# Direct Solver Based on FFT and SEL for Diffraction Problems with Distribution

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**Abstract.** A direct solver for diffraction problems is presented in this paper. The solver is based on the fast Fourier transform (FFT) and the successive elimination of lines which we call SEL. In the previous paper, we showed the numerical algorithm by use of SEL and proved that the limit function of approximate solutions satisfied the diffraction problem in the sense of distribution. In this paper, the above numerical algorithm is improved with FFT and we show that the calculation speed is faster than the previous one.

## 1 Introduction

This paper is devoted to study the construction of finite difference solutions based on the direct method which we call SEL and establish the numerical algorithm by use of FFT.

Let $\Omega$ be a rectangular domain in $R^2$, $\Omega_1$ be an open subset of $\Omega$ and $\Omega_2 = \Omega\backslash\overline{\Omega_1}$, the interface of them be denoted by $\Gamma(= \overline{\Omega_1} \cap \overline{\Omega_2})$. The diffraction problem considered here is the followings.

**Problem I.** For $f \in L^2(\Omega)$, $\sigma \in L^2(\Gamma)$ and $g \in H^{1/2}(\partial\Omega)$, find $\{u_1, u_2\} \in H^1(\Omega_1) \times H^1(\Omega_2)$ such that

$$-\epsilon_1 \triangle u_1 = f \quad \text{in} \quad \Omega_1 , \tag{1}$$

$$-\epsilon_2 \triangle u_2 = f \quad \text{in} \quad \Omega_2 , \tag{2}$$

$$u_1 = u_2 \quad \text{on} \quad \Gamma , \tag{3}$$

$$\epsilon_1 \frac{\partial u_1}{\partial \nu} - \epsilon_2 \frac{\partial u_2}{\partial \nu} = \sigma \quad \text{on} \quad \Gamma . \tag{4}$$

$$u_2 = g \quad \text{on} \quad \partial\Omega , \tag{5}$$

Here $\nu$ is the unit normal vector on $\Gamma$ directed from $\Omega_1$ to $\Omega_2$ and $\epsilon_i$ is a positive parameter $(i = 1, 2)$.

The sysytem consisting of equations (1)-(5) is called the diffraction problem ([4]) and the equation (3)-(4) imposed on the surface $\Gamma$ is also called the transmission conditions ([5]). The diffraction problems are arisen in various sciences. One of such examples can be found in the context of electricity and $\{\epsilon_1,\ \epsilon_2\}$ is corresponding to the dielectric constant of the material $\{\Omega_1, \Omega_2\}$.

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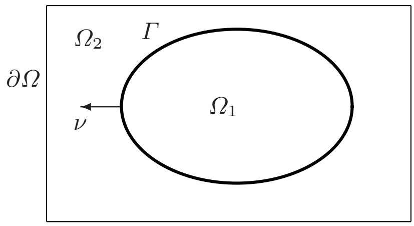


**Fig. 1.** Interface $\Gamma$ and unit normal $\nu$

From the view point of numerical analysis, one gives $w \in H^{1/2}(\Gamma)$, solves the Dirichlet problems in each domain $\Omega_i$ $(i = 1, 2)$:

$$\begin{cases} -\epsilon_1 \triangle u_1(w) = f \quad \text{in} \quad \Omega_1 , \\ u_1(w) = w \quad \text{on} \quad \Gamma , \end{cases} \tag{6}$$

$$\begin{cases} -\epsilon_2 \triangle u_2(w) = f \quad \text{in} \quad \Omega_2 , \\ u_2(w) = w \quad \text{on} \quad \Gamma, \\ u_2(w) = g \quad \text{on} \quad \partial\Omega , \end{cases} \tag{7}$$

and finds $w_0 \in H^{1/2}(\Gamma)$ satisfying

$$\epsilon_1 \frac{\partial u_1(w_0)}{\partial \nu} - \epsilon_2 \frac{\partial u_2(w_0)}{\partial \nu} = \sigma \quad \text{on} \quad \Gamma . \tag{8}$$

Hence introducing the Dirichlet-Neumann map $T$ defined by

$$T : H^{1/2}(\Gamma) \ni w \to \epsilon_1 \frac{\partial u_1(w)}{\partial \nu} - \epsilon_2 \frac{\partial u_2(w)}{\partial \nu} \in H^{-1/2}(\Gamma),$$

Problem I is reduced to find $w_0$ satisfying

$$Tw_0 = \sigma \tag{9}$$

(see also [6]). Therefore one of purposes of this paper is to present a direct method for solving (9).

This paper is organized as follows. Section 2 describes the finite difference scheme with distribution. Section 3 is devoted to study the construction of approximate solutions from the viewpoint of the successive elimination of lines. Finally we shall present a numerical algorithm by use of FFT and SEL in Section 4.

# 2 Finite Difference Scheme with Distribution

## 2.1 Distribution Formulation

Before proceeding to the finite difference scheme, we reform Problem I as follows:
**Problem II.** For $f \in L^2(\Omega),\ \sigma \in L^2(\Gamma)$ and $g \in H^{1/2}(\partial\Omega)$, find $u \in H^1(\Omega)$ such that

$$- \ \mathrm{div}\,(a(x,y)\,\nabla u) \ = \ f + \sigma\,\delta_\Gamma \quad \text{in} \quad D'(\Omega)\ , \tag{10}$$

$$u \ = \ g \quad \text{on} \quad \partial\Omega\ . \tag{11}$$

Here $a(x,y) = \epsilon_1\,\chi_{\Omega_1}(x,y) + \epsilon_2\ \chi_{\Omega_2}(x,y)$ where $\chi_{\Omega_i}$ $(i = 1,2)$ is defined by

$$\chi_{\Omega_i}(x,y) = \begin{cases} 1 & if\ (x,y) \in \Omega_i \\ 0 & if\ (x,y) \notin \Omega_i \end{cases}$$

and $\delta_\Gamma$ is the distribution with the support on $\Gamma$.
In fact, since the equation

$$\mathrm{div}\,(a(x,y)\,\nabla u) = a(x)\ \Delta u - (\epsilon_1\ \frac{\partial u_1}{\partial \nu}\ -\ \epsilon_2\ \frac{\partial u_2}{\partial \nu}) \quad \text{in} \ \ D'(\Omega)$$

holds for any $u \in H^1(\Omega)$, it follows that Problem I is equivalent to Problem II ([1], [7]).
Hereafter the discretization of Problem II will be used in stead of Problem I .

## 2.2 Finite Difference Scheme with Distribution

Without loss of generality we assume that $g = 0$ and that $\Omega$ is the unit square in $R^2$, i.e., $\Omega = \{(x,y)|\ 0 < x,y < 1\ \}$. Let $h \in R$ be a mesh size such that $h = 1/n$ for an integer $n$ and set $\Delta\ x = \Delta\ y = h$. We associate with it the set of the grid points:

$$\begin{aligned}\overline{\Omega}_h &= \{m_{i,j} \in R^2 \mid m_{i,j} = (i\ h,\ j\ h),\ 0 \le i,j \le n\},\\ \Omega_h &= \{m_{i,j} \in R^2 \mid m_{i,j} = (i\ h,\ j\ h),\ 1 \le i,j \le n-1\}.\end{aligned}$$

With each grid point $m_{i,j}$ of $\overline{\Omega}_h$, we associate the panel $\omega^0_{i,\ j}$ with center $m_{i,j}$:

$$\omega^0_{i,j} \equiv \Big(\ (i-1/2)h,\ (i+1/2)h\ \Big] \times \Big(\ (j-1/2)h,\ (j+1/2)h\ \Big], \tag{12}$$

and the cross $\omega^1_{i,j}$ with center $m_{i,j}$:

$$\omega^1_{i,j}\ =\ \omega^0_{i+1/2,j} \cup\ \omega^0_{i-1/2,j} \cup\ \omega^0_{i,j+1/2} \cup\ \omega^0_{i,j-1/2} \tag{13}$$

where $e_i$ denotes the $i$ th unit vector in $R^2$ and we set

$$\omega^0_{i\pm1/2,j}\ =\ \omega^0_{i,j} \pm \frac{h}{2}\ e_1, \quad \omega^0_{i,j\pm1/2}\ =\ \omega^0_{i,j} \pm \frac{h}{2}\ e_2. \tag{14}$$

Moreover using the datum in Problem I, we define

$$\begin{cases} a^E_{i,j} = \frac{1}{\Delta x \Delta y}\int_{\omega^0_{i+1/2,j}} a(x,y)\,dxdy, & a^W_{i,j} = \frac{1}{\Delta x \Delta y}\int_{\omega^0_{i-1/2,j}} a(x,y)\,dxdy, \\ a^N_{i,j} = \frac{1}{\Delta x \Delta y}\int_{\omega^0_{i,j+1/2}} a(x,y)\,dxdy, & a^S_{i,j} = \frac{1}{\Delta x \Delta y}\int_{\omega^0_{i,j-1/2}} a(x,y)\,dxdy, \\ f_{i,j} = \frac{1}{\Delta x \Delta y}\int_{\omega^0_{i,j}} f(x,y)\,dxdy, & \sigma_{i,j} = \frac{1}{\Delta\, l_{i,j}}\int_{\Gamma\,\cap\,\omega^0_{ij}} \sigma(s)\,ds, \\ \quad \Delta\, l_{i,j} = \int_{\Gamma\,\cap\,\omega^0_{i,j}} ds. \end{cases} \tag{15}$$

We then define the discrete equation of Problem II as follows.
**Problem F.** Find $\{u_{i,j}\}$ $(1 \le i,\ j \le n-1)$ such that

$$\begin{aligned} &-\frac{1}{\Delta x}\left( a^E_{i,j}\ \frac{u_{i+1,j} - u_{ij}}{\Delta x} - a^W_{i,j}\ \frac{u_{ij} - u_{i-1,j}}{\Delta x}\right) \\ &-\frac{1}{\Delta y}\left( a^N_{i,j}\ \frac{u_{i,j+1} - u_{i,j}}{\Delta y} - a^S_{i,j}\ \frac{u_{i,j} - u_{i,j-1}}{\Delta y}\right) \\ &\quad = f_{i,j} + \frac{\Delta l_{i,j}}{\Delta x\,\Delta y}\,\sigma_{i,j}, \quad 1 \le i,j \le n-1. \end{aligned} \tag{16}$$

Now introducing the step function $\theta_{i,j}$ :

$$\theta_{i,j}(x,y) = \begin{cases} 1, & (x,y) \in \omega^0_{i,j} \\ 0, & (x,y) \notin \omega^0_{i,j} \end{cases},$$

and let us define the piecewise functions $\sigma_h$ and $u_h$ by

$$\begin{aligned} \sigma_h &= \textstyle\sum_{i,j=1}^{n-1} \frac{\Delta l_{i,j}}{\Delta x\,\Delta y}\ \sigma_{i,j}\ \theta_{i,j}(x,y), \\ u_h &= \textstyle\sum_{i,j=1}^{n-1} u_{i,j}\ \theta_{i,j}(x,y) \end{aligned} \tag{17}$$

respectively. We then have ([3])

**Theorem 1.** *(i)* $\sigma_h \to \sigma\cdot\delta_\Gamma$ *in* $D'(\Omega)$,
*(ii)* $u_h \to u$ *weakly in* $L^2(\Omega)$, $u \in H^1(\Omega)$, *and*
*(iii)* $u$ *is the solution of Problem II in the sense of distrubution.*

## 3 Construction of Approximate Solutions (SEL)

### 3.1 Vector Valued Equations

In this section we state the direct method which we call the successive elimination of lines. Instead of the $(n-1)^2$ unknowns $u_{i,j}$ in the discrete equation (16), we introduce the line vectors such that

$$U_i = {}^t[u_{i,1},\ u_{i,2}, \cdots,\ u_{i,n-1}\ ]\ \ (1 \le i \le n-1).$$

Then Problem F *w.r.t.* $\{u_{i,j}\}$ is reduced to Problem M *w.r.t.* $\{U_i\}$ from the equations (16),
**Problem M.** Find $U_i$ $(1 \le i \le n-1)$ satisfying

$$A^\epsilon_i\, U_i = A^W_i\, U_{i-1} + A^E_i\, U_{i+1} + F_i \quad (1 \le i \le n-1) \tag{18}$$

where $U_0 = 0, U_n = 0$, $F_i$ is given by the data $\{f, \sigma, g\}$ , $A_i^\epsilon$ is a tridiagonal matrix defined by

$$A_i^\epsilon = \begin{pmatrix} a_{i,1}^\epsilon & -a_{i,1}^N & 0 & \cdots & \cdots & \cdots & 0 \\ -a_{i,2}^S & a_{i,2}^\epsilon & -a_{i,2}^N & 0 & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & -a_{i,n-2}^S & a_{i,n-2}^\epsilon & -a_{i,n-2}^N \\ 0 & \cdots & \cdots & \cdots & 0 & -a_{i,n-1}^S & a_{i,n-1}^\epsilon \end{pmatrix} \tag{19}$$

$A_i^W, A_i^E$ are the diagonal matrices given by

$$A_i^W = \mathrm{diag}[a_{i,j}^W]_{1\le j\le n-1} \text{ and } A_i^E = \mathrm{diag}[a_{i,j}^E]_{1\le j\le n-1} \tag{20}$$

and

$$a_{i,j}^\epsilon = a_{i,j}^W + a_{i,j}^E + a_{i,j}^S + a_{i,j}^N.$$

Moreover considering the geometry of the domain $\Omega$ and the interface $\Gamma$ we first introduce sets of interface lattice points $\Gamma_h$ and boundary lattice points $\partial\Omega_h$ as follows;
(i) $\Gamma_h = \{P_{i,j} = (ih, jh) \mid \Gamma \cap \omega_{i,j}^1 \neq \emptyset\}$,
(ii) $\partial\Omega_h = \overline{\Omega}_h \setminus \Omega_h$.
Using the above notion, we divide the unknown vector $\{U_i\}$ into two parts.

For each $U_i = \{u_{i,j}\}_{1\le j\le n-1}$, we define $U_i' = \{u_{i,j}'\}_{1\le j\le n-1}$ and $W_i = \{w_{i,j}\}_{1\le j\le n-1}$ as follows;

$$u_{i,j}' = \begin{cases} 0 & \text{if } P_{i,j} \in \Gamma_h \\ u_{i,j} & \text{if } P_{i,j} \in \Omega_h\backslash\Gamma_h, \end{cases} \qquad w_{i,j} = \begin{cases} u_{i,j} & \text{if } P_{i,j} \in \Gamma_h \\ 0 & \text{if } P_{i,j} \in \Omega_h\backslash\Gamma_h \end{cases} \tag{21}$$

and devide $U_i$ into two parts by

$$U_i = U_i' + W_i. \tag{22}$$

We then introduce the new vector $\{V_i\}$ defined by

$$V_i = A_i^W U_i' \ ( = A_i^E U_i' \ ) \qquad (1 \le i \le n-1). \tag{23}$$

From the definition of $\{U_i'\}$ and $\{V_i\}$, we get ([2])

**Lemma 1.** $A_i^\epsilon U_i' = B\, V_i$, $A_i^w U_{i-1}' = V_{i-1}$ *and* $A_i^E U_{i+1}' = V_{i+1}$ *hold* $(1 \le i \le n-1)$ *. Here $B$ is a block tridiagonal matrix in the discretization of the Laplace operator in $\Omega$ with homogeneous Dirichlet boundary conditions. i.e.,* $B = [b_{ij}]$ *is an* $(n-1)\times(n-1)$ *tridiagonal matrix such that* $B = \mathrm{tridiag}[-1,\ 4,\ -1]$.

Therefore the following equations are derived from (18),(22),(23) and Lemma 1.
**Problem PN.** Find $\{V_i\ ,\ W_i\}$ such that for $i(1 \le i \le n-1)$,

$$B\,V_i = V_{i-1} + V_{i+1} + \ F_i \ + \Big(\ A_i^W\, W_{i-1} - A_i^\epsilon\, W_i + A_i^E\, W_{i+1}\ \Big) \tag{24}$$

where $V_0 = V_n = W_0 = W_n = 0$.

### 3.2 Formulation of SEL

Applying the principle of the successive elimination of lines to (24), we have

**Theorem 2.** *$\{W_i\}_{1\le i\le n-1}$ in (24) is uniquely determined as follows.*

$$\begin{aligned} &\textstyle\sum_{i=1}^{k-1}\ {}^tP_l\ D(n-k,i)\ P\Big(\ -A_i^W\,W_{i-1}+A_i^{\epsilon}\,W_i-A_i^E\,W_{i+1}\Big) \\ +\ &\textstyle\sum_{i=k}^{n-1}\ {}^tP_l\ D(k,n-i)\ P\Big(\ -A_i^W\,W_{i-1}+A_i^{\epsilon}\,W_i-A_i^E\,W_{i+1}\Big) \\ =\ &\ {}^tP_l\Big(\ \textstyle\sum_{i=1}^{k-1}\ D(n-k,i)\ P\ F_i+\sum_{i=k}^{n-1}\ D(k,n-i)\ P\ F_i\Big) \end{aligned} \tag{25}$$

*for $(k,l)$ such that $m_{k,l}\in\Gamma_h$.*

Here $P(=(p_{i,j})_{1\le i,j\le n-1})$ is the othogonal matrix such that

$$p_{i,j}=\sqrt{\frac{2}{n}}\ sin\left(\frac{i\,j\,\pi}{n}\right)\ \ (1\le i,j\le n-1) \tag{26}$$

and $D(\ l,\ i)\ (1\le l,i\le n-1)$ is a diagonal matrix de$fi$ned by

$$D(l,i)=\text{diag}\left[\Big(\sinh(l\ \lambda_j)\ \sinh(i\ \lambda_j)\Big)/\Big(\sinh(n\ \lambda_j)\ \sinh(\lambda_j)\Big)\right]_{1\le j\le n-1} \tag{27}$$

where $\lambda_j=\text{arccosh}(2-\cos(j\pi/n))$.

*Remark 1.* $\{W_i\}$ corresponds to the approximate solution of (9).

*Remark 2.* Linear system w.r.t to $\{W_i\}$ is dramatically less than the total system w.r.t $\{u_{i,j}\}$. For example, let $\Omega=(-0.5,0.5)\times(-0.5,0.5)$ and $\Gamma: x^2+y^2=(1/4)^2$. Then using the notation $\{w_{ij}\}/\{u_{i,j}\}$ which means 'ratio' that is the percentage of the number of unknowns $\{w_{ij}\}$ to the total number of unknowns $\{u_{ij}\}$, we get the following table.

**Table 1.** Ratio

| Grids | $\{w_{ij}\}/\{u_{i,j}\}$ |
|---|---|
| $128^2$ | 2.68% |
| $256^2$ | 1.34% |

On the other hand, the remainder part $\{V_k\}_{1\le k\le n-1}$ of $\{U_i\}_{1\le i\le n-1}$ is calculated by the following linear algebra in stead of solving the linear systems.

**Theorem 3.** *$V_k$ is determined by*

$$\begin{aligned} V_k\ &=\textstyle\sum_{i=1}^{k-1}\ P\ D(n-k,i)\ P\Big(\ A_i^W\ W_{i-1}-A_i^{\epsilon}\ W_i+A_i^E\ W_{i+1}\Big) \\ &+\textstyle\sum_{i=k}^{n-1}\ P\ D(k,n-i)\ P\Big(\ A_i^W\ W_{i-1}-A_i^{\epsilon}\ W_i+A_i^E\ W_{i+1}\Big) \\ &+\ P\Big(\ \textstyle\sum_{i=1}^{k-1}\ D(n-k,i)\ P\ F_i+\sum_{i=k}^{n-1}\ D(k,n-i)\ P\ F_i\Big). \end{aligned} \tag{28}$$

From now on a new numerical algorithm using FFT is derived from Theorem 2 and 3.

# 4 Hybrid Numerical Algorithm Based on FFT and SEL

## 4.1 Hybrid Numerical Algorithm

Recalling Theorem 3 in the previous section, we notice that the essential part is the calculation of $P\alpha$ for any vector $\alpha =^t (\alpha_1, \cdots, \cdots, \alpha_{n-1})$ and that the row vectors of $P$ are bases of the discrete sine transformation.
In fact,

$$P\alpha = \sqrt{\frac{2}{n}} \begin{pmatrix} sin(\frac{1}{n}\pi) & sin(\frac{2}{n}\pi) & \cdots & \cdots & sin(\frac{n-1}{n}\pi) \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ sin(\frac{n-1}{n}\pi) & sin(\frac{2(n-1)}{n}\pi) & \cdots & \cdots & sin(\frac{(n-1)(n-1)}{n}\pi) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$$

from which the i-th component $\beta_i$ of $P\alpha$ has the form of

$$\beta_i = \sum_{j=1}^{n-1} p_{i,j}\alpha_j = \sum_{j=1}^{n-1} \sqrt{\frac{2}{n}} \sin\,(\frac{ij}{n}\pi) \cdot \alpha_j \ . \tag{29}$$

This means that $\{\beta_i\}$ is exactly the discrete sine transform for data $\alpha$.
Hence we are able to establish a numerical algorithm coupled with FFT and SEL since it is possible to make a program of the discrete sine transform via FFT( see [8]).
Therefore we summarize our numerical algorithm as follows.

**Hybrid Numerical Algorithm**
**1st step:** Calculate the solution $\{W_i\}$ on $\Gamma_h$ for the linear system (25).
**2nd step:** Compute $\{V_k\}$ on $\Omega_h \backslash \Gamma_h$ by use of the formulation in Theorem 3 and FFT we stated here.

## 4.2 Comparison of Calculation Speed

Let A and B denote the actual computing time using the matrix calculation in Theorem 3 and the above hybrid calculation respectively. Then we get the table under my computer condition as follows.

**Table 2.** Calculation speed

| Grids | A/B |
|---|---|
| $128^2$ | 4.1 |
| $256^2$ | 7.1 |

**Table 3.** Computer condition

| | |
|---|---|
| Pentium 4 | 2.4GHz |
| Memory | 512MB |
| Soft | Maple 8 |

## 5 Conclusions and Further Works

In this paper we described how to calculate numerically the diffraction problem by use of FFT. The formula of SEL was a mathematical approach and we showed that the hybrid calculation coupled with FFT and SEL was efficient to solve the diffraction problem.
In near future we will apply this method to the heat problem near earth's surface in environments which is described by the systems of diffusion equations with transmission conditions.

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