

# Chapter 4

## The Double Continuity of Algebra

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**Abstract** We consider Klein's double discontinuity between high school and university mathematics in relation to algebra as it is studied in both settings. We give examples of two kinds of continuities that might mend the break: (1) examples of how undergraduate courses in algebra and number theory can provide useful tools for prospective teachers in their professional work, as they design and sequence mathematical tasks, and (2) examples of how questions that arise in secondary pre-college mathematics can be extended and analyzed with methods from algebra and algebraic geometry, using both a careful analysis of algebraic calculations and the application of algebraic methods to geometric problems. We discuss useful sensibilities, for high school teachers and university faculty, that are suggested by these examples. We conclude with some recommendations about the content and structure of abstract algebra courses in university.

**Keywords** Klein · Double discontinuity · Algebra · Secondary school teaching

### 4.1 Introduction

The young university student found himself, at the outset, confronted with problems, which did not suggest, in any particular, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided,

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to discern any connection between this task and his university mathematics, he soon fell in with the time honoured way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching.

—Felix Klein, *Elementary Mathematics from an Advanced Standpoint* (1932, p. 1)

Does Klein's oft-quoted description of what he called the double discontinuity, given over 100 years ago, still hold today? The recommendations of the Conference Board on the Mathematical Sciences (CBMS 2012) for the mathematical education of teachers in the US were formulated on the premise that there was still a problem to be solved in at least one of the directions of the discontinuity: the transition from the university coursework of a prospective teacher to their practice in teaching secondary school. Chapter 2 of that report discusses some of the evidence for that point of view. In the other direction, de Guzmán et al. (1998) describe the discontinuity from the point of view of a secondary school student entering university, and find it particularly strong for prospective teachers.

Given the apparent persistence of the problem, it is reasonable to wonder if the double discontinuity results in part from a double discontinuity in the subject matter itself of the courses that students take. In the US, Wu (2015) has described what he calls *textbook school mathematics* as a subject alienated from genuine mathematics. The US Common Core State Standards for Mathematics were motivated in part by a desire to lessen the distance between school mathematics and university mathematics. Such projects need stories of continuity. In this paper we consider continuities between secondary school and university mathematics in both directions, focusing on the subject of algebra.

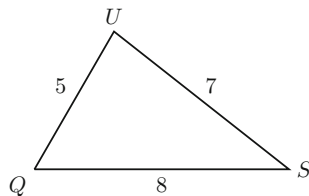
## 4.2 From University to Secondary School

In this section we follow mathematical threads from a course in abstract algebra or number theory to secondary school mathematics.

A common topic of discussion among teachers is finding tasks that can be used to launch a new topic. They want the answers to be simple, so that computational overhead does not get in the way of the ideas.

For example, consider the topic of the law of cosines, to be introduced by the task of finding the measure of  $\angle UQS$  in Fig. 4.1. This task has a particularly simple answer,  $60^\circ$ . Examples like this are prized by teachers; they are traded at department meetings and sought after online. The problem of finding such examples is a problem in task design. A teacher with university background in abstract algebra or algebraic number theory can apply that knowledge to task design and find general methods for constructing such examples. This can be viewed as a sort of applied mathematics for the profession of teaching, as described at the elementary level in Ball et al. (2008).

**Fig. 4.1** What is the measure of  $\angle UQS$ ?



### 4.2.1 Pythagorean Triples

Before considering the task design problem of finding nice triangles with a  $60^\circ$  angle, we consider the simpler problem of finding nice triangles with a  $90^\circ$  angle, that is, nice right triangles. The triangle with side lengths 3, 4, and 5 is such a triangle because the lengths satisfy the Pythagorean identity:

$$3^2 + 4^2 = 5^2.$$

Another example is the triangle with side lengths 5, 12, and 13, because

$$5^2 + 12^2 = 13^2.$$

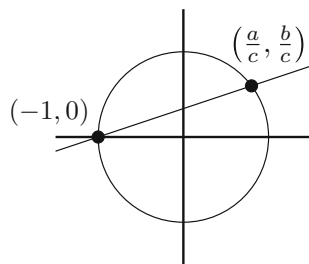
There are infinitely many such Pythagorean triples, that is, triples of positive integers  $(a, b, c)$  such that

$$a^2 + b^2 = c^2.$$

Diophantus of Alexandria developed, around 250 CE, a geometric method for generating such triples. In modern language, he realized that a rational point on the unit circle (the graph of  $x^2 + y^2 = 1$ ), when written in the form  $(\frac{a}{c}, \frac{b}{c})$ , produces a Pythagorean triple (Fig. 4.2):

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \quad \Rightarrow \quad a^2 + b^2 = c^2.$$

**Fig. 4.2** The method of Diophantus



One can get such a rational point by forming a line with positive rational slope through the point  $P = (-1, 0)$  and intersecting the line with the circle. The second intersection point will then be rational. Hence, it was known early on that there are infinitely many Pythagorean triples [for details, see Cuoco and Rotman (2013)].

In addition to this geometric method, there is an algebraic method for generating Pythagorean triples, using complex numbers and the observation that

$$x^2 + y^2 = (x + yi)(x - yi). \quad (4.1)$$

The sum of two squares can thus be written as the product of a complex number and its complex conjugate. So, if you want integers  $a$  and  $b$  so that  $a^2 + b^2$  is a perfect square, you might write the sum of these two squares as

$$a^2 + b^2 = (a + bi)(a - bi)$$

and try to make each factor on the right-hand side the square of a complex number with integer real and imaginary parts. And it is within the scope of secondary school mathematics to show that if  $a + bi = (r + si)^2$ , then  $a - bi = (r - si)^2$ . So, for example,

$$\begin{aligned} (3 + 2i)^2 &= 5 + 12i \quad \text{and} \\ (3 - 2i)^2 &= 5 - 12i. \end{aligned}$$

So,

$$\begin{aligned} 5^2 + 12^2 &= (5 + 12i)(5 - 12i) \\ &= (3 + 2i)^2(3 - 2i)^2 \\ &= ((3 + 2i)(3 - 2i))^2 \\ &= (3^2 + 2^2)^2 \\ &= 13^2, \end{aligned}$$

and we have the Pythagorean triple  $(5, 12, 13)$ .

### 4.2.2 *The Algebraic Method from a Higher Standpoint*

To extend the applicability of this method, it helps to look at it from a higher standpoint. Complex numbers of the form  $a + bi$  where  $a$  and  $b$  are integers are called *Gaussian integers*. The set of all Gaussian integers is denoted by  $\mathbb{Z}[i]$ , because  $\mathbb{Z}$  denotes the system of ordinary integers that is the focus of much of arithmetic in school; so  $\mathbb{Z}[i]$  is obtained from  $\mathbb{Z}$  by adjoining  $i$ . Both  $\mathbb{Z}$  and  $\mathbb{Z}[i]$  are

endowed with two operations—addition and multiplication. The properties of addition and multiplication that allow one to calculate with integers also hold in  $\mathbb{Z}[i]$ —order does not matter in addition or multiplication, multiplication distributes over addition, and so on. Formally, both systems are examples of *commutative rings*, and, in fact,  $\mathbb{Z}$  is a subring of  $\mathbb{Z}[i]$ .

The complex conjugate of a complex number  $z = a + bi$  is denoted by  $\bar{z}$ , so  $\overline{a + bi} = a - bi$ . This operation of multiplying a Gaussian integer by its complex conjugate is a map from  $\mathbb{Z}[i]$  to  $\mathbb{Z}$  called the *norm* and denoted by  $N$ :

$$N(z) = z\bar{z}.$$

The norm map has the following properties:

1.  $N(zw) = N(z)N(w)$  for all Gaussian integers  $z$  and  $w$ .
2. Hence, if  $z$  is a Gaussian integer, then

$$N(z^2) = (N(z))^2.$$

3. If  $z = a + bi$ ,  $N(z) = a^2 + b^2$ , a non-negative integer.

Put in the context of Pythagorean triples, item (3) shows that we are looking for Gaussian integers whose norms are perfect squares. Item (2) tells us how to do that:

*To make  $N(z)$  a square in  $\mathbb{Z}$ , make  $z$  a square in  $\mathbb{Z}[i]$ .*

This gives an easily programmed algorithm for generating Pythagorean triples, giving secondary school teachers a useful tool for their lesson planning. Table 4.1, generated in a computer algebra system, shows  $(r + si)^2 = a + ib$  and the corresponding norm  $c$ . The three integers  $a$ ,  $b$ , and  $c$  form a Pythagorean triple.

### 4.2.3 Using Norms to Construct Triangles with a $60^\circ$ Angle

We now return to the task design problem of finding examples like Fig. 4.1. The problem is to find a triple of positive integers  $(a, b, c)$  that are side-lengths of a triangle with a  $60^\circ$  angle.

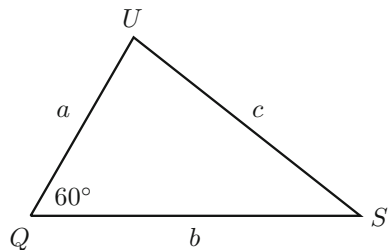
Applying the law of cosines to the triangle in Fig. 4.3, we have

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos 60^\circ \\ &= a^2 + b^2 - 2ab \frac{1}{2} \\ &= a^2 - ab + b^2 \end{aligned}$$

**Table 4.1**  $(r + si)^2$  and the resulting norm

	$s = 1$	$s = 2$	$s = 3$	$s = 4$
$r = 2$	$3 + 4i, 5$			
$r = 3$	$8 + 6i, 10$	$5 + 12i, 13$		
$r = 4$	$15 + 8i, 17$	$12 + 16i, 20$	$7 + 24i, 25$	
$r = 5$	$24 + 10i, 26$	$21 + 20i, 29$	$16 + 30i, 34$	$9 + 40i, 41$
$r = 6$	$35 + 12i, 37$	$32 + 24i, 40$	$27 + 36i, 45$	$20 + 48i, 52$
$r = 7$	$48 + 14i, 50$	$45 + 28i, 53$	$40 + 42i, 58$	$33 + 56i, 65$
$r = 8$	$63 + 16i, 65$	$60 + 32i, 68$	$55 + 48i, 73$	$48 + 64i, 80$

**Fig. 4.3** A nice triangle



So, we want integers  $(a, b, c)$  so that

$$c^2 = a^2 - ab + b^2. \tag{4.2}$$

Examples of such triples are  $(5, 8, 7)$  (corresponding to the example above) and  $(15, 7, 13)$ .

We will call such a triple an Eisenstein triple. We want to find a system analogous to  $\mathbb{Z}[i]$  in which the right-hand side of (4.2) is a norm. Such an expression arises naturally in number theory courses that treat roots of unity.

Just as we can form the ring of Gaussian integers  $\mathbb{Z}[i]$  by adjoining the fourth root of unity  $i$  to  $\mathbb{Z}$ , so we can form the ring of Eisenstein integers by adjoining the cube roots of unity, that is, the roots of  $x^3 - 1 = 0$ . Because

$$x^3 - 1 = (x - 1)(x^2 + x + 1),$$

the three roots are

$$\left\{ 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2} \right\}.$$

Let

$$\omega = \frac{-1 + i\sqrt{3}}{2},$$

and consider  $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$ . This is a ring (the Eisenstein integers) with structural similarities to  $\mathbb{Z}$  and  $\mathbb{Z}[i]$  [details are in Cuoco and Rotman (2013)]. In particular, because

$$\begin{aligned} \omega + \bar{\omega} &= -1 \quad \text{and} \\ \omega\bar{\omega} &= 1, \end{aligned}$$

a direct calculation shows that

$$\begin{aligned} N(z) &= (a + b\omega)(\overline{a + b\omega}) \\ &= (a + b\omega)(a + b\bar{\omega}) \\ &= a^2 - ab + b^2. \end{aligned}$$

Hence, the same mantra applies:

*To make  $N(z)$  a square in  $\mathbb{Z}$ , make  $z$  a square in  $\mathbb{Z}[\omega]$ .*

Once again, teachers have a method for generating Eisenstein triples (see Table 4.2).

### 4.2.4 What Is to Be Learned from This?

At one level, the methods given above could be viewed as no more than charming tricks of no great consequence in mathematics education. It is certainly true that

**Table 4.2**  $(r + s\omega)^2$  and the resulting norm

	$s = 1$	$s = 2$	$s = 3$	$s = 4$
$r = 2$	$3 + 3\omega, 3$			
$r = 3$	$8 + 5\omega, 7$	$5 + 8\omega, 7$		
$r = 4$	$15 + 7\omega, 13$	$12 + 12\omega, 12$	$7 + 15\omega, 13$	
$r = 5$	$24 + 9\omega, 21$	$21 + 16\omega, 19$	$16 + 21\omega, 19$	$9 + 24\omega, 21$
$r = 6$	$35 + 11\omega, 31$	$32 + 20\omega, 28$	$27 + 27\omega, 27$	$20 + 32\omega, 28$
$r = 7$	$48 + 13\omega, 43$	$45 + 24\omega, 39$	$40 + 33\omega, 37$	$33 + 40\omega, 37$
$r = 8$	$63 + 15\omega, 57$	$60 + 28\omega, 52$	$55 + 39\omega, 49$	$48 + 48\omega, 48$
$r = 9$	$80 + 17\omega, 73$	$77 + 32\omega, 67$	$72 + 45\omega, 63$	$65 + 56\omega, 61$
$r = 10$	$99 + 19\omega, 91$	$96 + 36\omega, 84$	$91 + 51\omega, 79$	$84 + 64\omega, 76$

secondary school mathematics teachers can get by without them. However, when viewed not as methods but as examples of a certain sensibility, they acquire greater significance. That sensibility is a tendency to view the mathematics learned in university as a useful tool in teaching secondary mathematics. This is a useful sensibility for high school teachers to have.

For example, the way of thinking about arithmetic in complex numbers as “algebra with  $i$ ” with an extra simplification rule—exemplified above—is often discouraged in secondary school mathematics. But it has quite a solid pedigree in modern algebra and can provide a glimpse of the reduction technique used to construct splitting fields for polynomials. Another example: In polynomial algebra, being explicit about the interplay between formal and functional thinking (something that is often blurred in secondary texts) helps students develop an appreciation for the “two faces” of algebra (Weyl 1995).

More generally, major themes in algebra—structure, extension, decomposition, reduction, localization, and representation—can help teachers bring coherence and parsimony to the entire secondary school curriculum.

### 4.3 From Secondary School to University

Now we look at a couple of mathematical threads that go in the opposite direction, from secondary school to university. Or, since the exact boundary between secondary school and university varies from country to country, it might be better to consider these simply as examples that go from some point in secondary school to a more advanced point, be it in secondary school, university, or beyond.

#### 4.3.1 Ptolemy’s Theorem

Consider the following secondary school mathematics problem.<sup>1</sup>

Given a cyclic quadrilateral whose sides are 2, 3, 5, 6, find the length of the square of the diagonal which makes a triangle with sides of length 2 and 3 (see Fig. 4.4).

Because the quadrilateral is inscribed in a circle,  $\angle ABC$  is supplementary to  $\angle CDA$ , so  $\phi = \theta - 180$ . Applying the Law of Cosines to both triangles formed by the diagonal  $AC$ , we get

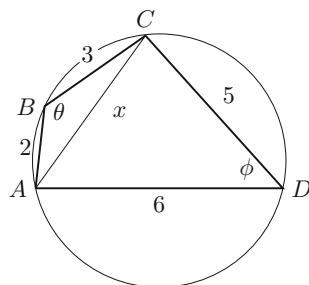
$$\begin{aligned}x^2 &= 4 + 9 - 2 \cdot 6 \cos \theta \\ &= 25 + 36 + 2 \cdot 30 \cos \theta\end{aligned}$$

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<sup>1</sup>We are indebted to Dick Askey for suggesting the sequence of ideas developed here.



**Fig. 4.4** A cyclic quadrilateral



Eliminating  $\cos \theta$  and solving for  $x$  yields  $x = \sqrt{21}$ . Now consider the same problem with a general quadrilateral, as in Fig. 4.5. The same method yields

$$x^2 = \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd}.$$

The expression on the right provides a wonderful opportunity for students to exercise what one might call algebraic insight. At first glance it is not obvious how to factor the numerator, but if one regroups the products in a way that shares the squared term with the other factors, it becomes easy to see:

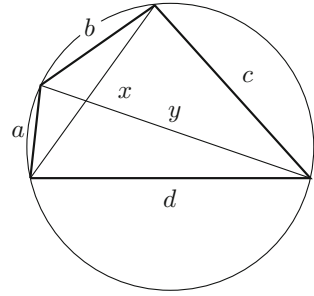
$$x^2 = \frac{(bc)(bd) + (ac)(ad) + (ac)(bc) + (ad)(bd)}{ab + cd} = \frac{(ac + bd)(ad + bc)}{ab + cd}$$

Another exercise in algebraic insight is to imagine what the corresponding expression for  $y^2$  would look like. One could repeat the calculation, or one could simply observe that  $y$  is in the same position with respect to  $(b, c, d, a)$  as  $x$  is with respect to  $(a, b, c, d)$ . Therefore the formula for  $y^2$  is obtained by performing the cyclic permutation  $a \rightarrow b \rightarrow c \rightarrow d \rightarrow a$ . Without actually writing the expression down, one can contemplate the effect of the permutation on the rightmost expression for  $x^2$ . The three parenthetical factors in that expression come in two types. Two of them,  $ab + cd$  and  $ad + bc$ , are obtained by multiplying pairs of adjacent sides and adding the resulting products. One of them,  $ac + bd$ , is obtained by multiplying pairs of opposite sides and adding the resulting products. The permutation is going to swap the first two types and leave the second type unchanged. This has the effect of causing a lot of cancellation when you multiply  $x^2$  and  $y^2$ . The swapped terms cancel each other out and we are left with

$$x^2y^2 = (ac + bd)^2,$$

or

**Fig. 4.5** A general cyclic quadrilateral



$$xy = ac + bd.$$

This is Ptolemy's theorem, a beautiful generalization of Pythagoras's theorem.

**Theorem (Ptolemy)** *In a quadrilateral inscribed in a circle, the product of the diagonals is the sum of the products of opposite sides.*

Pythagoras's theorem is the special case where the quadrilateral is a rectangle.

### 4.3.2 A Question from a Secondary School Class

This section is inspired by a story from the PROMYS for Teachers program at Boston University, recounted in Rosenberg et al. (2008). It starts with a question that could come up in a secondary school geometry class:

If two triangles have the same perimeter and same area, are they congruent?

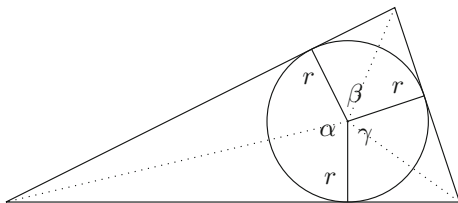
It is natural to assume the answer is no, if only on the grounds that if the answer were yes it would be a well known theorem. However, it turns out to be surprisingly difficult to come up with counterexamples. From an advanced point of view, the reason for this is that the counterexamples live on a curve which, unlike the circle in Sect. 4.2.1, is not easy to parameterize. We briefly sketch the derivation of that curve here.

We want to parameterize the family of triangles with fixed perimeter  $p$  and fixed area  $A$ . The radius  $r$  of the inscribed circle of such a triangle is related to  $A$  and  $p$  by the equation

$$A = \frac{p}{2}r. \tag{4.3}$$

This can be seen by decomposing the triangle into three triangles with bases on the sides of the triangle and vertices at the incenter, and adding up their areas, taking note of the fact that the altitude of each of the smaller triangles is  $r$  (see Fig. 4.6). It follows from (4.3) that all the triangles in the family have the same in

**Fig. 4.6** Triangle and inscribed circle



radius  $r$ , and they can all be circumscribed around a fixed circle, as in Fig. 4.6. We consider the space of all triangles circumscribed around this circle, obtained by varying the angles  $\alpha$ ,  $\beta$ , and  $\gamma$  and keeping the base horizontal. This is a two parameter space, since the angles are constrained by the condition that they must add up to  $2\pi$ . Every triangle is similar to a triangle in this space, since every triangle can be scaled to have inradius  $r$ . Our family of triangles with perimeter  $p$  and area  $A$  is a curve within that space, defined by a constraint on the angles  $\alpha$ ,  $\beta$ , and  $\gamma$ , which we now derive.

Another way of decomposing the triangle is to break it into 3 quadrilaterals formed by the radii and the segments into which the sides are divided by the perpendiculars from the incenter. Since the center of the inscribed circle is the intersection of the angle bisectors, these quadrilaterals are divided into pairs of congruent right triangles by the lines from the vertices to the center of the inscribed circle (dotted in Fig. 4.6). Therefore these lines also bisect the angles  $\alpha$ ,  $\beta$ , and  $\gamma$ . Adding up the lengths of the 6 line segments around the perimeter we get

$$p = 2r \left( \tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} \right). \quad (4.4)$$

From Eqs. (4.3) to (4.4) we see that, in our family of triangles with fixed area  $A$  and fixed perimeter  $p$ , the sum of the tangents is also fixed;

$$\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} = k, \quad \text{where } k \text{ is the constant } \frac{p^2}{4A}. \quad (4.5)$$

We can get an algebraic equation out of this by choosing parameters  $x = \tan(\alpha/2)$  and  $y = \tan(\beta/2)$ . Since  $\alpha + \beta + \gamma = 2\pi$ , we have

$$\frac{\gamma}{2} = \pi - \frac{\alpha}{2} - \frac{\beta}{2},$$

so

$$\tan \left( \frac{\gamma}{2} \right) = \tan \left( \pi - \frac{\alpha}{2} - \frac{\beta}{2} \right) = -\tan \left( \frac{\alpha}{2} + \frac{\beta}{2} \right) = -\frac{x+y}{1-xy}.$$

Referring back to Eq. (4.5) we obtain, for fixed  $k$ , the equation

$$x + y - \frac{x + y}{1 - xy} = k,$$

or equivalently

$$x^2y + xy^2 - kxy + k = 0. \quad (4.6)$$

This defines a curve in the  $xy$ -plane called an elliptic curve. Our original triangle gives a point on this curve; conversely, given a point on the curve with  $x > 0$  and  $y > 0$ , we can reconstruct a triangle circumscribed around a circle of radius  $r$  with area  $A$  and perimeter  $p$ . Moreover, one can verify that if  $x$  and  $y$  are rational numbers, then  $A$  and  $p$  are also rational numbers.

The method of finding rational points on elliptic curves using the secant method is well-developed in number theory and has a venerable history. We won't describe it further here, referring the interested reader to (Silverman and Tate 1994). We conclude with an example which is enjoyable to carry out by hand. The right triangle with sides 3, 4, and 5 corresponds to a point on the curve defined by (4.6) with  $k = 6$ . In fact there are six points, depending on which side you choose as base and how you orient the triangle:  $(1, 2)$ ,  $(2, 1)$ ,  $(1, 3)$ ,  $(3, 1)$ ,  $(2, 3)$ , and  $(3, 2)$ . Using the secant method one can find the rational point  $(\frac{54}{35}, \frac{25}{21})$  on this curve, which corresponds to the triangle with side lengths  $\frac{41}{15}$ ,  $\frac{101}{21}$ , and  $\frac{156}{35}$ . Our method shows that this triangle has perimeter 12 and area 6, just like the  $(3, 4, 5)$  triangle.

The journey does not stop here. The family of elliptic curves described here is closely related to the elliptic surfaces studied in (van Luijk 2007). Thus a journey that started in high school leads to the frontiers of research today.

### 4.3.3 *What Is to Be Learned from This?*

Again, we present these two examples as examples of a sensibility. Just as it is useful for high school teachers to view the mathematics learned in university as a useful tool in teaching secondary mathematics, it is useful for faculty at universities teaching prospective high school teachers to have the sensibility that the mathematics of high school can be mined for advanced examples in their courses. These two sensibilities are intertwined; together they could resolve the double discontinuity.

## 4.4 Implications for Teaching Abstract Algebra

The examples in Sect. 4.2 illustrate some of the many concrete applications of algebra, algebraic geometry, and number theory to the work of teaching mathematics at the secondary level, applications that are often missed in undergraduate courses and professional development programs. The examples in Sect. 4.3 illustrate ways in which high school mathematics can be applied to deep questions that show the utility of abstraction and of seeking structure in expressions. Courses in abstract algebra, in particular, would be much more useful to prospective teachers (and all undergraduate students, we claim) if they incorporated examples like these, examples that show how abstract methods provide useful tools for the day-to-day work of teaching and how questions and methods that live in high school mathematics can motivate some of those abstract methods.

We conclude with some suggestions for preservice courses in abstract algebra that we propose would contribute to closing the distance between school mathematics, university mathematics, and mathematics as it is practiced by mathematics professionals.

1. Abstractions should be capstones, not foundations—they should be motivated with concrete examples whenever possible. This “experience before formality” is one of the hallmarks of mathematical work, and it is sometimes missing from dogmatic expositions of established mathematical results.
2. Groups should be introduced in an historically faithful way, as part of an introduction to the Galois theory of polynomial equations.
3. The structural similarities between  $\mathbb{Z}$  and  $k[x]$  ( $k$  a field) should be a major focus. These are the two major systems developed in school mathematics, and their underlying structure (that of a principal ideal domain) can form a bridge between arithmetic and algebra.
4. The development should follow the historical evolution of the ideas, showing how algebra evolved from techniques for solving equations to a study of systems in which the “rules of algebra” hold.
5. Applications should include those that are foundational for high school teaching. First, such applications can enrich and bring coherence to the mathematics that students study; second, they can help teachers in their professional work, such as designing lessons and tasks and sequencing topics; finally, to quote from (CBMS 2012, p. 54), they can help teachers understand that

“the mathematics of high school” does not mean simply the syllabus of high school mathematics, the list of topics in a typical high school text. Rather it is the structure of mathematical ideas from which that syllabus is derived.

Earlier texts, like the celebrated (Birkhoff and MacLane 2008), met many of these principles, except for item 5 above. The text (Cuoco and Rotman 2013) is one example of a course that attempts to meet all of them.<sup>2</sup> Some of the applications included in that course are:

- Pythagorean triples and the method of Diophantus.
- A historical development of  $\mathbb{C}$ .
- The mathematics of task design.
- Periods of repeating decimals.
- Cryptography.
- Lagrange interpolation and the Chinese Remainder Theorem.
- Ruler and compass constructions.
- Gauss' construction of the regular 17-gon.
- The arithmetic of  $\mathbb{Z}[i]$  and  $\mathbb{Z}[\omega]$ .
- Solvability by radicals.
- Fermat's Last Theorem for exponents 3 and 4.

These are a mere sample of the ideas that have direct application to the work of teaching. Again, from (CBMS 2012, p. 66), "It is impossible to learn all the mathematics one will use in any mathematical profession, including teaching, in four years of college—teachers need opportunities to learn further mathematics throughout their careers." Professional development programs that developed other applications—in geometry, analysis, and statistics, for example—could carry this program forward for career-long learning for practicing teachers.

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<sup>2</sup>Rotman died in 2016. Joe was a mathematician and teacher *extraordinaire*.

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