (right pictures).

the hyperbolic problem with $\epsilon = 0$, $c = 1$; each problem with $\nu = 0$ and $\tau_0 = 0$ for non-delay case and with $\nu = 5$ and $\tau_0 = 0.1$ for delay case.

To integrate the systems (2.3) in time we use the BDF3 method with the time step $\Delta t = 10^{-2}$. Integration of the systems (2.3) results in the approximations $U_{i,n}^k \approx U_i^k(n\Delta t)$, $n = 1, 2, \ldots$. We study the errors

$$\max_{i=0,\ldots,N} |U_{i,n}^k - u(y_i^\alpha, n\Delta t)| \tag{3.7}$$

measured at $n\Delta t = T = 1$ which can be compared with the upper bounds (2.5) plotted in Figures 1.

To compare the errors (3.7) with the error bounds from Figure 1 we present the errors (3.7) with $N = 32$ in Figure 2. They are presented as functions of the iteration index k for $\alpha = 0$, $\alpha = 0.9$, $\alpha = 1$ and as functions of the parameter α for fixed values of k. The fixed values of k are chosen to be the values where the error curves as functions of k become horizontal.

Figure 2 shows that for a fixed k the error (3.7) decreases as α increases with the smallest value at $\alpha = 1$. Figure 3 presents the errors (3.7) as functions of k for fixed values of α and for different values of N. The errors (3.7) are presented for the extreme values of parameter choices $\alpha = 0$ and $\alpha = 1$. Pictures for delay case are presented in [7].

4 Concluding Remarks

We applied the ChPS spatial discretization with the KTE transformation to delay and non-delay partial differential equations. Jacobi WR was then applied to the resulting semi-discrete systems. Our method is new for both kinds of equations. We conclude that the method works equally well for delay and non-delay equations.

Since Jacobi WR was used, our method can be efficiently used in parallel computing environments. We studied the relation between the WR convergence and the parameter α used for the spatial discretization. Using error bounds we conclude that WR converges more quickly as α increases from 0 to 1. This conclusion is confirmed by numerical experiments with delay and nondelay equations.

Since our method is successful for the test problem (1.1), our future work will address the numerical solution of the delay differential problem from [1].

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