Chapter 19
Natural Differentiation—An Approach to Cope with Heterogeneity

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Abstract Teachers in their classes always have to cope with heterogeneity, and that by no means is a new problem. In Germany e.g. plenty of (mostly pedagogical) publications from the midst 1970s until today offer brilliant advice for several kinds of differentiation. How then can it be that after forty years, heterogeneity and differentiation are still called a ›mega issue‹? Could it be that those traditional kinds of differentiation are admittedly to be considered or necessary, but not sufficient—and if: why? This paper will discuss questions like these aiming to bring together crucial issues for (primary) math education in heterogeneous classes, like standards for mathematical practice, standards for mathematical content, social learning with and from each other, and heterogeneity. Main theoretical concepts are substantial learning environments (Wittmann in Educational Studies in Mathematics 15(1):25–36, 1984; Wittmann in Educational Studies in Mathematics 48(1):1–20, 2001a; Wittmann in Proceedings of the Ninth International Congress on Mathematical Education. Kluwer Academic Publishers, Norwell, MA, 2004) and natural differentiation (Wittmann and Müller in Grundkonzeption des Zahlenbuchs. Klett, Stuttgart, 2012; Krauthausen and Scherer in Ideas for natural differentiation in primary mathematics classrooms. Vol. 1: The substantial environment number triangles. Wydawnictwo Uniwersytetu Rzeszowskiego, Rzeszòw, 2010a; Krauthausen and Scherer in Motivation via natural differentiation in mathematics. Wydawnictwo Universytetu Rzeszowskiego, Rzeszów, pp. 11–37, 2010b; Krauthausen and Scherer in Natürliche Differenzierung im Mathematikunterricht – Konzepte und Praxisbeispiele aus der Grundschule. Kallmeyer, Seelze, 2014).

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19.1 Heterogeneity and Differentiation—A Traditional Problem with Traditional Answers?

Math teachers in their classes have to cope with heterogeneity every day of their professional lives. It is by no means a new problem. And it is still as much a theoretical challenge as it is a practical one. This paper tries to illuminate the perspective, concepts and experiences from Germany. Since the early 1970s there have been plenty of publications in that country addressing differentiation (e.g. Bönsch 1976; Geppert and Preuß 1981; Klafki and Stöcker 1976; Winkeler 1976), when it was already called a ›mega issue‹, just as it is still called these days. For primary schools, normally just ›inner differentiation‹ has been taken into account. That means methods to be used within a classroom and not splitting up the class in (supposed) homogeneous groups for a longer time. Today's references in Germany often quote the same theoretical and methodical concepts of inner differentiation, though new terms may have been created (cf. Bönsch 2004; Paradies and Linser 2005). Some of those traditional methods are:

- social differentiation: single work, partner work, group work, …
- differentiation by teaching methods: course-like formats, projects, …
- differentiation by media: textbook, worksheets, manipulatives, digital media, …
- quantitative differentiation: same amount of time for different workload/amount of content, or different amount of time for identical workload/amount of content
- qualitative differentiation: objectives and tasks with different levels of difficulty.

This list looks like a current offer of in-service courses. Actually, it dates back to a booklet from Winkeler (1976), which may raise two questions: (1) Is there no namable progress since then? Why else would differentiation still be so prominent on the agenda? (2) And why is that so?

For sure, those traditional recommendations should not be devaluated per se, nor can be claimed that they are non-effective. But obvious problems can possibly hinder or prevent what is actually intended. Four examples for that:

The idea of ›difficulty‹

Declaring a task as difficult/moderate/easy—a common practice found in workbooks from publishers or on self-made worksheets by teachers—necessarily comes up against limiting factors:

(a) A level of difficulty varies not just between different students, but also with the same student, at different times, and even with the same task (cf. Selter and Spiegel 1997). Difficulty is a question of subjective valuation and not an objective concern.

(b) A *felt* grade of difficulty depends on diverse considerations:

   - complexity of the demands of calculation (kind and size of numbers etc.),
   - involved arithmetic operations (addition and subtraction often seem to be easier to do than multiplication and division),

- demands of cogitation, strategic comprehension, process-related competency, linguistic understanding of the task, amount of required (oral or written) text production or documentation, etc.

(c) The level of a task's difficulty cannot be measured just by the formal-syntactic steps that the solution requires.

All these aspects clearly relativize the sometimes stated claim (by workbooks and even more by digital media) that a learning offer would *automatically* adjust its level of difficulty to the capability or demands of the individual learner.

Individualization and social learning

Put to practice, the postulate of individualization sometimes even results in the abolition of social learning, the learning with and from each other. But individualization does not mean that each student should deal with his very own, individual, and different tasks or even topics. That kind of misunderstanding leads to scenarios that evokes pictures of open-plan offices: Students working at their desks, dispatching different things, and no substantial communication about the things they individually deal with. In this case, social learning is often seen in a mainly pedagogical sense or as a question of classroom management, aiming at implementing effective rules and rituals within and for lessons in order to make classroom relations affable, friendly and non-threatening. No doubt that this all is important, too.

But if restricted to that, argumentation—that is the *communication of minds on shared contents*—turns to become nearly impossible, because there *are no* shared experiences with a common content. Some teachers even take pride in that by declaring: »We abolished those common plenary phases in favor of a thorough individualization«. In doing so, individualization is made absolute and actually leads to the isolation of the learners.

But social learning is not independent of *contents*, and it is reliant on *communication*. And that, according to Bakhtin (1981, 1986), means at least two voices engaging in persuasive discourses about shared contents. Teaching mathematics means to foster the internalization of multivoiced dialogical thinking. In contrast to transmission models (Shannon and Weaver 1949), Bakhtin postulates that *multivoicedness* of communication. And then, heterogeneity comes into play not as an obstacle, but as a source of *cognitive pluralism* to evoke multivoiced discourses (Wertsch 1991).

The importance of collaborative working in groups is a well known basic assumption. But more than students with essentially similar ways of thinking and contributing each a piece of the whole, here it is a matter of students with *truly different* ways of thinking. A heterogeneous classroom in a *natural* way can provide qualitatively different voices. In addition to that, Bakhtin's (1981) rent metaphor may be helpful: A voice, an utterance can just ›rent‹ meaning instead of owning a fixed meaning as it is assumed in an authoritative discourse (Wertsch 1991). Tenants are individuals (students), renter is the community they are part of (e.g. a classroom). And progressing the metaphor (cf. Hollenstein 1997): Renting implies options of influence. What is used cannot remain unchanged. In a multivoiced discourse meanings are steadily modified. In that sense providing and using meanings can create new meaning.

Arbitrariness and wasted thoughtfulness

›Open learning‹ and ›free work‹ are sometimes interpreted as leaving it to the students themselves which contents they would like to deal with. This may harbor the risk of arbitrariness, namely if learning needs are confused with students' desire to deal with whatever they fancy. Instead, the *teacher* him- or herself is responsible for …

- identifying and choosing mathematically substantial contents,
- the didactic design of so called *substantial learning environments* (sensu Wittmann 2001a) and
- keeping in mind far-reaching didactic and subject-matter goals as well as process-related competency (communication, argumentation, problem solving, representation, modelling; cf. KMK 2005).

This requires specific professional competency and cannot just be handed over to elementary students. Even an autonomously learning and high-performing child needs sound support when (s)he meets the zone of his or her proximal development (Vygotsky 1978). The teacher, on the one hand, is responsible for leading the child to its individual limits. On the other hand, (s)he must offer the child sound impulses in order to push those limits more and more forwards. Delving into mathematical structures of the learning contents in that sense does not happen automatically, rather a well-considered encouragement is necessary.

This does not deny the requirement to gradually qualify children for autonomy and self-reliance regarding their own learning process. But about it is necessary to contemplate *where, when* and with *which prerequisites* which degrees of freedom are meaningful, important, and rational for the child.

What about mathematics …?

In Germany, theoretical and conceptual discussions concerning heterogeneity and differentiation were mostly dominated by organizational and methodical questions. In addition to that, most publications originated from a pedagogical point of view. This neglects the essential importance of the subject matter, in this case mathematics, and its specifics.

Meanwhile, several proposals for learning environments in mathematics education were developed where desirable forms of differentiation can take effect—because, in a sense, it is implemented in the topic *itself* (e.g. Hengartner 2006; Hirt and Wälti 2009; Wittmann and Müller 1992, 2017).

19.2 Modified Requirements and Potential Risks

Traditional approaches as a matter of principle are limited. They come along with modified or increased requirements regarding education, school, or society:

- *Increased range of heterogeneity*: The gap between low and high achievers has perceptibly expanded over the years. In one and the same classroom there may sit students whose proficiency may spread over three school years.
- *Inclusion*: In 2008, the Convention of the United Nations on the rights of persons with disabilities came into effect. Implementing its demands for schools and mathematics education is far from being trivial. Until this very day it requires development efforts in order to provide »[e]ffective individualized support […] in environments that maximize academic and social development, consistent with the goal of full inclusion« (UN-Convention 2014, p. 36).
- Traditional kinds of inner differentiation may be helpful and needed. But evidently they are not sufficient. It still lacks a crucial element in order to make perceptible progress in coping with heterogeneity.

The limited range and efficiency of traditional kinds of inner differentiation as well as the varied requirements mentioned above, and finally yet importantly, the demands of actual Common Core State Standards (NGA Center/CCSSO 2010; KMK 2005) involve potential risks: Teachers can feel left alone when trying to bridge the gap between fitting learning processes individually for *all* students *as a basic principle* and fulfilling the requirements of the standards. More than a few teachers in Germany complain that they do not have convincing and effective tools for that.

Consequences can be observed in classes, when the terms differentiation or individualization as popular catch cries are understood rather ambiguously. Their meaning remains mostly unexamined in terms of effective classroom practice, while being still expected to serve as a universal secret weapon for optimizing students' proficiency in math classes.

Additionally, the wide-spread availability and the familiarity with *traditional* kinds of differentiation may cause their application for anything and everything, even in situations where they come up against limiting factors. In cases of perceived helplessness teachers may settle for the mere semblance of what can be called ›modern teaching‹—potentially ending in »open communication with closed mathematics« (Steinbring 1999). In that case, differentiation and individualization become mere labels. But because that practice at least looks ›modern‹, teachers can live with it—more or less, despite an awkward aftertaste.

19.3 Natural Differentiation—A Redefined Answer

The concept of natural differentiation (cf. Wittmann 2001a; Krauthausen and Scherer 2014) intends to fill the gaps of traditional inner differentiation, in particular by …

- orientating actions of differentiation explicitly towards the *specifics of mathematics*,
- doing justice to the different areas of *responsibilities* for teachers and for students,
- ensuring degrees of *freedom* for individual learning processes,
- laying great emphasis on guaranteeing common *social learning* in the sense mentioned above (multivoiced discourses).

There is no unambiguous and comprehensive ›definition‹ of what natural differentiation encompasses. The constitutive characteristics of natural differentiation are embedded in theoretical concepts of math education which are widespread (not only) in German schools and teacher education, namely discovery learning (Bruner 1961; Winter 2016), productive practicing (Winter 1984; Wittmann 1992), and substantial learning environments (Wittmann 1984, 2001a, 2004).

19.3.1 Constitutive Features

Due to text length constraints, they can just briefly be sketched here (cf. Fig. 19.1), followed by a short example (Figs. 19.2 and 19.3; cf. more concrete examples in Krauthausen and Scherer 2010a, b, 2014):

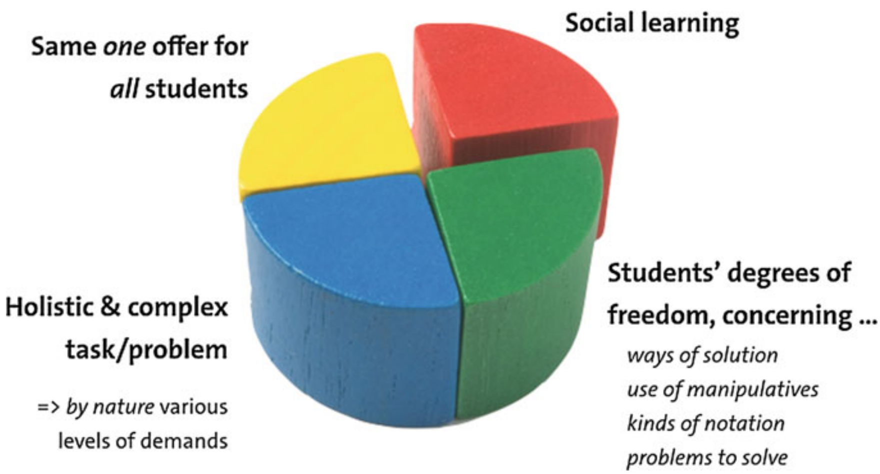
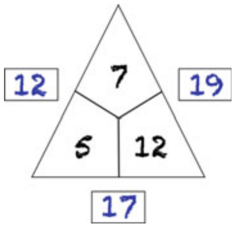


Fig. 19.1 The constitutive components of natural differentiation

Fig. 19.2 Number triangle



1. *Same one offer for all*: All students get the same offer like e.g. a common task or problem. In contrast to traditional differentiation, there is no need for a vast number of additional worksheets or ›special tasks‹ for different levels of capabilities.
2. *Holistic and sufficiently complex*: Because holistic content includes more meaning than isolated parts, this offer must not be split up into several isolated pieces. It must be *holistic* with regards to the content (so, not meant here is the pedagogical sense of ›head, heart and hand‹). This facilitates access for learners of *all* capabilities. The task or problem may not fall below a specific amount of complexity and mathematical substance. This may startle teachers at first or make them sceptical—especially with respect to low achievers who (in traditional differentiation) have been fostered by applying the principle of small and smallest steps and the principle of isolated difficulties. But holistic and complex problems *naturally* allow to develop a momentum of their own, giving room for the inherent dynamism of the topic.
   It is helpful to carefully distinguish between *complex* and *complicated*, which are not necessarily identical! Complexity does not by nature make things more complicated; but complexity can rather facilitate an overview, allow assorting (despite not yet mastering all the details of the content), and a personal access at a certain point for the individual student. On the contrary, if the learning environment or the task is too narrow, the accesses are too limited, possibly to just the one which successfully leads to the (one and only) solution. This kind of challenging and complex learning environments (in contrast to common isolated tasks) are not only an advantage for better learning students (cf. Scherer 1999).

It is important to emphasize in particular: The sound realization of these first two features of natural differentiation is specifically the *teacher's* responsibility. It cannot be delegated to elementary students, as some questionable teaching methods may suggest. Because what is needed here, is a professional background in several respects: Knowledge of mathematical content and mathematics pedagogical content in accordance with the design of well-considered substantial learning environments (cf. Wittmann 2004) as well as of goals concerning the content and the process for elementary math education (and beyond). A substantial learning environment, developed on the basis of a structural-genetic didactic analysis (Wittmann 2013), *then* offers a sufficiently complex frame, including meaningful, reasonable and beneficial degrees of freedom, that means for …

3. *Students' degrees of freedom*: Those first two features mentioned above, *by nature* (naturally) imply different levels of aspiration and difficulty within such a learning environment, without determining them in advance. It is not the teacher who decides about the grade of difficulty to actually work on, but the student, asking him-/herself: Which ways could I follow for a solution? Which aids or manipulatives may be helpful? How could I argue? Which kinds of documentation are at my disposal? Which levels of argumentation are plausible or adequate? (cf. Example in Sect. 19.3.2)
4. *Social learning*: The postulate of social learning from and with each other is fulfilled in a *natural* way as well, since it makes sense *by the content itself*: Because if the whole group has worked on the same problem (though on different levels), then it is obvious to share the various approaches, experiences and solutions. Everybody knows what is on the agenda and what is talked about. Everybody has the opportunity to link his/her own experiences with those of others. And the multivoiced discourse (Bakhtin 1986; Wertsch 1991) can serve as a thinktank to create meaning. Compared with that, traditional differentiation with separated worksheets requires that each student at once makes him-/herself familiar with the pretty different topics presented … if (s)he is at all motivated and capable of that in a final plenary.
   »All students will be confronted with alternative ways of thinking, different techniques, variable conceptions, independent from their individual cognitive level. Rigid inner differentiation is more likely to just complicate this opportunity. […] So, the various, individually organized ways of solution also have an impact on affective, emotional areas. They leave a cognitive scope to students which can facilitate their identification with the learning demands. In this way, the direct experience of autonomy can lead to motivation and interest« (Neubrand and Neubrand 1999, p. 154 f., transl. GKr; also cf. Freudenthal 1974, p. 66 ff.).

19.3.2 An Example: Number Triangles

A well-known topic in nearly every German mathematics textbook for primary schools (grade 1–4) are number triangles. They consist of three interior fields, filled with one number each, and three exterior fields with the particular sum of the corresponding adjacent interior fields (cf. Fig. 19.2).

Number triangles could be used just as a container for any addition or subtraction tasks, simply chosen by chance. Traditional differentiation mostly offered different worksheets with easy/medium/difficult number triangles, allocated to the students by the teacher or chosen by the students themselves. After completing the fields the results just were compared, and the task was done. But this would not at all savor what is actually inherent in number triangles (cf. Wittmann 2001b).

Here are just some tasks or problems around number triangles going beyond simple addition and subtraction (cf. Krauthausen and Scherer 2010a, b, 2014):

- *Practicing the number triangles rule*: In order to make oneself acquainted with the rule, students have to fill in some number tringles. This is more interesting with an additional focus: »Make number triangles with your own numbers—three number triangles you would call easy, three you would call difficult, and three ›special‹ ones« (The latter turned out to be a very interesting question because the term special is so vague!). »And in each case write down why you think so« (to be discussed in the plenum …).
- *Discovering and describing pattern*s: Three filled out number triangles with an inherent pattern are given, two empty ones left to be filled. »Work out and continue! What do you discover? Write down your explanation« (to be discussed …).
- *Generating own patterns*: »Make number triangles with your own patterns and describe them.«
- *Generating patterns for others*: »Design number triangles with different patterns to be continued and described by your partner.« And: »Write down descriptions of different patterns for your partner to be filled into and continued in number triangles.«
- *Moving counters*: »Place a counter into one of the interior fields of a completed number triangle (= increasing this field by 1). Now let the counter move conjointly around the interior fields. What do you discover?« Or: »Two of those counters move clockwisely, one counter moves counter clockwisely. What do you discover?«
- *Number triangles with numbers from the multiplication tables*: Partly completed number triangles just contain numbers from the multiplication tables. »What do you discover? Explain …« (The inherent distributive law can be justified by patterns of counters.)
- *Even/odd exterior fields*: This example will be explained below (cf. Fig. 19.3).
- *Three exterior fields given*: »How can you find the numbers for the interior fields?« (there are different ways, with and beyond just trial and error)
- *Sums of the interior/exterior fields*: »Have a closer look at sums of interior fields and sums of exterior fields. What do you discover? Explain …« (This problem also offers some hints for the task mentioned before.)

How do number triangles with problems like these serve the constitutive features of natural differentiation? As an example, Fig. 19.3 shows the upper part of a corresponding worksheet which was tested in many classes (the lower part just offered empty number triangles for investigations).

Same one offer for all students

The whole class got this same worksheet with claims from Mandy and John. No different tasks and ›special‹ worksheets for ›special‹ needs.

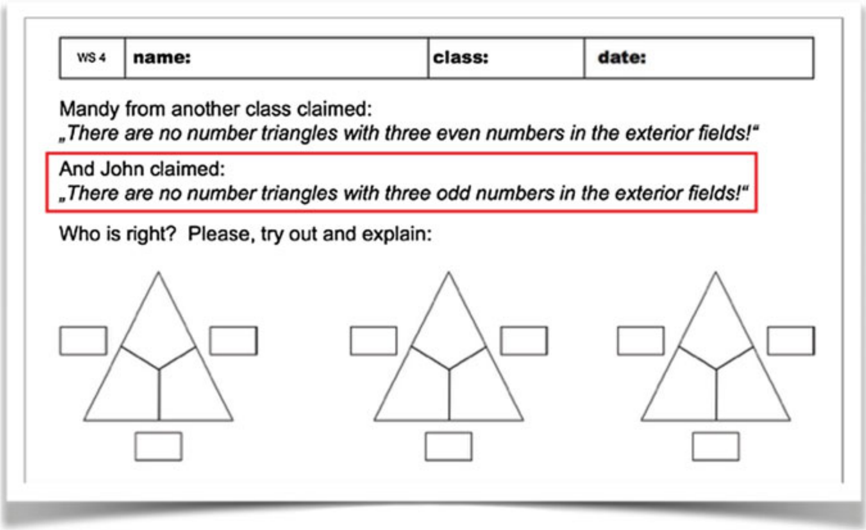


Fig. 19.3 The even/odd-problem

Holistic and sufficiently complex

Mandy's and John's claims by nature offered various levels of demands, ranging from more or less trial-and-error approaches (calculating several number triangles) to arguing with number properties in a more general way.

Students' degrees of freedom

It was possible for the students to choose own numbers for their investigations (arithmetical approach). Or they used abbreviations like *e*(ven) or *o*(dd), a pre-algebraical approach. Mandy's utterance could be disproved by just one counter-example. This may be found with or without a case discrimination (three even numbers or three odd numbers in the interior fields). John's claim could also be investigated with concrete numbers or pre-algebraically. Several bidirectional transitions between those approaches could be observed within classrooms.

Some students used counters or other manipulatives to explain their arguments, others worked just on the symbolic level. Some fourth-graders were not settled for the fact that there are no triangles with three odd exterior numbers (John's claim). So they felt free to change the triangle format to a number square with four interior fields—and for that John was proved right. Then they investigated more cases, ending up in the general utterance that John's claim is true for all cases where there is an even number of interior fields. Others argued that John is right because they used rational numbers in the interior fields.

Social learning

All those different approaches and argumentations *by nature* suggested common discussions in a plenary phase: Is it allowed to use rational numbers? Why or why not, who tells? What is the definition of even numbers—a number that can be split up evenly, into two equal halves? »But that is the case with 7 = 3.5 + 3.5! So seven is even?!« This utterance of a fourth-grader opened a substantial discussion among the students of that class …

19.4 Demands for Teacher Education

Natural differentiation is not a magic wand, evolving its efficiency automatically or in the sense of self-evident, unstudied etc. The term ›natural‹ (in its colloquial meaning) might connote this, but in clear contrast to that the concept of natural differentiation is understood in the sense explained above. And then it can be a pretty powerful tool for math educators. But it is reliant on diverse prevailing circumstances. Just two of them will be shortly introduced here as they emerge for teacher education.

Mathematics content knowledge

Due to text length constraints it cannot explored here in detail what kind of content knowledge is needed for elementary teachers (cf. e.g. Loewenberg Ball 2003; Osana et al. 2006). But for sure, the content knowledge of elementary teachers has to be expanded, as TEDS-M has shown (cf. Blömeke and Delaney 2012). For a general characterization a postulate from Freudenthal can be helpful. For teachers' mathematics content knowledge he required the same as for the mathematics that they have to teach: It has to be diversely related (Freudenthal 1978, p. 71 f.). Mathematics content knowledge for prospective teachers should not just consist of isolated collections of facts (cf. above: ›Even numbers can be divided into two equal parts‹), but rather make the manifold interconnections and structural relations transparent and available. And it includes *attitudes* as well.

Math education has to enable students to realize the demands of the subject matter. Teachers might succeed in this even better, the more they themselves feel an aptitude to the contents of their teaching. »The art of teaching has to convey the claim of the content. It is generally agreed that we also have to know something about the learners […]. It is a given that we have to reflect the order of presentation […], the arrangement of our teaching. But all of this remains hollow without love for the content, without a steady effort to do justice to it […]. The shift of pedagogical interest, away from contents and instead to psychology or methods, seems questionable to me. I ask myself, how can contents be imparted by people who know how to present them, but who themselves do not feel the demands involved. How can somebody who is not interested in a topic, make this topic interesting for somebody else?« (Schreier 1995, p. 14 f.; transl. GKr).

Methodical competency

A fundamental aspect is the ability to stimulate and to maintain a shared communicative exchange among the learners in the sense of mathematical discourses. This »is not an easy task—neither for the teacher as a moderator, nor for the students, who at first will have to learn a more self-directed communication; and for that they are entitled to professional support by their teacher« (Krauthausen and Scherer 2014, p. 82; transl. GKr). Some teachers may confuse a plenum with rigid ›chalk and talk method‹, a teacher-centered approach from the front of the classroom and generally associated with an antiquated understanding of classroom practice. But in fact, there are several good reasons for a plenary phase, e.g. content-related ones.

A common plenum by no means involves a revitalization of an outdated traditional method. Instead it gets *a new function* with the main goal of deepening the content-related demands. It is just this *newly customized* plenary phase which first and foremost allows a deeper incursion into the mathematical core. Because students hardly will and can do that by themselves (e.g. in the range of traditional differentiation), a higher point of view is needed for that. In other words: A professional moderation by the teacher is as a matter of fact not just indispensable, but also much more possible than at other times when students work actively (alone or in small groups) on a problem. A sound moderation in that sense certainly belongs to the most demanding tasks of teaching. Because a deep understanding of content-knowledge is needed, as well as a sure instinct for the right moment and the appropriate impulse—and all that in real-time, spontaneously, and without time to contemplate. The teacher in the example mentioned above did that quite well (cf. transcripts in Krauthausen and Scherer 2010a).

Another reason would be a sociological perspective on the role of mathematical discourses about shared contents. Miller (2006, p. 200 ff.) considers social discourses a compulsory factor in modern learning (cf. Bakhtin). According to that, learning can only happen as desired (that means: effectively and sustainably), if learners enter a shared argumentation—about the *process of generating knowledge*, and not first and foremost about the completed products (cf. Krauthausen and Scherer 2014). »Only in collective discourses the learners involved will be able to develop argumentative contexts for generating new insights and new knowledge […] and that by moments of reciprocal differences, misunderstandings and irritations« (Schülke and Söbbeke 2010, p. 21, transl. GKr; cf. also Schülke 2013).

Therefore both the following demands are essential for teaching:

(a) Content must be expressed in <u>*language*</u>—*as a matter of principle* there must be communication (= emphasis on *language*); and …
(b) <u>*Content*</u> must be expressed in language—so, there must be content involved (= emphasis on *content*), not just talking about anything.

Connecting both meanings in a fruitful way is one of the special and most demanding tasks for teachers moderating such kind of argumentative discourses in a sound and child-oriented way—mathematically substantial, elaborately expressed and effective.

Moderation competence, too, is a complex concept and cannot be discussed here in its entirety. That's why just a few catch words will be mentioned in order to mark the direction and to hint at some facets of the bundle of skills (cf. examples in Krauthausen and Scherer 2014):

- *Imperative of influence*: It is a misunderstanding (and in a sense a failure to render assistance) that students should discover all and everything just by themselves, and teachers would have nothing to do but observe. Didactic responsibility includes exerting influence. The question indeed is what that means, and how it is done. In his famous paper ›Taboos of the Teaching Profession‹ Adorno says: »Success as a teacher is apparently due to the absence of any kind of predictive influence and relinquishing persuasion« (Adorno 1965, p. 491; transl. GKr). Adorno does not argue against influence, but against predictive influence, e.g. by too deterministic and prescribing lesson plans which then are strictly executed. Possible deviations from prescription are rebalanced by means of Bauersfeld's funnel pattern (Bauersfeld 1983; Voigt 1984). This, of course, is not what Adorno had in mind.
- *Reserve*: Teachers have to control their own ›missionary enthusiasm‹. They must not tell and explain their students everything immediately. »To reveal something to a child what it could find out by himself is not just bad teaching, it is a crime« (Freudenthal 1971, p. 424; transl. GKr).
- *Monitoring the learning process and analytical listening*: Once again Freudenthal explains the difference to just occasionally watching students' activities: »I called it intelligent observing. Not recording photographically. Before you start observing you have to know what to pay attention to. On the other hand, you must not know this too exactly, because then you will *just* see what you want« (Freudenthal 1978, p. 162; transl. GKr).
- *Authentic curiosity*: Genuine curiosity for what a child knows and how (s)he thinks as well as authentic, true and no staged enthusiasm are a fundamental tenor of teachers who want to foster and support mathematical discourses with and among their students. Their comments and answers are not just classified as wrong or right or (counter-)productive, but helpful for the teacher to understand even better what and how the students think.
- *Encouragement to express oneself and turn towards others*: All students must have secure confidence that they can express all their thoughts, assumptions, even ventured ideas, free of sanctions and without the prospect of hasty evaluations.
- *Manifold repertoire of questions*: Especially valuable are ›higher order‹ questions and impulses which initiate new/distinct/variable thinking as well as autonomous/reasoning/inferential thinking.
- *Probing into the subject matter again*: This is to encourage students to dig deeper into the subject. To repeat questions does not at all mean that an answer must have been wrong.

- *Having a break*: Productive discourses sometimes need a break. Not in order to interrupt the thinking process, but to pause for thought. Short moments of silence—caused e.g. by speechless astonishment, by surprise, by hesitation, by skepticism—should be experienced as productive, not as embarrassing blankness which ought to be filled as soon as possible with a strange comment or a displacement activity.

19.5 Conclusion

Natural differentiation as a specific kind of inner differentiation offers opportunities to design the learning of heterogeneous students in a way that is more productive and more sustainable for all. It is natural, because heterogeneous groups of learners *by nature* evoke and foster multivoiced discourses expressing truly different ways of thinking on truly different levels. And it is natural, because complex and holistic problems, like substantial learning environments, *by nature* allow a momentum of their own, giving room for the inherent dynamism of the *content*. And it is natural because it is the *learner* who can make use of his/her degrees of freedom in several respects in a designated frame of a mathematical substantial learning environment.

The special prospects in particular lie in the following attributes of the concept:

- Emphasizing the *specifics of mathematics* and consciously valuating the demands of the content as well as the *social learning* postulate, especially via moderated discourses about shared experiences with working on a common learning environment.
- Emphasizing an *integrative* access to content-related and process-related mathematical competency (KMK 2005).
- No claim as an *all-in-one* tool for the whole range of mathematical learning and teaching (though rather likely for its major part). Practicing basic facts or introducing a specific procedure may require other methods.
- Availability of numerous *appropriate learning environments* (sensu Wittmann 2004) for substantial hands-on activities that meet the demands of proper natural differentiation.

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