

Chapter 17

Powering Knowledge Versus Pouring Facts

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Abstract Many problems related to the real world admit a mathematical description (i.e., a mathematical model) based on what is studied at school. Solving the mathematical model, however, often requires a higher level of mathematics, and this is the reason for not including such problems in the curriculum. We present several problems of this kind and propose solutions to their mathematical models by means of widely available dynamic mathematics software (DMS) systems. For some of the problems, it is possible to directly use the in-built functionalities of the DMS and to construct a computer representation of the problem that allows exploring the situation and obtaining a solution without developing a mathematical model first. Using DMS in this way can broaden the applicability of school mathematics and increase its appeal. The ability of students to solve problems with the help of DMS has been tested by means of two types of competitions.

Keywords Mathematical modelling · Inquiry education · Computational thinking

17.1 Introduction

There are two partially contradicting trends in high school mathematics education. On one hand, we want mathematical knowledge to be based on a solid logical base (rigor). On the other hand, we want this knowledge to be rich both in content and applications. These two trends cannot always (and easily) be reconciled (De Lange 1996). One of the reasons for this contradiction is the fact that only a few problems related to practice allow mathematically pure and complete treatment with the traditional rigor. The demonstration of patterns of logical thinking is time consuming and often related to simplified mathematical content that does not properly reflect the unavoidable complexity of the real world. The formulation of a

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mathematical model for a real-life situation cannot be based on rigor only. Dropping out some features and keeping only the most essential ones in the mathematical model requires skills that have little to do with rigor, and this is an obstacle for the inclusion of complex real-life situations in the mathematics curriculum. Furthermore, there are many problems related to practice (some of which will be considered below) that can be equipped with a reasonable mathematical model based on what is studied at school. The corresponding model may be a system of equations, an optimization problem, or something else of a mathematical nature. Solving this mathematical model, however, with the traditional rigor within the frame of the school mathematics is not always possible. It may require a higher level of mathematical knowledge, for instance, advanced calculus and/or numerical methods for approximation of the exact solution. This is another reason for avoiding the consideration of genuine real-life applications within the school mathematics. However, with the appearance of powerful and widely accessible dynamic mathematics software (DMS) systems it became possible to reduce, at least partially, the mentioned contradiction between rigor and applications. Solving a model can be performed by means of DMS. As mentioned in Hegedus et al. (2017) “This leaves more time for essential mathematical skills, e.g., interpreting, reflecting, arguing and also modeling or model building for which there is mostly no time in traditional teaching” (p. 20). With the help of technology, it is possible to offer to students much more demanding mathematical content and interesting applications (Hoyles and Lagrange 2009). Such a change would drastically increase the realm of real-life problems that can be considered in school. We do not have in mind only the traditional application of computers where a mathematical model of the problem is solved by a computer; in addition, some examples will be described below where the standard in-built operations (“buttons”) of the DMS system can be used directly to make a computer representation of the problem without first writing the formulas of a mathematical model. This DMS representation of the problem will be called a “computer model of the problem.” By means of this model and the in-built functionalities of DMS (such as dragging, measuring distances, and areas), the solution of the problem can be found with a reasonable degree of precision. This direct DMS modelling of the problem as well as the mathematical modelling of the problem, followed by a DMS-assisted solution, are in the focus of this paper, which is mainly oriented toward problem solving. Both types of modelling support the most natural way of knowledge acquisition: by experimenting, by formulating and verifying conjectures, by discussing with peers, and by asking more experienced people. In a nutshell, the technology provides the opportunity to learn mathematics by inquiry. This refers not only to what happens (and how it happens) in class but also to extracurricular activities that provide a fruitful playground for building mathematical literacy and cultivating elements of computational thinking (Freiman et al. 2009). Another advantage of using technology in this way is that much larger and more operational mathematical content could be given to the students at an earlier age.

Later in the paper, several problems are considered for which it is easy to assign a proper mathematical model based on school mathematics but whose solution with the necessary rigor while remaining in the frame of school mathematics is relatively

difficult (or at least not easy). On the other hand, these models are easily solvable by means of DMS systems. This way of problem solving opens further opportunities for inquiry and cultivates the elementary computational thinking skills of the students, thus *powering* (in the sense of “adding power to”) their existing knowledge and skills. Problems such as the ones considered below and the inquiry-based approach to their solving can make the mathematics studied in school more applicable and more appealing in contrast to the now prevailing *pouring* of mathematical facts. The ability of school students of different ages to solve such problems has been tested by means of two online competitions called VIVA Mathematics with Computer and Theme of the Month. The participants’ scores show that the use of DMS for problem solving is gradually gaining popularity in Bulgaria. The students are interested in this approach and many are capable of using it. The problems considered next have been used in these competitions.

17.2 The Sample of Problems

The problems in this section illustrate the differences in the uses of the models we consider in this paper: a *mathematical model* which can (or cannot) be solved in the traditional way, a *mathematical model* allowing a simple DMS supported solution, and a *computer model* (direct DMS representation of the problem). Each of the problems is easy to formulate as a mathematical model but not so easy to solve with the usual rigor within school mathematics. On the other hand, an approximate DMS-assisted solution is readily available, or a computer model of the problem is easy to construct by means of which the problem can be solved even at the earlier stages of secondary education.

The Parking Entrance Problem

This problem is a further elaboration of one of the *Problems of the Month* used in the European Project MASCIL (<http://www.mascil-project.eu/>). We present both the computer modeling, which is amenable for younger students using DMS, and the pencil-and-paper mathematical modeling, which requires rather advanced knowledge of mathematics.

Problem 1 A vehicle (car, baby carriage, or wheel-chair) with a wheelbase b (the distance between the centers of the wheels) and clearance (ride height) c is to be moved from the street to the basement of a house over a slope of γ degrees (Fig. 17.1; $\gamma = 20^\circ$). Is this possible without damaging the bottom of the vehicle? (Fig. 17.2)

The answer to this problem depends on the concrete values of the parameters b , c , and γ . A steeper slope γ is more likely to cause damage to the vehicle. Damage will occur also if b is big enough. The clearance c is also decisive. The interplay between these parameters is not simple and the usual intuition does not help much. The heavy scratches on the surface of many “sleeping policemen” (speed bumps)

Fig. 17.1 Moving a vehicle from the street to the basement

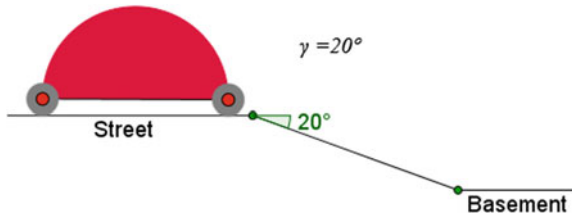
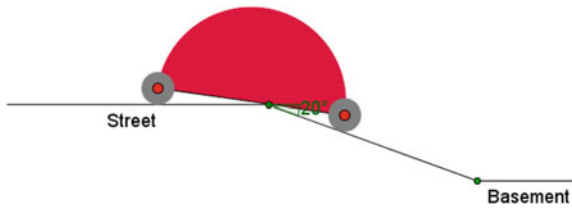


Fig. 17.2 The lowest flat part of the vehicle can hit the “vertex” at the beginning of the slope



on the streets indicate that problems similar to this one are important. Further, for the sake of simplicity, we will depict the vehicle only by its two wheels (circles of radius c centered at A and B respectively) and the segment AB (the wheelbase) connecting the wheels. Both the computer model of this problem and its mathematical model rely on the very basic geometric fact that the opposite angles formed by two intersecting lines are equal (angles α in Fig. 17.3). Figure 17.3 shows the collision situation when the vertex at the beginning of the slope hits the bottom of the vehicle at some point C from the segment AB . The second arm of the angle β on Fig. 17.3 is the tangent from C to the front wheel.

A collision occurs only if $\alpha + \beta < \gamma$ (the front wheel is no longer rolling on the slope). This suggests the idea for the computer model that is visualized on Fig. 17.4. The numbers b , c , and γ are entered in the model as parameters (sliders in GeoGebra). Using the built-in operations of GeoGebra, one constructs a segment AB of length b and two circles (the wheels) of radius c centered at A and B and takes an arbitrary point C on AB that is outside the two wheels. Further, tangents from C to these circles are drawn as shown in Fig. 17.4 and, finally, the angles α and β are measured by the corresponding operation in GeoGebra.

Fig. 17.3 Collision situation

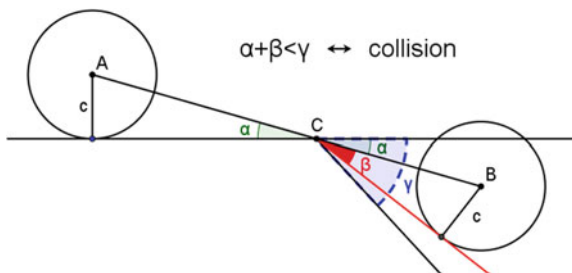
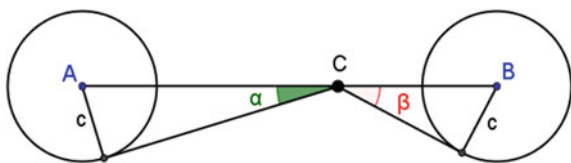


Fig. 17.4 A computer model for the parking problem



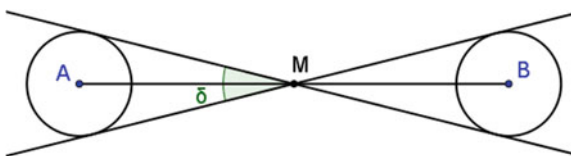
The sum $\delta = \alpha + \beta$ is a function of the position of the point C . By moving (dragging) point C along AB and observing the change of δ , one can establish experimentally that the function δ attains its minimum at the point M , which is the middle of AB . If this minimum is bigger or equal to γ , the vehicle could be parked safely in the basement. This observation confirms the intuitive expectation that the middle M of the segment AB is the critical and most vulnerable point. If it passes above the slope vertex, the vehicle can be parked safely in the basement. This observation also shows that even a simpler computer model can solve the problem. Note that if C and M coincide, then $\alpha = \beta$, and the condition for non-collision takes the form $2\alpha \geq \gamma$. Given the numbers b , c , and γ , one finds the middle M of the segment AB , draws the tangents from M to the two wheels, and measures the angle δ between these tangents (Fig. 17.5). If $\delta \geq \gamma$, the vehicle can be moved safely. If $\delta < \gamma$ there will be a collision and moving it without damage becomes impossible.

The second computer model solution of this problem is completely amenable for students at earlier stages of secondary education. In contrast, as we will now see, the mathematical model of the problem requires knowledge of inverse trigonometric functions, and the classical solution uses some elements of calculus. Denote by x the length of the segment CA in Fig. 17.4. Then $\alpha = \arcsin \frac{c}{x}$ and $\beta = \arcsin \frac{c}{b-x}$. One has to find the minimum of the function $\delta(x) = \arcsin \frac{c}{x} + \arcsin \frac{c}{b-x}$ in the interval $[c, b - c]$ (this is the interval where the function $\delta(x)$ is well-defined; we implicitly assume here that $b > 2c$). By finding the zeros of the derivative of $\delta(x)$, one can derive that the minimum of this function is attained for $x = \frac{b}{2}$ and solve the problem.

Here are some tasks for further inquiry with the computer or the mathematical model of this problem:

Problem 1.1 What is the steepest slope (in degrees) that a baby carriage with $b = 130$ cm and $c = 12$ cm can overcome without troubles?

Fig. 17.5 A simpler computer model for the parking problem



Problem 1.2 If the slope to the basement is 20° and the wheelbase of the car is $b = 290$ cm, what is the smallest radius of the wheels such that moving the car to the basement will not be a problem?

Problem 1.3 For some vehicles, the bottom line is different from the line connecting the centers of the wheels. Also, the front wheels and the rear wheels are not always of the same radius (Fig. 17.6). Develop a computer and a mathematical model for the exploration of the dangers for moving such vehicles down slopes.

One could further explore the parking problem by means of the more realistic computer model developed by Toni Chehlarova. The corresponding GeoGebra file is available at <http://cabinet.bg/content/bg/html/d22178.html> (last visited December 2016).

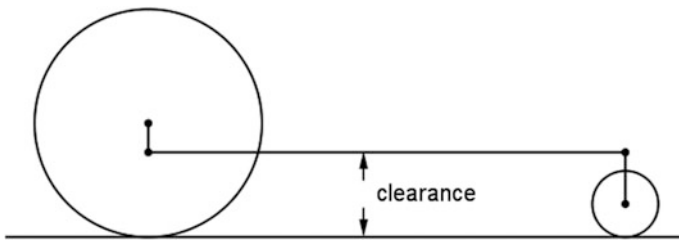


Fig. 17.6 A model with different wheels

The Cylindrical Container Problem

Problem 2 Two thirds of the volume of a closed cylindrical can of radius 5 cm (Fig. 17.7) is filled with some liquid. What is the height of the liquid if the can is laid horizontally?

The problem seems to be three dimensional but could be easily reduced to a two dimensional one. In the horizontal position, two thirds of the circle area of the can base are covered by the liquid. Hence, the problem is reduced to finding a horizontal chord AB (Fig. 17.8) in a circle of radius 5 cm with center at O that cuts off a circular segment (slice) of area one third of the total area of the circle.

Fig. 17.7 The cistern problem

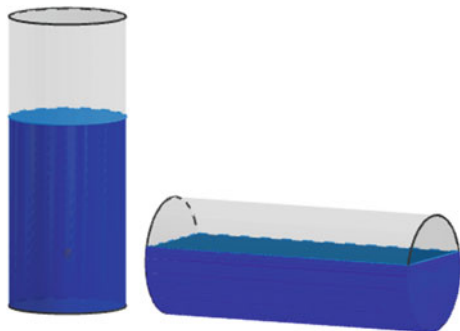
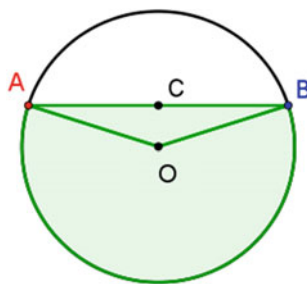


Fig. 17.8 Cutting a circular segment with an area one third of the circle area



This can be done in different ways. The in-built operations of the DMS can be used to find the area of the circular sector outlined by the segments OA , OB , and the arc from B to A (in the counterclockwise direction) and the area of the triangle AOB . The difference between the two areas is the area of the circular segment we are looking for. If the horizontal chord AB is made movable (the DMS takes care of the dynamics and automatically re-calculates the areas), a position for the chord AB can be found such that the area of the circular segment is one third of the area of the entire circle. If C is the middle of the chord AB at this position, then the height of the liquid in the horizontal can is equal to the radius of the can base (5 cm) plus the length of the segment CO (which can be measured by the functionalities of the DMS). In our case, an approximate value for the height of the liquid is 6.32 cm. The computer model just developed allows exploration of similar situations with other cylindrical cans (the radius of the can could be made changeable, the part of the can volume which is filled with liquid in vertical position can change, etc.).

We will now proceed to a mathematical model of the problem. For the sake of generality (and since this will not introduce further complications), we will denote the radius of the can base by r . Let α be the measure (in radians) of the angle in the circular sector considered above. The area of this sector is $\frac{\alpha}{2}r^2$. The area of the triangle OBA is $\frac{1}{2}r^2 \sin \alpha$. Hence, the angle α that corresponds to a circular segment with area equal to one third of the area of the circle has to satisfy the equation $\frac{\alpha}{2}r^2 - \frac{1}{2}r^2 \sin \alpha = \frac{1}{3}\pi r^2$. Equivalently, $\alpha - \sin \alpha - \frac{2}{3}\pi = 0$. As we see, the mathematical model of this problem is an exotic equation. School mathematics does not deal with such equations, and this seems to be the reason for not including this important cistern problem in the curriculum. The numerical/graphical solution of this model by DMS, however, is available. The graph of the function $f(x) = x - \sin x - \frac{2}{3}\pi$ is depicted in Fig. 17.9. The point A has been constructed as the intersection of the graph of f and the x -axis. The first coordinate of A gives the angle we are looking for: $\alpha = 2.60533$ (the precision of 5 digits after the decimal point is taken here arbitrarily; it can be increased or decreased).

The length of the segment OC corresponding to this α and $r = 5$ can be calculated: $OC = r \cos \frac{\alpha}{2} = 1.32465$. For the height of the liquid in the horizontal position of the can, we obtain 6.32465.

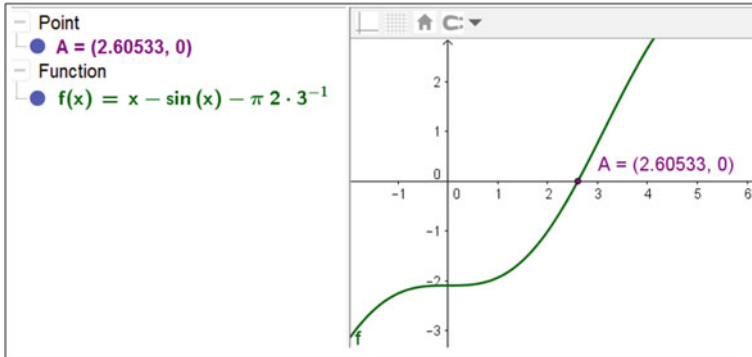


Fig. 17.9 Graphical presentation of the function $f(x)$

If the angle α is measured in degrees, the area of the circular sector is $\frac{\alpha}{360} \pi r^2$. Correspondingly, the equation from which the angle α will be determined has the following appearance:

$$\frac{\alpha}{180} \pi - \sin \alpha - \frac{2}{3} \pi = 0$$

For further inquiries with either the computer model or with the mathematical model, one could consider the following related problems:

Problem 2.1 A horizontally laid cylindrical tank with diameter 200 cm and length 500 cm is partially filled with petrol so that the level of the petrol is 80 cm. How many liters of petrol are there in the tank?

Problem 2.2 If the height of the can from Problem 2 is 24 cm, how much additional liquid should be poured into it in a horizontal position so that the level of the liquid is elevated by 1 cm? If after the addition of the liquid the can is turned into vertical position, what is the height of the liquid level?

Problem 2.3 If the height of the can from Problem 2 is 24 cm, how much liquid should be removed from it so that in a horizontal position the liquid level drops down by 1 cm?

Problem 2.4 A heavy metal ball of radius 4 cm is placed into an empty vertically placed can of radius 5 cm and height 25 cm. Then liquid is poured into the can until its level reaches 20 cm and then the can is sealed. What would the liquid level be, if the can is laid horizontally (see Fig. 17.10)?

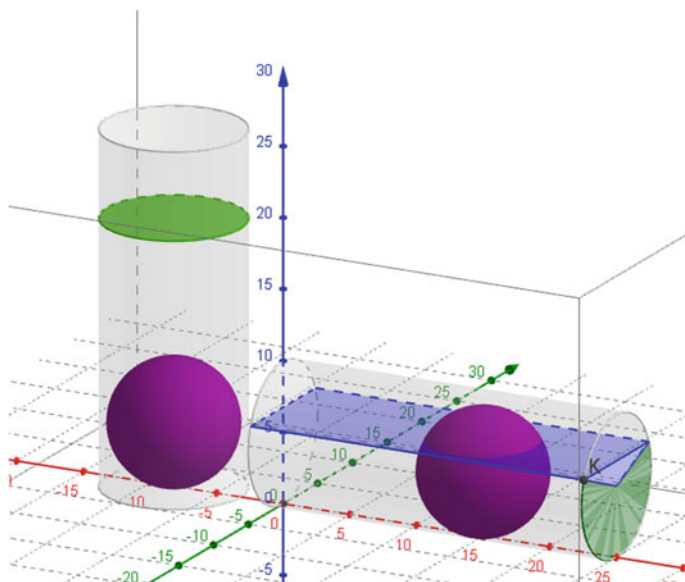


Fig. 17.10 In a horizontal position, a part of the ball might be above the liquid surface

The Conical Container Problem

This problem is a well-known mathematics exercise for university students. It can be settled by means of calculus or by a mathematical trick with inequalities. We present the mathematical model and demonstrate that by means of a DMS the problem can be considered and solved in school.

Problem 3 A circular sector of measure α (in degrees) has been cut out from a circular plastic sheet of radius l with center O (Fig. 17.11). From the remaining part, a right circular cone is made by sticking (gluing) the cuts (Fig. 17.12). What is the size of angle α (in degrees) for which the volume of the resultant cone is maximal?

The mathematical model of this problem is based on the well-known formula for the volume V of the cone: $V = \frac{\pi R^2}{3} h$. Here R is the radius of the cone base and h is

Fig. 17.11 Cutting a sector

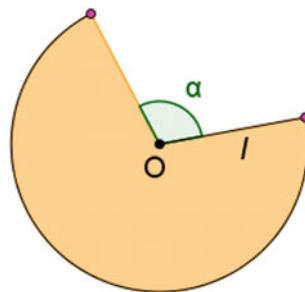
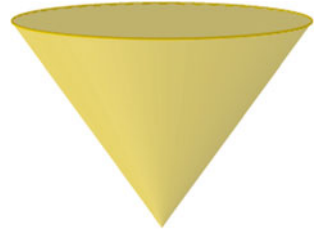


Fig. 17.12 Gluing a cone



cone's height. Since α is measured in degrees, the length of the arc of the removed circular sector is $\frac{\alpha}{360} 2\pi l$. Therefore, the length of the cone base circumference is what remains after the cutting: $2\pi l - \frac{\alpha}{360} 2\pi l$. Hence, $2\pi l - \frac{\alpha}{360} 2\pi l = 2\pi R$. It follows that the radius R can be expressed as function of $x = \frac{\alpha}{360}$: $R = l(1 - x)$. Further, it follows from Pythagoras's theorem that $h^2 = l^2 - R^2 = l^2(1 - (1 - x)^2)$. i.e.,

$h = l\sqrt{1 - (1 - x)^2}$. Thus, the volume of the cone is $V = \frac{1}{3}\pi l^3(1 - x)^2\sqrt{1 - (1 - x)^2}$. The essence of the problem, its mathematical

model, is to find a number x , $0 \leq x \leq 1$, for which the function $f(x) = (1 - x)^2\sqrt{1 - (1 - x)^2}$ attains its maximal value. Once again we see that the derivation of the mathematical model is based on school mathematics. Solving this model however requires more advanced mathematics. Using calculus one can find the extremal values of this function f by finding the zeros of its derivative. These zeros are $x = 1 - \frac{\sqrt{2}}{\sqrt{3}}$, $x = 1$ and $x = 1 + \frac{\sqrt{2}}{\sqrt{3}}$. The last of these numbers is outside the interval $[0, 1]$ and is not relevant for our considerations. The value $x = 1$ corresponds to a minimum for f because $f(1) = 0$. Therefore the maximum of f is attained at $x = 1 - \frac{\sqrt{2}}{\sqrt{3}}$ and the value of f at this point is equal to $\frac{2}{3}\sqrt{\frac{1}{3}}$.

There is a nice trick which allows solution of this mathematical model by means of the well-known inequality between the arithmetic mean and the geometric mean of any non-negative numbers a , b , and c : $\sqrt[3]{abc} \leq \frac{a+b+c}{3}$. It is known also that equality is attained in this inequality if and only if $a = b = c$. Applying this inequality for $a = b = \frac{(1-x)^2}{2}$, $c = 1 - (1 - x)^2$, we get

$$\begin{aligned} f(x) &= \sqrt{(1-x)^4(1-(1-x)^2)} \\ &= 2\sqrt{\frac{(1-x)^2}{2} \frac{(1-x)^2}{2} (1-(1-x)^2)} \leq 2\sqrt{\left(\frac{1}{3}\right)^3} = \frac{2}{3}\sqrt{\frac{1}{3}}. \end{aligned}$$

The equality will be reached when $\frac{(1-x)^2}{2} = (1 - (1 - x)^2)$. This again yields $x = 1 - \frac{\sqrt{2}}{\sqrt{3}}$.

If at all, calculus and the mentioned trick with the inequality are available only at the last stages of school mathematics. With the help of DMS, however, the mathematical model of this problem can be solved by younger students. It is possible to draw the graph of the function $f(x)$ and see where its maximum is. The graph of the function $f(x)$ can be seen in Fig. 17.13.

It is clear from this picture that the function f has two maxima. Only the one in the interval $[0, 1]$ on the x -axes is of interest for us. The DMS (GeoGebra) allows observation of the coordinates of a point A , which moves along the graph of the function. When A is dragged to the highest point in the graph, its first coordinate will be equal to the value of x we are looking for. In Fig. 17.13, this is the point $A = (0.18, 0.38)$. If the precision of calculations is increased, one gets $x = 0.1835$, which is a very good approximation of $x = 1 - \frac{\sqrt{5}}{3}$. This value of x corresponds to $\alpha \approx 66.06^\circ$, and the latter value could be accepted as a reasonable solution to Problem 3.

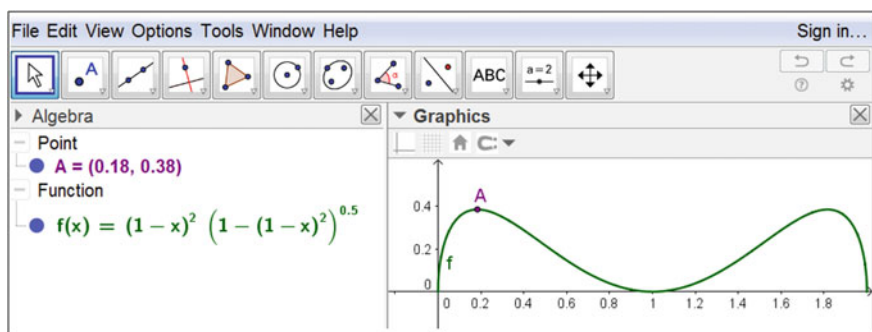


Fig. 17.13 The graph of the function $f(x)$

The Ice Cream Container Problem

The next problem is a challenge for pencil-and-paper technology, even for university students. With the help of DMS it is completely amenable for school students.

Problem 4 An ice cream container (as depicted in Fig. 17.14) is to be made of a circular plastic sheet of radius l with center O by cutting and gluing (sticking). The cutting and gluing operations allowed and the order in which they are performed are:

- Cut a circular sector of measure α (in degrees) from the plastic sheet (Fig. 17.15) and, by gluing, make from it a cone that will serve as the lower part of the ice cream container.
- Cut off from the remainder (Fig. 17.15) a full circular sector of radius t (this number t is to be specified later) and glue a cut cone (truncated cone) that will serve as the upper part of the ice cream container.

For what size of α will the ice cream container have largest volume?

Fig. 17.14 The ice-cream container

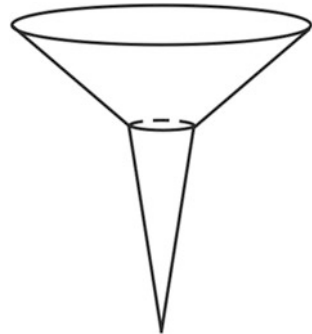
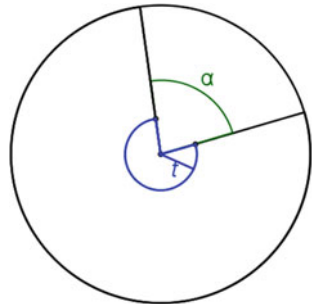


Fig. 17.15 The cutting and gluing process



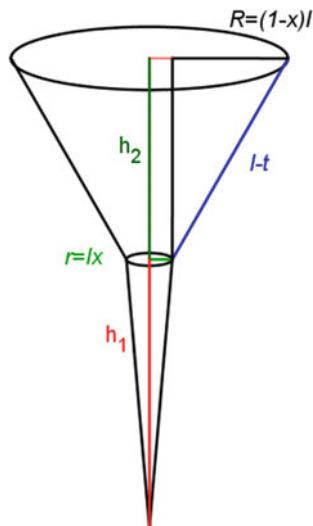
The length of the arc of the circular sector of measure α is $\frac{2\pi l\alpha}{360}$. The cone made of this sector will have a radius r of the base determined from the equation $2\pi r = \frac{2\pi l\alpha}{360}$, i.e., $r = lx$ where $x = \frac{\alpha}{360}$.

The radius t of the full circular sector mentioned in (b) is determined in such a way that the upper circle of the lower cone fits the lower circle of the upper truncated cone: $\frac{(360-\alpha)}{360} 2\pi t = 2\pi r$. Hence $t = \frac{r}{1-x}$. Note that the length of the generatrix of the truncated cone obtained in (b) is $l - t$. The resultant container is depicted in Fig. 17.16. As in Problem 3, we see that the radius R of the upper circle of the truncated cone is $R = (1 - x)l$. The altitude h_1 of lower cone is determined by Pythagoras's theorem: $h_1^2 = l^2 - r^2 = l^2(1 - x^2)$. The volume of the lower cone is

$$V_1 = \frac{\pi}{3} l^3 x^2 \sqrt{1 - x^2}.$$

The altitude h_2 of the truncated cone is determined similarly (using Pythagoras's theorem):

Fig. 17.16 Geometry behind the mathematical model



$$h_2 = \frac{l(1-2x)}{1-x} \sqrt{2x-x^2}.$$

The volume V_2 of the truncated cone is $V_2 = \frac{\pi}{3} h_2 (R^2 + Rr + r^2)$ where $R = (1-x)l$, $r = lx$, $R^2 + Rr + r^2 = l^2 \left((1-x)^2 + (1-x)x + x^2 \right) = l^2 \left((1-x)^2 + x \right)$.

Hence $V_2 = \frac{\pi}{3} l^3 \frac{1-2x}{1-x} \sqrt{2x-x^2} (1-x+x^2)$. The volume of the ice cream container is $V = V_1 + V_2$. We note here that x must belong to the interval $[0, \frac{1}{2}]$. This follows from the fact that the number $t = \frac{r}{1-x} = \frac{lx}{1-x}$ cannot be bigger than l .

Finding the maximum of V by means of calculus is a challenge. With the help of a DMS it can be found, as in the previous problem, that the maximal value of V is attained for $x \approx 0.23088$, which corresponds to $\alpha \approx 83.12^\circ$.

Here are some problems for further inquiry:

Problem 4.1 What is the minimal radius l of the initial circle from which the ice cream container is produced in the above way so that its volume is at least 200 cm^3 ?

Problem 4.2 A bucket (the far right of Fig. 17.17) with a circular base of radius $r = 10 \text{ cm}$ has to be made from a circular plastic sheet of radius $l = 60 \text{ cm}$ with center O by cutting and gluing (sticking). The cuts that are allowed and the order in which they are performed are:

- (a) Cut circles centered at O (i.e., concentric with the initial circle).
- (b) Cut from the remainder a radial segment of measure α (in degrees).

For what size of α will the volume of the bucket be the largest?

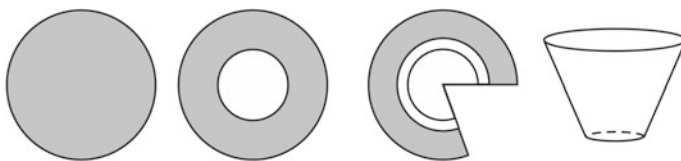


Fig. 17.17 The construction of a bucket

What is the largest possible volume of the bucket?

A computer model for Problem 4 was developed by Toni Chehlarova. It can be found at <http://cabinet.bg/content/bg/html/d22582.html> (visited December 2016).

A geometrical problem

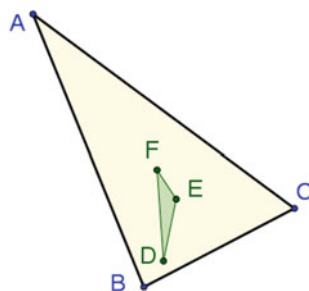
This is the last of the sample problems:

Problem 5 For an arbitrary triangle ABC , denote by D , E , and F its orthocenter, incenter, and the centroid, correspondingly (Fig. 17.18). Are there triangles ABC for which the area of the triangle DEF is bigger than the area of the triangle ABC itself?

This problem deviates in style from the previously considered problems. It contains a research-like component that is suitable for work on a project by the students. The computer model for this problem is easy to construct. The in-built operations of GeoGebra can be used to construct the orthocenter, the incenter, and the centroid of an arbitrary triangle. Using the “finding area of a polygon” command, the areas of the triangles ABC and DEF are calculated and displayed on the monitor. Due to the dynamic functionalities of GeoGebra, this computer model of Problem 5 allows to explore many triangles (by dragging some of the vertices A , B , and C). Playing with the vertices can experimentally establish that for some obtuse triangles ABC the answer to the question in Problem 5 is positive.

Note that this computer model solution of the problem does not require knowledge of more advanced mathematics (trigonometry, analytical geometry, etc.). It relies on the knowledge of the basic notions involved (orthocenter, incenter,

Fig. 17.18 The orthocenter D , the incenter E , and the centroid F of the triangle ABC



and centroid), on acquaintance with the functionalities of GeoGebra, and on some modeling skills.

Problem 5.1 For an arbitrary triangle ABC , find the area of the triangle with vertices at the orthocenter, the circumcenter, and the centroid of ABC .

Exploring this task with the corresponding computer model can show that the required area is always zero and, therefore, the three points are collinear (they lie on the famous Euler line of the triangle ABC).

The following simplified form of Problem 5 was given as one of the tasks in the competition VIVA Mathematics with Computer.

Problem 5.2 Given is a triangle ABC (by its sides or by the coordinates of its vertices; see Fig. 17.18). Find the area of the triangle with vertices at the orthocenter D , the incenter E , and the centroid F of the triangle ABC .

17.3 The Competitions Viva Mathematics with Computer and Theme of the Month

In order to examine the attitudes of Bulgarian students to problems like those in the previous section and to test the students' ability to solve such problems, two online competitions named VIVA Mathematics with Computer (VIVA MC) and Theme of the Month (TM) were launched in 2014 with the financial support of VIVACOM, a major telecommunication operator in the country (<https://www.vivacom.bg/bg>). The VIVA MC competition is for students from Grade 3 to Grade 12 and has two rounds. The first round is conducted twice during the academic year (in December and April) and is with open access. The second round takes place in September or early October and is only for the best performers in the December and April editions of the first round from the previous academic year. Pre-registration is needed at the VIVAcognita portal (<http://vivacognita.org/>) for participation in VIVA MC. Each registered student chooses how to participate in the competition: from any place with internet access by desktop, tablet, or laptop. On a fixed day and time every participant gets access for 60 min to a worksheet that contains 10 tasks corresponding to the participant's age group. The easier tasks are equipped with several possible answers. i.e., these are multiple-choice questions. The participant is expected to select the correct answer on the basis of performing some mathematical operations. The majority of the remaining tasks require a decimal number (usually up to two digits after the decimal point) as an answer that has to be entered in a special answer field. To find this answer, the student has to make a computer model of the task and explore it with the functionalities of DMS. Some of the most difficult tasks are accompanied by a file (a computer model) that solves a similar problem, and participants must modify the files accordingly in order to solve the tasks assigned to them. The number of points given for the answer to a task depends on both how close the student's answer is to the one calculated by the jury and/or by

the author of the task and the difficulty of the problem. The maximum possible score is 50 points. There are no restrictions concerning the use of resources: books, internet search, advice from specialists, etc. More information about this competition can be found in Chehlarova and Kenderov (2015). In April 2016, there were 474 participants while in December 2016 the number of participants was 1321. In both cases there were five age groups (two grades per group). An impression of the degree to which the participants were capable of solving problems with the help of DMS can be gained by the overview of their scores presented in Tables 17.1 and 17.2.

Students' scores in solving the problems from Sect. 17.2 were similar. Problem 5.2 from Sect. 17.2 was proposed as a last (presumably most difficult) task in the very first edition of VIVA MC (December 2014) to 207 students from Grades 8 to 12. The lack of experience with such problems and the short time to work on the problems (60 min) is clearly seen from the obtained results: About half of the students (48%) did not enter any answer for this task, 13% provided precise answer, and 2% gave an answer with satisfactory precision. The cylindrical container problem (Problem 2 from Sect. 17.2) was given to 317 students from Grade 8 to Grade 12 at the December 2015 edition of VIVA MC. An auxiliary DMS file was provided in order to facilitate the exploration of the problem. Only 13% provided an answer with sufficiently high precision. The answers of a further 37% were given with satisfactory precision. The general feeling has been that with every new edition of VIVA MC the performance of the participants improves, though rather gradually.

The other competition, TM, is conducted monthly. A theme of five tasks related to a common mathematical idea is published at the beginning of the month on the abovementioned portal (vivacognita.org). The tasks are arranged in the direction of increasing difficulty. The participants are expected to solve the problems and send responses online by the end of the month. Some of the problems are accompanied by auxiliary DMS files which allow the students to explore the mathematical

Table 17.1 Scores of participants in April 2016 competition VIVA MC

Grades in a group	3 and 4	5 and 6	7 and 8	9 and 10	11 and 12
Number of participants	146	142	79	67	40
Participants with 35–50 points	80	49	2	9	1
Participants with 20–34 points	44	59	15	21	16
Participants with 10–19 points	19	24	22	23	5

Table 17.2 Scores of participants in December 2016 competition VIVA MC

Grades in a group	3 and 4	5 and 6	7 and 8	9 and 10	11 and 12
Number of participants	449	385	268	123	86
Participants with 35–50 points	180	27	7	12	11
Participants with 20–34 points	147	146	24	29	28
Participants with 10–19 points	75	114	84	37	20

problem, find suitable properties, try out different strategies, and find a (usually approximate) solution. To solve the more difficult tasks from the theme, the students have to adapt the auxiliary files from previous problems or to develop their own files for testing and solving the problem. Each problem brings at most 10 points (depending on the degree of preciseness of the answer). The maximum total score is 50 points. Usually there are hundreds of visits to the site where the theme is published. Only dozens, however, submit solutions. The theme for February 2015 was related to the parking problem (Problem 1 from Sect. 17.2). Seventeen participants submitted their solutions. Seven received between 35 and 50 points and two received between 20 and 34. Much better were the results from the theme from September 2015, which was related to conical containers (Problems 3, 4, and 4.2 from Sect. 17.2). Sixteen students submitted their solutions, with 14 scoring between 41 and 50 points and one scoring 34 points. The results of the first several runs of TM are published in Kenderov et al. (2015) and Chehlarova and Kenderov (2015).

After the April 2017 edition of VIVA MC, the participants (more than 500) were asked to fill in a questionnaire and submit it to organizers. Of the 143 participants who returned the questionnaire, 95.51% said they liked the event. Here are some of their responses:

The problems are interesting because they require logical thinking.

I like it because I could use GeoGebra for each problem.

The contest is nice since I don't feel pressed when solving the problems.

The questions are at the right level for me.

It is interesting and helps me develop.

I find the problems entertaining.

It was easy for me to understand the formulation of the problem by means of the dynamic file I could use.

Every problem is interesting in its own way.

I like the fact that I can explore while solving the problem.

I like the parking entrance problem because it is something you could face in the real world.

This relatively modest feedback confirms the expectation that providing the students with appropriate exploration tools can increase their awareness of both the beauty and the applicability of mathematics.

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