Chapter 2
Mathematical Experiments—An Ideal First Step into Mathematics

Albrecht Beutelspacher

Abstract Since the foundation of the Mathematikum, Germany, in 2002 and Il Giardino di Archimede, Florence, Italy, in 2004 there have been many activities around the world to present mathematical experiments in exhibitions and museums. Although these activities are all very successful with respect to their number of visitors, the question arises what is their impact for "learning" mathematics in a broad sense. This question is discussed in the paper. We present a few experiments from the Mathematikum and shall then discuss the questions, as to whether these are experiments and whether they show mathematics. The conclusion will be that experiments provide an optimal first step into mathematics. This means in particular that they do not offer the whole depth of mathematical reasoning, but let the visitors experience real mathematics, insofar as they provide insight by thinking.

Keywords Mathematical experiments · Science centers · Learning by experience Mathematics in leisure time

In the last years, quite a few mathematical exhibitions have been developed and mathematical museums ("science centers") have been opened. In these, mathematics is typically not presented in the traditional way using the mathematical language. On the contrary: visitors find "exhibits", in which they may see or explore mathematics. In other words, visitors are challenged to perform "mathematical experiments". In addition, also several books with easy-to-perform experiments have been published, which aim at teachers, students or the general public.

In this article we look at mathematical experiments, and investigate their potential for formal and informal learning of mathematics. The basic reason for the success of science centers in general is expressed in the slogan "hands-on, minds-on, hearts-on". In other words, in performing the experiments, visitors get

A. Beutelspacher (✉)
Giessen, Germany
e-mail: albrecht.beutelspacher@mathematikum.de

G. Kaiser et al. (eds.), *Invited Lectures from the 13th International Congress on Mathematical Education*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-319-72170-5_2

experience. This experience leads to understanding, and understanding gives pleasure.

2.1 Mathematical Experiments and Science Centers

Probably the first man-made experiments are due to the time of Galilei (for instance experiments with pendula). In mathematics, models and instruments became important in the 19th century. The book of Dyck (1892) shows an impressive collection of mathematical models, apparatuses and instruments.

Some mathematical experiments have been known for a long time, mostly under the name of "mathematical games". Famous games are for instance Hamiltons's Icosian Game (1857), the "Tower of Hanoi" (Lucas 1883), and the Soma cube (Hein 1934).

The first initiative to collect and develop mathematical experiments as such was undertaken by the Italian professors Franco Conti and Enrico Giusti, who very successfully developed and organized the exhibition "Oltre iI compasso—the mathematics of curves", which was first shown in 1992. Since 2004 it has been enlarged to form the "Giardino di Archimede" in Florence. Nearly at the same time, the first step towards the Mathematikum was taken: in 1994 the first German exhibition under the name "hands-on mathematics" ("Mathematik zum Anfassen") was shown in Giessen, Germany. This exhibition was a work of a group of students, who organized this exhibition as a follow-up of a mathematical seminar. Mathematikum, the world's first mathematical science center, was opened in 2002. Since then, quite a few institutions of different size followed these ideas, for instance "Adventure Land Mathematics" in Dresden, "MoMath" in New York, and "Maison des Maths" in Mons, Belgium.

The idea of all these institutions is basically that the combination "interactive exhibits and visitors" works. It is fascinating to observe that in all science centers visitors start working, without a guide, without a teacher, even without reading the label, and have lots of fun. In most science centers, certainly in all mathematical science centers, the responsible people take science serious. "Fun" should not arise from strange colors, noise, fog and so on, but from insight into the phenomena. Looking at the visitors, we see experience, understanding and pleasure. In the science center-terminology: hands-on, minds-on, hearts-on.

2.2 Mathematikum Giessen

The Mathematikum in Giessen, Germany (near Frankfurt) is a mathematical science centre founded in 2002. It aims to make mathematics accessible to as many people as possible, in particular to young people. On its 1200 m^2 exhibition area it shows about 180 interactive exhibits. From the very beginning, it was a great success.

Between 120,000 and 150,000 people visit the Mathematikum each year. About 40% are group visitors, mainly school classes, 60% are private visitors, mainly families.

Visitors like the Mathematikum. In particular they like the way mathematics is presented. They are entertained by performing the experiments and trying to understand what they have experienced. The Mathematikum is a house full of communication. When one listens to what people are talking about, one notices that it is always about the exhibits.

The permanent exhibition of Mathematikum is complemented by several other formats, which address different target groups.

- *Temporary exhibitions* on special topics, such as randomness, calculating devices, mathematics in everyday life, mathematical games, etc.
- *Popular lectures* on special topics such as cryptography, astronomy, etc.
- *Lectures for children* on topics as, for instance, mathematics and—the bicycle, the bees, the heaven, the kitchen, the Christmas tree, and so on.

2.3 Some Experiments

The experiments in Mathematikum cover many mathematical disciplines, such as geometry (shapes and patterns), arithmetic (numbers and calculating), calculus (functions), probability (randomness and statistics), algorithms, and history of mathematics. No mathematical discipline is generally excluded.

We shorty describe some exhibits; more can be found in Beutelspacher (2015).

Figure 2.1 shows an invention of John H. Conway. It is a puzzle consisting of three small cubes of side length 1 and six $2 \times 2 \times 1$-cuboids, which should be

Fig. 2.1 Conway's cube

Fig. 2.2 Lights on!

assembled to form a cube. One first calculates how big the cube will be. Even with this knowledge, most people struggle—until they get the idea where to locate the small cubes in the big cube.

In Fig. 2.2 we see seven lamps in a circle. To each lamp a switch is attached. When trying the switches, one notes that each switch affects three lamps, precisely the lamp is attached to the switch and the lamps on the right hand side and on the left hand side of the switch. When activating a switch, the status of these three lamps changes: those which have been off, are on now, and those which have been on, are off now.

The task, which is already included in the title of the experiment, is to put all lamps on.

Fig. 2.3 Tetrahedron in the cube

Many people start by randomly pressing the switches. Also in this way, we arrive at situations which are promising. For instance, if four lamps in a row are on, then it is easy to switch on the remaining lamps. Also, if only one lamp is on, one has a promising situation. By pushing one switch one gets four enlightened lamps in a row and one can proceed to finish as above.

The experiment shown in Fig. 2.3 consists of two parts.This experiment consists of two parts. One part is a cube made of glass with its upper face removed. The other part is a rather big tetrahedron which is supposed to be put inside of the cube. Most likely, first attempts will fail. Describing failed attempts, one gets an idea of how to succeed. If one holds the tetrahedron so that one vertex points downwards, it won't work. Also, if one vertex points upwards (and its face downwards), it will not work. Now, one could think of trying to let an edge point downwards. In fact, putting edge on a diagonal of the cube's upper square the tetrahedron automatically slides inside.

The experiment shown in Fig. 2.4 provides a challenging task. There is a poster showing a pattern of equilateral triangles. Following the task, one has to hold a framed irregular triangle in-between the lamp and the poster. Of course, we see a shadow. Moreover, the shadow is an irregular triangle. The task is now to put the triangle in a position so that its shadow perfectly fits onto one of the smaller equilateral triangles. For this, one has to move the triangle; back and forth, rotating in all possible ways. Eventually, the perfect shadow is found.

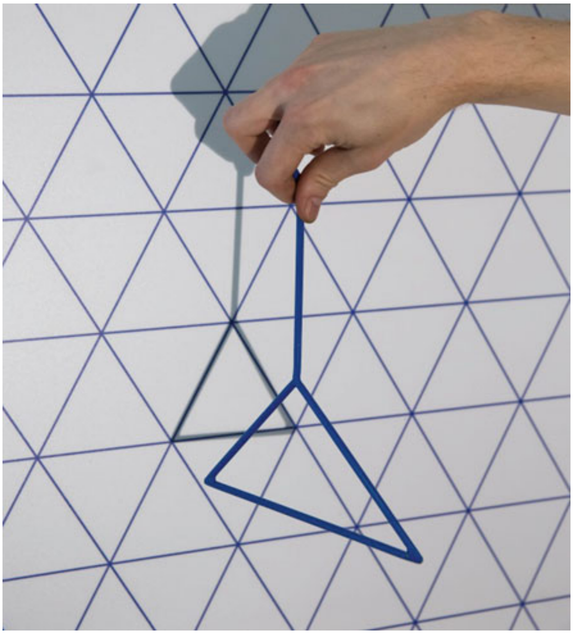

Fig. 2.4 All triangles are equal

Fig. 2.5 The smarties

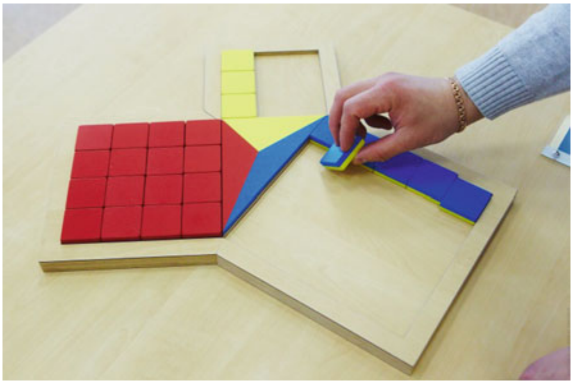

Fig. 2.6 Pythagoras puzzle

In the experiment shown in Fig. 2.5 the vistor is confronted with a poster, where one sees an incredible number of smarties, far too many to be counted. If you want to know how many smarties there are, you have to rely on estimation strategies. Estimations nevertheless are not blindly guessing a number but using the method of a random sample. Next to the picture, you find a square frame. Holding the frame onto the picture, it is easy to count the smarties within the frame. Now, you only have to know how often the frame fits into the picture. You find this number for example by how many times the frame fits into the upper side of the picture and how many times it fits into the vertical side.

Figure 2.6 shows an experiment related to Pythagoras' theorem. In front, there is a triangle the longer side of which is blue and the shorter sides are red and yellow. On either side, there is a square which can be filled with coloured plates. The yellow square can be filled with 3×3 yellow plates, the red square can be filled

Fig. 2.7 Two in a row

with 4 × 4 red plates so that all plates are used. By turning these 9 + 16 plates, the 25 blue plates perfectly fit into the 5 × 5 square above the triangle's blue side.

This experiment illustrates the Pythagorean theorem which states that in a rectangular triangle, the size of the legs' squares ($a^2 + b^2$) equals the size of the hypotenuse's square (c^2). In short, $a^2 + b^2 = c^2$.

In the experiment shown in Fig. 2.7 six wheels invite us to turn them. Each wheel has colorful pieces on it, which vary in form and color. Four different shapes (triangle, square, star, and circle) occur in four different colors, so that we have in total 16 symbols. Each wheel is adorned with these 16 symbols in some random order.

Now we let the wheels rotate. The wheels come to a standstill at random positions. The question is, whether "by accident" two equal symbols (shape and color identical) are at the same line.

Naively, we would conjecture that this will be a rare event, since it is no problem at all to put the six wheels in a position where no equal symbols are at the line. But when performing the experiment, we often see the bewildering situation that two equal symbols are in the same row.

2.4 Books and Easy-to-Built Experiments

In recent years, quite a few books have appeared which contain experiments with cheap material. Many of them are based on paper folding, assembling objects with sticks, and so on. Classical books on this subject are van Delft and Botermans (1978), and, on a higher level, Cundy and Rollett (1952), and Wenninger (1974).

Most of these books aim at the leisure market (for instance Beutelspacher and Wagner 2008), but some are explicitly meant for teachers (e.g. Schmitt-Hartmann and Herget 2013).

The experiments in science centers and the models which can be built using these books share several properties.

– Everybody can perform it. The experiments are deliberately simply to perform. In a strong way it is “mathematics for everybody”.
– People like it. One reason why people like the experiments is their success. Each experiment has the possibility of a positive ending, and “all’s well that ends well”. What is more: the success is undoubtable. When I have composed the pyramid, it stands there and nobody can question it.
– On the other hand, from a mathematician’s point of view, people often stop at an early stage and are satisfied with a superficial effect.

2.5 Two Critical Questions

2.5.1 Are These Experiments at All?

One of the main features of mathematics is that the truth of an assertion is obtained by a proof, that is by purely logical arguments, and not, for instance, by experiments. This distinguishes mathematics from sciences such as physics or chemistry, where experiments are used to verify a theory or to falsify a wrong hypothesis.

Also, mathematical experiments are not used to simply illustrate a definition or a theorem.

The role of a mathematical experiment is quite different. Its basic property is to stimulate thinking. In science centers, experiments do not come second (after a theory), but experiments come first. They provide a strong impulse. Basically, a person working with a mathematical experiment is challenged by a mathematical problem. As in research, one has to get the right conception, the right idea of what’s going on. And sometimes, after a while of thinking, and sometimes with luck, one finds the solution.

A big advantage of such experiments is the fact that the solution is beyond any doubt, because it is materialized: the cube is there, the bridge is stable, the pattern is correct.

To put it short, a mathematical experiment works “bottom-up”: starting from experience, leading to insight. It is an impulse. If the experiment is good, this impulse is so strong that it enables the visitor to ask the right questions, to get the right conceptions and, finally to get by an “Aha-moment” the right insight.

2.5.2 Is This at All Mathematics?

Certainly, it does not look like mathematics, in particular not like school mathematics. In fact, in Mathematikum we explicitly stated at the beginning that we want to make a place that doesn't look like school. Mathematical experiments do not show the mathematical language: no point is called "P", no variable is called x, in fact, there are no formulas. Also, no definitions, no theorems, no proofs.

On the other hand, an important part of mathematical activity is clearly present, namely problems. And, if visitors solve the problems, they activate mathematics-related competences, such as arguing, and communicating.

Mathematical experiments have two main target groups. (a) School classes, (b) private visitors.

When a school class visits Mathematikum, the students may deal with experiments closely related to the topics in math education. For instance, they may look at experiments dealing with the theorem of Pythagoras, or with number systems or with randomness. The teacher then can talk with the students the next day in school about their experiences and insights.

Private visitors, in particular families, behave quite differently. First of all, they have no idea, whether an exhibit represents important mathematics or mathematics at all. They do not care whether the formal mathematics behind it is difficult or easy.

For all visitors it is true that when they start to deal with an experiment, they have a chance to perform a first step into mathematics. The most important aspect is that they think. In fact, they automatically start thinking, for instance asking questions and making conjectures. They try out ideas to solve the problem and eventually they experience the Aha!-moment, in which the whole situation becomes clear.

In addition, when trying to solve a mathematical experiment, the visitors concentrate on important mathematical notions such as edge, angle, it fits, etc. and also they get acquainted with important mathematical concepts, such as patterns, correspondence, infinity, etc.

Finally, they meet not only mathematics taught in school but many aspects which go far beyond school, for instance the travelling salesman problem, minimal surfaces, prime numbers, conic sections, etc.

To sum up, working with mathematical experiments is *a first step into mathematics*. This statement has two sides.

Firstly, it is a step into mathematics. In fact, the problems posed by the experiments can only be solved by thinking, by carefully observing, by looking for the right idea.

On the other hand, dealing with experiments provides only a first step into mathematics. Many more steps could follow. In particular, in this context, there is no formal description of the mathematical phenomena.

In other words, mathematical experiments offer extremely good possibilities to "do" mathematics, but have also clear limitations: they stimulate enthusiasm and true motivation, but also they neither give formal arguments nor can replace a course in a mathematical subject.

Working with mathematical experiments goes far beyond "learning mathematics". It *empowers* people: When visitors see that they have achieved something by thinking by themselves, they become more self-confident.

2.6 Effects and Impact on the Visitors

The main effect of all science centers is *experience*. Visitors experience real phenomena. This is also what visitors like. It is not a virtual experience, which we have by working with computer programs. When we feel real physical objects and work with them, it is clear that we cannot be cheated.

Mathematical experiments stimulate *thinking*. One has to consider several possibilities, one has to develop the right idea for a solution and one verifies whether a solution is correct.

The unquestionable experience of many years of Mathematikum is that dealing with mathematical experiments makes the visitors happy. They become happy because they have *understood* something, which is very satisfying (see also Beutelspacher 2016).

The fact that experiments activate people's brain can be seen—or heard—by the noise in the exhibition. Sometimes it is really loud. But in fact, it is *communication*. People talk to each other, ask questions, give advice—and enjoy the common solution.

A final point: if mathematics is interesting, then it is also interesting outside school. In mathematical science centers as the Mathematikum, mathematics is part of the visitor's leisure time. Adult people and whole families spend hours to experience the power of mathematics. Thus, mathematical experiments and mathematical science centers have a great impact on a *mathematical education of the general public*.

References

Beutelspacher, A. (2015). *Wie man in eine Seifenblase steigt. Die Welt der Mathematik in 100 Experimenten*. Munich: C.H. Beck.

Beutelspacher, A. (2016). What is the impact of interactive mathematical experiments? In W. König (Ed.), *Mathematics and society* (pp. 27–35). Zurich: European Mathematical Society.

Beutelspacher, A., & Wagner, M. (2008). *Wie man durch eine Postkarte steigt.* Freiburg: Herder.

Cundy, H. M., & Rollett, A. P. (1952). *Mathematical models.* Oxford: Clarendon Press.

Dyck, W. (1892). *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente.* München: Wolf.

Schmitt-Hartmann, R., & Herget, W. (2013). *Moderner Unterricht: Papierfalten im Mathematikunterricht.* Stuttgart: Klett.

van Delft, P., & Botermans, J. (1978). *Creative puzzles of the world.* New York: H.N. Abrams.

Wenninger, M. J. (1974). *Polyhedron models*. Cambridge: Cambridge University Press.