Chapter 34
Enactive Metaphorising in the Learning of Mathematics

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Abstract We argue that an approach to the learning of mathematics based on enactive (bodily acted out) metaphorising may significantly help in alleviating the cognitive abuse millions of children worldwide suffer when exposed to mathematics. We present illustrative examples of enactive metaphoric approaches in the context of problem posing and solving in mathematics education, involving geometry and randomness, two critical subjects in school mathematics. Our examples show to what extent the way a mathematical situation is metaphorised and enacted by the learners shapes their emerging ideas and insights and how this may help to bridge the gap between the 'mathematically gifted' and those apparently not so gifted or mathematically inclined. Our experimental background includes a broad spectrum of prospective secondary math teachers, in-service primary teachers and their pupils, first-year university students majoring in social sciences and humanities and university students majoring in mathematics.

Keywords Metaphor · Enacting · Enactivism · Learning · Mathematics

34.1 Introduction

In our view, the most critical issue in mathematics education is the fact that millions of schoolchildren worldwide are exposed to mathematics in a way that turns out to be an inescapable torture for most of them. This phenomenon has been rather recently acknowledged as 'cognitive abuse' or 'cognitive bullying' in the English literature (Johnston-Wilder and Lee 2010; Watson 2008) and has been described as a practice that is 'at best marginally productive and at worst emotionally damaging' (Watson 2008, p. 165).

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Indeed a consequence of this abuse, besides stress, frustration, math anxiety and phobia, to begin with, is that many children and adolescents experience mathematics as a senseless ritual and remain mathematically maimed and crippled for the rest of their lives. Illusory understanding and amnesia follow, as pointed out by Shulman (1999). The 'fatal pedagogical error', denounced by theologian Tillich (cited by Weissglass 1979) as 'throw[ing] answers like stones at the heads of those who have not yet asked the questions' (p. 59), is ubiquitous and recurrent. The same point is made by Freire in his criticism of a pedagogy based on answers to non-existent questions (Freire and Faúndez 2014). Moreover, many teachers unwittingly, or unwillingly under systemic pressure, are functional to this often unseen and unacknowledged situation. Hall's (1959) famous saying fully applies here: 'Culture hides more than it reveals, and strangely enough what it hides, it hides most effectively from its own participants' (p. 39).

From our perspective, there is an urgent need to democratize, even to humanize, the learning of mathematics (Cantoral 2013; Freire 1970; Gattegno 1971), and we hypothesise that an approach that takes advantage of metaphorising and acting out —natural cognitive mechanisms evolved in our species, but thwarted by traditional teaching—may significantly help in alleviating the current cognitive abuse and its sequels.

In this paper then, we intend to investigate from an enactivistic perspective to which extent the way a mathematical situation is metaphorised and enacted (i.e., acted out) by the learners shapes the ideas and insights that may emerge in them. Also, how metaphorising and enacting may help to bridge the gap between the 'mathematically gifted' and those apparently not so gifted or mathematically inclined and facilitate sense making of mathematics for the latter.

After recalling below the basics of (conceptual) metaphorising and enactivism in cognitive science, and arguing about their implications for mathematics education, we discuss some down-to-earth illustrative classroom examples of what we call *enactive metaphorising* in the context of mathematical problem posing and problem solving involving geometry and randomness, two especially critical subjects in contemporary school mathematics and beyond.

34.2 Theoretical Background and Research Questions

34.2.1 Metaphorising in Mathematical Education

Increasing awareness has been emerging during the last decades in the mathematics education community that metaphors are not just rhetorical devices but powerful cognitive tools that help us in building or grasping new concepts, as well as in solving problems in an efficient and friendly way (Chiu 2000; Díaz-Rojas and Soto-Andrade 2015; English 1997; Lakoff and Núñez 2000; Libedinsky and

Soto-Andrade 2015; Sfard 2008, 2009; Soto-Andrade 2006, 2007, 2015; and many others).[1]

In a broader perspective, increasing agreement has arisen in cognitive science that metaphorising (looking at something and seeing something else) serves as the often unknowing foundation for human thought (Gibbs 2008). As suggested by Johnson and Lakoff (2003), our ordinary conceptual system, in terms of which we think and act, is fundamentally metaphorical in nature. Lakoff and Núñez (2000) highlight the intensive use we make of conceptual metaphors that appear—metaphorically—as inference-preserving mappings from a more concrete 'source domain' into a more abstract 'target domain', enabling us to fathom the latter in terms of the former.

Elementary examples of (conceptual) metaphors in mathematics education are the two foremost metaphors for multiplication, to wit, the 'area metaphor' and the 'grafting metaphor' (Soto-Andrade 2014), illustrated in Fig. 34.1 for the case of 2 times 3 and 3 times 2.

Notice that the area metaphor allows us to see commutativity of multiplication as invariance of area under rotation. We 'see' that $2 \times 3 = 3 \times 2$, without counting and knowing that it is 6. On the contrary, the grafting (or concatenated branching) metaphor does not allow us to 'see' at a glance the commutativity of multiplication. As realised by Lakoff and Núñez (Núñez, personal communication, December 2012) this fact suggests that multiplication is not really commutative. In more precise terms, there might be 'multiplications', inspired by the grafting metaphor, that are not commutative! Indeed, think of *composition* of permutations, of matrices and of operators. We have here then two different metaphors for the 'same' mathematical object, each with a different scope. The first one, the area metaphor, which is quite close to East Asian crossing metaphor for multiplication, where you count the number of crossings of, in this case, 2 lines and 3 lines, is quite friendly and lets us see immediately the commutativity of multiplication. The second one (multiplication is concatenation), points in a different direction, does not exhibit commutativity (Mac Lane 1998) as an obvious property and is in fact more profound: It reshapes our understanding of multiplication and it unfolds into category theory in contemporary mathematics. A case of a felicitous metaphor opening up the way to deep and far-reaching generalisations of a seemingly innocent elementary concept (see Manin 2007)!

Notice that, as argued by Sfard (1997), metaphorising appears here as a circular autopoietic process (Maturana and Varela 1980) rather than as a unidirectional mapping. So a more appropriate metaphor than the arrow metaphor to describe metaphorising would be the ouroboros (the snake eating its own tail), an outstanding metaphor of circularity, self-reference and organisational closure in living systems (see Soto-Andrade et al. 2011). The ouroboros indeed plays an important role in Maturana and Varela's theory of autopoietic systems (Maturana and Varela 1980), appearing even on the cover of the first Spanish edition of their work

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[1]For a recent survey of the role of metaphor in mathematics education, see Soto-Andrade (2014).
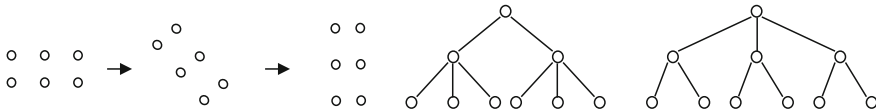
Fig. 34.1 Area metaphor (left) and grafting metaphor (right) for multiplication

(Maturana and Varela 1973; see also Soto-Andrade et al. 2011, Fig. 34.1), and in cognition as enaction (Varela et al. 1991). Recall that the latter was also metaphorised by Varela by the famous *Drawing Hands* lithograph by Escher, where each hand draws the other into existence.

Our approach to the learning of mathematics emphasises the poietic (from the Greek *poiesis* = creation, production) role of metaphorising, which brings concepts into existence. For instance, we bring the concept of probability into existence when, while studying a symmetric random walk on the integers, we look at the walker (a frog, say) and we see it *splitting* into two equal halves that go right and left instead of jumping equally likely right or left (Soto-Andrade 2007, 2014, 2015). This 'metaphoric sleight of hand' that turns a random process into a deterministic one allows us to reduce probabilistic calculations to deterministic ones where we just need to keep track of the walker's splitting into pieces: The probability of finding the walker at a given location after n jumps is just the portion of the walker landing there after n splittings.

We remark that a different metaphoric way of bringing mathematical notions into existence, called *reification* by Sfard (2008), where a *process* is seen as an *object*, is exemplified by the case of fractions: Splitting a whole into 3 equal parts and keeping 2 of them becomes the number 2/3. Of course, splitting the whole into 6 equal parts and keeping 4 is a different (but equivalent) process whose reification is the same number, 4/6 = 2/3. Saying that 4/6 and 2/3 are just *equivalent* fractions instead of *equal* fractions is here a sign of incomplete reification.

Although in the literature metaphor and representation are often used as synonyms, we draw here a distinction: we *metaphorise* to construct concepts (as in the above example) and we *represent* to explain concepts. Typically, metaphors are arrows going *upwards*, from a down-to-earth domain to a more abstract one, and representations are arrows going *downwards,* i.e., the other way around. In this connection, it is pertinent to recall that in the German school of didactics of mathematics, originally mostly concerned with primary mathematics education and going back to Pestalozzi (Herbart 1804; vom Hofe 1995), representation and metaphor were quite present: as *Darstellung*—representation aiming at explaining something to others—and *Vorstellung*—a personal way to figure out or fathom something, operationally equivalent to metaphor (Soto-Andrade and Reyes-Santander 2011). So metaphorising was already recognised and appreciated at the beginning of the 19th century in German didactics of mathematics, well before its irruption from cognitive psychology and linguistics into mathematics education (Lakoff and Núñez 2000).

The ubiquity of metaphor in mathematics education should not be underestimated: Besides bringing into existence mathematical concepts or objects or helping learners to fathom them, unconscious metaphorising often dramatically shapes the way teachers teach, for instance. A foremost example is afforded by the metaphor 'teaching is transmitting knowledge'. Indeed, when confronted with it, many teachers reply: This is not a metaphor, teaching *is* transmitting knowledge! What else could it be? Unperceived here is the 'acquisition metaphor' (Sfard 2009; Soto-Andrade 2007) for learning, dominant in mathematics education, that sees learning as acquiring an accumulated commodity. The alternative, complementary metaphor is the 'Participation Metaphor': learning as participation (Sfard 2009). This dichotomy is well expressed in Plutarch's metaphor: 'A mind is a fire to be kindled, not a vessel to be filled' (Sfard 2009, p. 41).

Paraphrasing Bachelard (1938), who advocated epistemological vigilance, we suggest nowadays to practise metaphorical vigilance, i.e., the art of noticing (Mason 2002) our unconscious or implicit metaphors, that shape our way of interacting with the world and particularly our approach to teaching and learning.

Last but not least, metaphorising plays also a key epistemological role: We have claimed elsewhere (Díaz-Rojas and Soto-Andrade 2015) that—metaphorically—a theory is in fact just the 'unfolding' of a metaphor (the involved unfolding process, however, may be laborious and technical).

A paradigmatic example is the 'tree of life' metaphor in Darwin's theory of evolution. Also, Brousseau's theory of didactical situations (Brousseau and Warfield 2014) may be seen as an unfolding of the 'emergence metaphor' that sees mathematical concepts emerging in a situation instead of being parachuted from Olympus as in traditional and abusive teaching. The 'grafting' metaphor above for multiplication (Soto-Andrade 2014) unfolds into category theory in mathematics Mac Lane (1998). We use in fact the metaphorical approach as a meta-theory to describe other theories relevant to us in terms of their generating metaphors, something more helpful to fathoming how they arise than just describing them a posteriori. We exemplify this below in the case of Varela's enaction.

34.2.2 Enactivism in Mathematics Education

An unfolding metaphor for enaction is Antonio Machado's famous poem (Machado 1988, p. 142; Thompson 2007; Malkemus 2012): 'Caminante, son tus huellas el camino, y nada más; caminante, no hay camino, se hace camino al andar' ['Wanderer, your footsteps are the path, nothing else; there is no path, you lay down a path in walking.'], cited by Varela (1987, p. 63) himself when he introduced what he called the *enactive approach* in cognitive science (Varela et al. 1991). In his own words: 'The world is not something that is given to us but something we engage in by moving, touching, breathing, and eating. This is what I call cognition as enaction since enaction connotes this bringing forth by concrete handling' (Varela 1999, p. 8).

Notice *en passant* to what extent the 'laying a path in walking' metaphor is transversal to the traditional one for learning as following a well-marked path given in advance.

Before proceeding any further, however, to avoid confusion given the somewhat polysemic current status of the terms *enactivism*, *enactivist*, *enaction*, *enact*, *enacting* and *enactive*, we will adhere to the following usage.

The now prevalent terms *enactivism* and *enactivist* will always refer to Varela's anti-representationalist 'enactive program' (Varela et al. 1991, p. xx), which sees cognition as embodied action, more precisely, cognition as enaction, as metaphorised by Machado's verse. Key aspects of enaction are: perceptually guided action, embodiment and structural coupling through recurrent sensorimotor patterns (Varela et al. 1991; Reid and Mgombelo 2015). In an aphorism: 'All doing is knowing, and all knowing is doing' (Maturana and Varela 1992, p. 26). We will speak then of an 'enactivist approach' to problem solving or to mathematics education. We will also use the term 'enaction' exclusively in Varela's sense (Maturana and Varela 1992).

On the other hand, unless otherwise explicitly stated, 'enact', 'enacting' and 'enactive' are to be understood in the sense of everyday language and also in the sense of Dewey (1997) and Bruner (1966), i.e., as synonyms of 'acting out' or 'acted out', in an embodied way. So 'enacting a metaphor' just means 'to act it out', with your body (see Example 34.4.1). This fully coincides with the use of 'enactive' in Gallagher and Lindgren (2015), where they refer to 'enactive metaphors' (metaphors in action, that we act out bodily) as opposed to what they call 'sitting metaphors'. We use 'enactive metaphorising' below in this sense.

As mentioned above, in mathematics education the term *enaction* may be traced back to Bruner (1966), who was following the traces of Dewey's (1997) 'learning by doing'. Bruner's enaction, which means essentially acting out or doing, is however far less radical than Varela's, in that it does not challenge the notion of a given reality 'out there' that we perceive or represent more or less successfully. Dewey, however, already emphasised the role of sensorimotor coordination in perception, acknowledging that movement is primary and sensation is secondary (Dewey 1896; Gallagher and Lindgren 2015).

In particular, the enactivist notions of structural determinism and structural coupling (Maturana and Varela 1992; Varela 1999; Varela et al. 1991) have provided new insights on learning, problem solving and problem-posing processes: Learning is not determined by a didactical environment but arises from the interaction of the learner's structure and environment, which plays at most the role of a 'trigger'. Traditionally, however, problem solving entails problems given beforehand, lying 'out there' in the world, waiting to be solved, independently of us as cognitive agents. In the enactivist perspective, because of our structural coupling with the world (Varela 1996; Varela et al. 1991), we bring forth emergent problematic situations instead. This is what Varela calls problem posing. This diverges from the usual gas fitter metaphor for problem solving, where solvers look into their toolboxes of predefined strategies and choose the appropriate one for solving the problem at hand (Soto-Andrade 2007; Proulx 2008). In the enactivist perspective,

mathematical strategies emerge continually in the interaction of solver and problematic situation (Proulx 2013; Thom et al. 2009).

At present, an enactivist didactics of mathematics unfolds where the teacher is an enactivist practitioner acting in situation and learning appears as an emergent, situated and embodied process (Brown 2015; Brown and Coles 2012; Proulx 2008, 2013; Proulx and Simmt 2013). For a recent survey of enactivist theories, see Goodchild (2014).

According to Varela, we are always 'enacting' a world, most of the time unconsciously. So we cannot choose to enact or not to enact (in Varela's sense); enaction is just the way we cognise as living beings. We may nevertheless entertain the 'representationalist illusion' (a privilege of humankind!) that we are perceiving and representing an objective reality 'out there'. Also, we can choose to enact (in the everyday sense of the word of bodily acting out) a given metaphor or situation or not, for instance. Paradoxically, we are definitely able to teach in a way that ignores enaction (in Varela's sense) and does not allow for enacting (as bodily acting out): a non-enactivist stance that paves the way for cognitive bullying. Our enactivist approach to education, distilled in the 'lying down a path in walking' metaphor for cognition and learning, leads us on the contrary to foster metaphor enacting among the learners.

34.2.3 Research Questions

Along the lines of our stated research aim in the Introduction, we intend to address here the following research questions:

– When and how does metaphorising arise from learners in a problem-solving situation, particularly idiosyncratic metaphorising?
– How does metaphorising correlate with the emergence of new ideas or insights to tackle challenging situations?
– Is metaphorising enactive most of the time? How relevant for learning are action-based enactive metaphors?
– What is the influence of learners' non-metaphoric enacting in mathematical problem-solving situations?

34.3 Methodology

Our methodology adheres to the enactivist perspective, where we focus on the learners doing and knowledge is not metaphorised—by the researcher—as an object to be captured or held by a learner (Sfard 2008, 2009).

According to this and our research objectives, we suggest and propose various challenging situations to the learners and observe how they tackle them. We pay attention to the whole spectrum of emerging strategies, to whether they metaphorise or enact and how the emergence and the quality of their ideas and insights correlate with their metaphorising and enacting. We do not focus on trying to measure their knowledge over time but on monitoring their being mathematical as a means to tackle a challenge co-emergent with their doing.

Our experimental methodology relies on qualitative approaches and field observation, especially multiple case study (Yin 2003); participant observation techniques and ethnographic methods (Spradley 1980; Brewer and Firmin 2006). Our experimental background includes a broad spectrum of learners (seven cohorts) with whom we carried out didactical experiences based on a metaphor-intensive enactivist approach in 2015 and 2016 that included the following:

A. Fifty students in a one-semester first-year mathematics course in the social sciences and humanities option of the Baccalaureate Programme of the University of Chile (two cohorts: 2015 and 2016).
B. Thirty-five prospective secondary school physics and mathematics teachers in a one-semester course in probability and statistics at the same university (two cohorts: 2015 and 2016).
C. Twenty (5 graduate and 15 undergraduate) students majoring in mathematics in an optional course on random walks at the University of Chile (2015).
D. Fifty participants in a two-session workshop, each session consisting of 1.5 h, on enactive metaphoric approaches to mathematical problem solving, held at the annual meeting of the Chilean Mathematics Education Society, in Valparaiso (2016). Participants included in-service secondary and primary school teachers, prospective secondary teachers, post-graduate students in mathematics education, researchers in mathematics education and some undergraduate and graduate math students.
E. Twenty in-service primary school teachers engaged in a 15-month professional development programme (mathematics option) at the University of Chile at Santiago (2016).

These cohorts were chosen because they constituted a rather broad spectrum of learners with whom our overarching approach could be tested while performing our usual teaching duties at the university and facilitating invited workshops elsewhere.

Regarding data recollection, learners, working most of the time in random groups of three to four, were observed by the teacher or facilitator and an assistant, the latter assuming the role of participant observer or ethnographer. Field notes and transcripts of the generated dynamics were taken (especially of critical moments of the work sessions, such as emergence of metaphors, horizontal confrontations between the students, and didactic tension build up), snapshots of their written output (on paper or whiteboard) in problem-solving activities were taken,

short videos of their enacting moments were recorded. We recall that all these data are also acts of interpretation, where a researcher learns in co-emergence with a research situation (Reid 1996).

Regarding data analysis, categories involved in the initial phase of the observation included:

– Learners' participation and engagement (level estimates).
– Questions and answers (from teacher and learners, frequency, relative weight, spaces for pondering).
– Horizontal (peer) interaction (level estimate).
– Metaphors, especially idiosyncratic ones (emergence, spontaneously or under prompting, variety).
– Arising of gestural language of learners and teacher.
– Expression and explicit acknowledgement of affective reactions from the learners.
– Enacting (acting out) of metaphors and situations by the learners.

Recall that in an enactivist methodological framework the initial categorical grid evolves according to the flow of activity in the classroom and the reactions of learners and teachers in an autopoietic way (Reid 1996; Maheux and Proulx 2015).

To address our research objective we kept track of ideas and actions emerging after either spontaneous or prompted metaphorising (see examples below).

From data analysis, we compared cognitive and affective reactions of the different cohorts and inferred the profile and strength of the prevailing didactical contract (Brousseau et al. 2014), usually installed during secondary math education for most learners.

Moreover, learning in Cohorts A, B and E was assessed through monthly tests (where students had to solve contextual problems in a limited time), compulsory and optional exercises and challenges as homework (Cohorts A, B, C and E). Process assessment was also done by observing their acting and behaviour and recording their production during individual and group work sessions and lectures (all cohorts).

34.4 Illustrative Examples of Enactive Metaphorising

We report and discuss below two types of examples of enactive metaphorising in challenging mathematical situations that we experimented with in the above cohorts.

34.4.1 The Sum of the Exterior Angles of a Polygon and the Inner Acute Angles of a Star (Cohorts D and E)

34.4.1.1 Problem: Everybody Knows About the Inner Angles of a Triangle and Their Sum. But What About the Exterior Angles? And Their Sum? Also What About the Same for a General Polygon?

As suggested by our enactivist perspective, we prompt the learners to notice and to voice their reactions, cognitive and affective, to this 45-min challenge. We intend in this way to facilitate a circular interaction between the problem and the learners that could trigger a reshaping of the challenge. Most of them, however, have trouble in acknowledging a negative reaction. After renewed prompting some dare to ask: Why should we be interested in the exterior angles of a polygon and their sum? A few (in-service and prospective teachers alike) complain about the prescriptive way this sort of geometry is usually taught. After a while, most of them agree that one needs to re-signify exterior angles: What are they good for?

After some polling, we found that a majority of learners (students as well as teachers) prefer inner angles to exterior angles and wonder about the meaning, usefulness or relevance of exterior angles.

We observed that to tackle this problem almost every in-service and prospective secondary mathematics school teacher in our country calculates first the sum of the inner angles (usually by triangulating), finding that it depends on the number n of sides of the polygon, as $(n − 2)$ times 180°. Then, replacing each interior angle by 180° minus the corresponding exterior angle and calculating, they wind up discovering that the sum of all exterior angles of a (convex) polygon is 360°, independently of its number of sides! This is surprising for most of them! At this point some students (more often girls than boys, in our courses) ask: Isn't there a simpler way to get this? Others feel frustrated because they have 'calculated blindly' and without insight.

This is the usual way in which teachers and students 'get into' the proposed task (Proulx 2013). Unfortunately, this is *the only* approach found in almost every textbook, where the mathematical content 'exterior angles of a polygon' is then checked as having been covered.

Our metaphoric approach suggests, however, prompting the students to metaphorise a polygon first (not just recite its definition) to help them to get into the problem in more transparent ways. Their metaphorising will depend, of course, on their previous history and experiences. We observed the rather slow emergence of two main competing metaphors:

– A polygon is an enclosure between crossing sticks (most popular among in-service primary teachers).

– A polygon is a closed path made out of straight segments. Interestingly, some learners *say* that a polygon is a closed plane figure, while *drawing* a circuit in the air with their index!

Notice that the first metaphor carries the viewpoint of the eagle (who sees from above) and the second one, the viewpoint of the ant (who crawls down to earth). A high-speed version of the second one is quite familiar to children nowadays in video games.

Among primary school teachers (Cohorts D and E), enacting the first metaphor triggers the idea of manipulating the sticks, translating them in convenient ways, so as to make clearly visible the exterior angles first, and then shifting them parallel to themselves to shrink the polygon to a minimum, preserving its shape. So in fact they zoom out the polygon! In this way they *see* that the sum of all exterior angles is clearly 360° instead of getting this value by blind calculation.

Enacting the second metaphor also allows the learners to *see* that the sum of exterior angles is a whole turn, when 'laying down a polygonal path in walking'.

Indeed, we observed that the in-service primary teachers in Cohort E, working in groups, had one of them (who had trouble *seeing* in this way that the sum of exterior angles was a whole turn) 'lay down a polygon in walking', following the instructions of his peers: Walk 5 steps, stop, turn 45° to your left, walk 7 steps, and so on. In this way they realized that exterior angles rather than inner angles were the necessary and convenient data to inflect or bend the path of the walker as desired. Addition of all exterior angles occurred when the walker made a complete circuit and came back to his starting point with his nose pointing in the same initial direction. Learners also noticed that this metaphor suggests a natural generalisation, involving a signed sum, for the case of a non-convex polygon!

Recently a third metaphor was suggested by one of our former mathematics students:

– A polygon is a wheel of the Flintstones' car.

Learners realised quickly that when the Flintstones' car runs, its wheels turn, and when they complete a whole turn, their exterior angles (arising as the successive angles between the wheel's sides and the ground) add up to a whole turn!

We see in this example that metaphorising and enacting can make a dramatic difference in understanding that is within the reach of 'everybody', as opposed to the 'blind' unappealing calculation found everywhere. Our appraisal of inner and exterior angles also changes: Exterior angles appear now to be more natural and friendlier than inner angles: a dissident view indeed! In particular, learners realised that it is smarter to figure out first the sum of all exterior angles of the polygon and then deduce the value of the sum of the inner angles, which is contrary to the usual procedure and a valuable idea for the next challenge.

34.4.1.2   The Five-Pointed Japanese Star Problem, The Enactive Way

A typical challenge in Hosomizu's little red book (Hosomizu 2008) is to calculate the sum of the inner acute angles of a five-pointed star (eventually non-regular). Several clever approaches are discussed there, although none of them are enactive or metaphoric. We posed this problem to Cohorts D and E. When posed from scratch, before Problem 34.4.1.1, everybody tackled it in a geometric-algebraic way: Some learners in Cohort D even wrote down a whole system of equations, taking advantage of the inscribed pentagon in the star. Most took the star to be regular and computed dutifully the value of each inner acute angle. Then they conjectured that the total sum would remain constant if the star were deformed. So they more or less converged to the approaches illustrated in Hosomizu (2008), although less clever on the average. Nobody thought of 'laying down a star in walking' (following the circuit usually used to *draw* the star) to instantly see that the sum of all exterior angles at the points of the star equals 2 whole turns and from there get the sum of all inner acute angles (as 5 half turns minus 4 half turns = 1 half turn). Several in-service secondary teachers avowed nevertheless that they preferred the algebraic approach that they felt more at home with.

Learners in Cohort E, who worked on this problem after having worked out Problem 34.4.1.1, wondered for a while which closed path to walk to lay down the star before settling for the one they use to *draw* the star. After learners in Cohort D solved this problem using 'angular yoga', as in Hosomizu (2008), we proposed to them Problem 34.4.1.1, which they discussed and finally solved metaphorically and enactively. They went then back to the five-pointed star to find a friendly circuit to walk and solve the problem. Learners in Cohorts D and E noticed that this enactive metaphoric approach worked equally well for irregularly drawn seven-pointed stars and more generally for stars with an odd number of points.

34.4.2   Probabilistic Enacting

34.4.2.1   Falk's Urn and Fischbein's Test

The following challenging question (Falk and Konold 1992; Fischbein and Schnarch 1997) was proposed to learners in Cohorts A, B and C.

> John and Mary each receive a box containing 2 black marbles and 2 white marbles.
>
> John extracts a marble from his box and finds out that it is white. Without replacing this marble, he extracts a second marble. Is the likelihood that this second marble is also white smaller than, equal to or greater than the likelihood that it is black?
>
> Mary extracts a marble from her box that she puts in her pocket without looking at it. Now she extracts a second marble that turns out to be white. Is the likelihood that the marble in her pocket is white smaller than, equal to or greater than the likelihood that it is black?

Learners had no trouble with John's drawings, but roughly 60% of learners in Cohort A and 40% of learners in Cohorts B and C thought that the fact that Mary's second marble be white had no effect whatsoever on the likelihood of the first one being black. The remaining learners thought intuitively that since the second one was white it was more likely that it was drawn from a box with more white than black marbles, so it was more likely that the first marble was black. Just a few learners in Cohorts A and B had the idea of simulating many times to figure out what would be more likely. Others (learners in Cohort C included) metaphorised the whole process as a two-step random walk on a binary tree, or better on a grid, and computed diligently the non-required probabilities (the question was qualitative and couched in everyday language, not in probabilistic language). They found correctly that the probability of the marble being black in both cases is 2/3. They realised, however, that they did not really *see* why probabilities were the same and why 'there was no time arrow'.

Following our enactivist perspective, we prompted the students to enact (act out) the experiment. Extracting the marbles, they ended up with two marbles by the box, in the first case the first one being white and the second one being hidden under a cap, in the second case, the first one being hidden and the second one being white. They realised then that they had just extracted two marbles from the box and hidden one of them, the other one being white!

A variant of this enactment that we suggested to the students, inspired from a remark by M. Borovcnik (personal communication at ICME 13, July 27, 2016), starts by grabbing a marble from the box with one hand and then another one with the other hand, keeping both fists closed. They realise then by themselves that it is just a matter of opening first one fist or the other and that they could have also grabbed the two marbles simultaneously.

34.4.2.2 Drawing Balls from an Urn Without Replacement: Metaphorising as a Random Walk and Enacting (Proposed to Learners in Cohorts B and C)

Problem: From an urn containing 3 red balls and 5 blue balls, 5 balls are drawn one after another at random without replacement. How likely is that the 5th ball drawn is red?

We have discussed this problem, proposed to learners in Cohort C, in detail elsewhere (Soto-Andrade et al. 2016) in the simpler case of the 3rd ball drawn from a (2, 3) urn instead of a (3, 5) urn. We comment here on further experimentation with learners in Cohort B in 2016 and new ways of enacting it (acting it out).

We observed that most students in Cohort B, when exposed to the problem in a test, dutifully calculated the requested probability with the help of a lush possibility tree with probabilities assigned stepwise using a hydraulic metaphor (that sees probabilities as portions of one litre of water that drained downwards from the 'root' of the tree). Nevertheless most of them did not realise that the probability of a

red ball at any drawing was always 3/8, because they did not even notice that for the second drawing it was 21/56 = 3/8! One or two intuited that order did not matter, but most of them were quite surprised in a subsequent stage, when working in groups in the classroom they finally found that the probability of the nth ball drawn being red was the same for all n up to 8. We then prompted them to enact the process by actually drawing marbles from a (3, 5) bag. Some were a bit reluctant to do so. A good performing student said bluntly:

> I do not see how enacting can help me to solve the problem. What else do I get from enacting that I do not get from thinking? I just need to think about it!

Nevertheless, afterwards he gave the following intuitive argument to see that all probabilities were the same. Keeping the first 4 drawn marbles in his closed left hand, he said:

> Now I have to choose a 5th one from the 4 marbles remaining in the bag. But it is clear that this is equivalent to choosing one of the hidden 4 marbles in my left hand! So it amounts to choosing 1 marble from the whole bunch of 8 marbles!

All other students put the drawn balls carefully in a line, one by one (they did not throw them away!). This helped several of them to see the invariance of the probability of drawing a red marble. No one put them insightfully in a circle, as an undergraduate female student[2] in Cohort C did for the (2, 3) bag in 2015, but they really appreciated the idea when told.

Now, a new enaction, suggested by M. Borovcnik (personal communication at ICME13, July 27, 2016) is to grab sequentially first, five marbles from the bag, with five hands (of three students) keeping the five fists closed, and subsequently opening them in the same sequence, or in another one, e.g. the fifth fist first. Eventually the grabbing could be also simultaneous! Enacting in this way all students *saw* the invariance of the probability of 'red', not just a few clever eidetic students.

To get a better grasp of the drawing process, learners also metaphorised it as a 2D random walk—from the source (3, 5) to the sink (0, 0) or from the source (8, 3) to the sink (0,0)—that in turn they metaphorised as a splitting process, whose transition probabilities they calculated with the help on a hydraulic metaphor. They realized then that the associated (deterministic) 'barycentric walk' provides a friendly metaphor for the 'expected walk' of the walker. They intuitively guessed that the barycentric walk should proceed geodesically along a line whose slope corresponds exactly to the probability of red at any drawing in the case where they represent the initial state of the urn by (8, 3).

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[2]Notwithstanding that Chile's boys-girls PISA math performance gap is extreme among OECD countries (OECD 2016, p. 198).

34.5   Discussion and Conclusions

Motivated by an enactivist perspective, we have shown by way of illustrative examples how metaphorising and enacting (acting out) mathematical objects, processes and situations can make a significant difference in the ideas and insights that may emerge from learners tackling a mathematical challenge. In the cases considered (34.4.1 and 34.4.2) concerning geometry and probability, we observed notably that there was a dramatic contrast between blind calculation before metaphorising and sudden insight when metaphorising or enacting. We also saw how different insights were triggered by different metaphors or enactings. In Problem 34.4.1.1, for instance, we collected in all one blind calculation and three different insights leading to the answer triggered by three different metaphors for a polygon with different levels of enactivity (bodily engagement), the foremost one being 'laying down a polygon in walking'.

Very concretely, we observed that when they enact, learners have to make up their minds: What do I do with the balls I draw from the urn? Throw them away? Keep them in my hand or in my closed fists? Put them carefully in a row or a circle on the table? Each way of enacting—determined primarily by the solvers' structures and histories—suggests various different ideas and insights that do not emerge so easily when they just think about a problem. Our learners working on the challenges in Cases 34.4.1 and 34.4.2 indeed discovered unforeseen mathematical relations or facts in their bodily actions (see Abrahamson and Trninic 2015).

We nevertheless found that, surprisingly, metaphorising and enacting were quite difficult for most of the observed learners. Persistent prompting and plenty of time was often needed to elicit both among them. Notice that learners in Cohort A, for instance, came straight from secondary school (where cognitive bullying prevails). Even so, students in Cohort A and in-service primary school teachers were more prone to metaphorise than prospective or in-service secondary school math teachers.

Particularly, we noticed that metaphorising a polygon, for instance, was a very unusual challenge for students, prospective and in-service teachers alike: a violation of the prevailing didactical contract. But once they felt they were allowed to, even prompted to, metaphors began to arise among them, shyly and slowly at first. They came later to gradually appreciate the operational virtues of metaphorising.

In fact, we observed chains of metaphors emerging that completely transformed a given problem (e.g., Sects. 34.4.1.1 and 34.4.2.2) and allowed learners to better fathom what was going on. From an enactivist perspective, they were not just reacting to a problem out there or looking for a solving strategy that had been stocked beforehand in their personal toolkit but rather shaping and transforming the problem, eventually because they did not like it (see Proulx 2013). Moreover, metaphorising a mathematical object, such as a polygon, may show them the way to guess and discover meaningful or significant properties amidst the huge set of properties entailed by its formal definition.

Interestingly, no more than one student out of 20 on the average tried spontaneously to enact (act out) an opaque problematic situation. Some prospective teachers even voiced their disbelief regarding the usefulness of enacting, because math problems are just a matter of thought!

Recalling the well-documented strong negative feelings of school children towards mathematical content taught the traditional way, it seemed to us a bit paradoxical to observe widespread 'emotional anaesthesia' in most of our learners, who had trouble in acknowledging and expressing their emotional reactions, especially negative ones, towards mathematical content. Only primary school teachers and students in Cohort A escaped this condition to some extent. We interpret this syndrome as a consequence of the didactic contract (Brousseau et al. 2014) associated to the prevailing cognitive abuse in our culture, where students are expected to understand a mathematical content or not, but not to like or dislike it. Expression of affect is then ignored and repressed.

This phenomenon seemed important to us because we observed that metaphorisation, for instance, may be often triggered by disliking of a proposed problem: The learner tries to metaphorise it in order to see it otherwise, wearing friendlier attire. So in fact negative emotions may foster creativity!

We noticed a remarkable convergence of our claims and experimental findings regarding the positive incidence of enacting in the arising of new insights in problem-solving situations with very recent research in cognitive science (e.g., Glenberg 2015; Vallée-Tourangeau et al. 2016; Abrahamson and Trninic 2015).

We may conclude that metaphorising and enacting (in the sense of bodily acting out) play indeed a key role in the learning of mathematics, especially for non-mathematically inclined learners who have been cognitively abused by traditional learning. Since cognitive bullying is to a great extent institutionalised by the prevailing unspoken didactic contract that is functional in thwarting metaphorising, enacting and affect in teaching and learning contexts, it seems urgent to reshape this contract to allow for and foster these processes. This endeavour deserves further research, taking into account the relevance of collaborative group work and learners' horizontal interaction and participation.

As an open end, we would like to extend longitudinally our study to involve the pupils of in-service and pre-service teachers we have worked with, to further investigate the incidence of metaphorising and enacting in their learning and their role as an antidote to cognitive bullying.

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